Symmetry and Separability In Spatial-Temporal Processes

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In general, environmental data have very complex spatial-temporal dependency structures. Standard assumptions are not enough to capture these dependency structures. Nonseparability, even nonstationarity may not be helpful. New solution may be needed. Here we develop new models and methods to estimate and predict air-pollution data.
Research Objectives

1. New Class of Asymmetric Spatial -Temporal Covariance Models
   • Define Symmetry under the Space-Time Setting
   • Propose New Classes of Asymmetric Covariance Models

   • Introduce New Formal Tests for Lack of Symmetry

3. Simulation Study & Real Application
   • Study the Performance of the Testing Methods
   • Apply to Real Air-Pollution Datasets
Literature Review

1. Tests for Separability and Stationarity


- Fuentes (2005): spectral methods for testing separability and stationarity in space
2. New class of Nonseparable Covariance Models

- Cressie and Huang (1999), Gneiting (2002)
- Fuentes et al. (2005): a mixture of local spectrums

\[ C(s_i - s_j; t_p - t_q) = \sum_{n=1}^{M} K(s_i - s_n)K(s_j - s_n)C_n(s_i - s_j; t_p - t_q) \]
3. Symmetry

- Scaccia and Martin (2005)
  - Introduce Symmetry and Separability in Spatial (Lattice) Process
    * Axial Symmetry: $C(h_1, h_2) = C(h_1, -h_2)$
    * Diagonal Symmetry: $C(h_1, h_2) = C(h_2, h_1)$
    * Separability: $C(h_1, h_2) \propto C_1(h_1, 0) \cdot C_2(0, h_2)$
  - Propose new tests for Symmetry and Separability based on Periodograms

- Lu, and Zimmerman (2005), Stein (2005)
Limitations

- No generalization of symmetry in spatial-temporal processes
- No formal tests for symmetry in spatial-temporal processes
- No class of asymmetric spatial-temporal covariance models
Definition: Stationarity

For \( \{Z(s; t) : s = (s_1, s_2)' \in \mathcal{D} \subset \mathbb{R}^2 ; t \in \mathcal{T} \subset \mathbb{R} \} \) with

\[
\text{Cov}\{Z(s_i; t_k), Z(s_j; t_l)\} = C(s_i - s_j; t_k - t_l),
\]

- Weak (Second-Order) Stationarity
  1) \( E\{Z(s_i; t_k)\} = \mu \), where \( \mu \) is constant
  2) for \( h \in \mathbb{R}^2 \) and \( u \in \mathbb{R} \),

\[
\text{Cov}\{Z(s_i; t_k + u), Z(s_i; t_k)\} = C(h; u) < \infty
\]

- Strong Stationarity
  1) for any finite \( N \) and domain \( \mathcal{D} \times \mathcal{T} \), with \( \{S_1, \cdots, S_N\} \subset \mathcal{D} \times \mathcal{T} \) and \( z_1, \cdots, z_N \in \mathbb{R} \),

\[
\text{Pr}(Z(S_1 + \mathbf{S}) \leq z_1, \cdots, Z(S_N + \mathbf{S}) \leq z_N) = \text{Pr}(Z(S_1) \leq z_1, \cdots, Z(S_N) \leq z_N)
\]
Definition: Separability

For \( \{Z(s; t) : s = (s_1, s_2) \in \mathbb{D} \subset \mathbb{R}^2 ; t \in \mathbb{T} \subset \mathbb{R} \} \) with

\[
\text{Cov}\{Z(s_i; t_k), Z(s_j; t_l)\} = C(s_i - s_j; t_k - t_l),
\]

- Separability

\[
\text{Cov}\{Z(s_i + h; t_k + u), Z(s_i; t_k)\} = C_s(s_i + h, s_i) \cdot C_T(t_k + u, t_k),
\]

- \( C_s(\cdot) \) is a spatial covariance and \( C_T(\cdot) \) is a temporal covariance

- Separability under Stationarity

\[
\text{Cov}\{Z(s_i + h; t_k + u), Z(s_i; t_k)\} = C_s(h) \cdot C_T(u)
\]
Figure 1: separability

- there are ridges along the lines where $\|h\| = 0$ or $u = 0$
Definitions: Symmetry

For \( \{ Z(s; t) : s = (s_1, s_2, \cdots, s_d)' \in \mathbb{D} \subset \mathbb{R}^d; t \in \mathbb{T} \subset \mathbb{R} \} \) with

\[
\text{Cov}\{Z(s_i; t_k), Z(s_j; t_l)\} = C(s_i - s_j; t_k - t_l) \equiv C(h; u),
\]

**Definition 1.** A process is called axially symmetric in time if

\[
C(h; u) = C(h; -u),
\]

where all possible \( h = (h_1, h_2, \cdots, h_d)' \neq 0 \) for \( u \neq 0 \).
Figure 2: Axial Symmetry in Time
Definition 2. A process is called axially symmetric in space if

\[ C(h; u) = C(\hat{h}; u), \]

where \( \hat{h} = (h_1, \ldots, h_{k-1}, -h_k, h_{k+1}, \ldots, h_d)' \) for \( k \) fixed.
Figure 3: Axial Symmetry in Space
Definition 3. A process is called diagonally symmetric in space if

\[ C(h; u) = C(\bar{h}; u), \]

where \( \bar{h} = (h_1, \ldots, h_{k-1}, h_l, h_{k+1}, \ldots, h_{l-1}, h_k, h_{l+1}, \ldots, h_d)' \) for \( k \neq l \).
Figure 4: Diagonal Symmetry in Space
Symmetry (2-Dim Spatial Domain)

For \( \{Z(s; t) : s = (s_1, s_2)' \in D \subset \mathbb{R}^2; t \in T \subset \mathbb{R}\} \) with

\[
\text{Cov}\{Z(s_i; t_k), Z(s_j; t_l)\} = C(s_i - s_j; t_k - t_l) \equiv C(h; u),
\]

- Axial Symmetry in Time:
  \[
  C(h; u) = C(h_1, h_2; u) = C(h_1, h_2; -u) = C(h; -u)
  \]

- Axial Symmetry in Space:
  \[
  C(h; u) = C(h_1, h_2; u) = C(h_1, -h_2; u) = C(\hat{h}; u)
  \]

- Diagonal Symmetry in Space:
  \[
  C(h; u) = C(h_1, h_2; u) = C(h_2, h_1; u) = C(\ddot{h}; u)
  \]
Background: Matern-type Covariance

\[ C(u) = \frac{\pi \gamma}{2^{\nu-1} \Gamma(\nu)} (\alpha |u|)^\nu K_\nu (\alpha |u|) \]
\[ , \quad f(\omega) = \gamma (\alpha^2 + |\omega|^2)^{-\nu - 1/2} \]

- \( \nu \) measures the degree of smoothness
- \( \alpha \) explains the rate of decay of correlation
A New Class of Asymmetric Stationary Models

- Propose the spatial-temporal spectral density function given by

\[
    f_\mathbf{v}(\omega; \tau) = \gamma \left( \alpha^2 \beta^2 + \beta^2 \| \omega + \tau \mathbf{v}_1 \|^2 + \alpha^2 (\tau + \mathbf{v}'_2 \omega)^2 \right)^{-\nu} \\
    = f_0(\omega + \tau \mathbf{v}_1; \tau + \mathbf{v}'_2 \omega),
\]

where \( f_0(\omega, \tau) = \gamma (\alpha^2 \beta^2 + \beta^2 \| \omega \|^2 + \alpha^2 \tau^2)^{-\nu} \) and \( \mathbf{v} = (\mathbf{v}'_1, \mathbf{v}'_2)' \)

- \( f_\mathbf{v}(\omega, \tau) > 0 \) everywhere
- \( f_\mathbf{v}(\omega, \tau) < \infty \) for \( \forall \mathbf{v} \)
A New Class of Asymmetric Covariance Models

- Asymmetric Stationary Covariance Function:

\[
C_{v}(h; u) = \int_{\mathbb{R}^d} \int_{\mathbb{R}} \exp\{ih'\omega + iu\tau\}f_0(\omega + \tau v_1; \tau + v_2'\omega) \, d\tau \, d\omega
\]

\[
= \frac{1}{1 - v_1'v_2} \, C_0 \left( \frac{\tilde{h} - uv_2}{1 - v_1'v_2}, \frac{u - h'v_1}{1 - v_1'v_2} \right),
\]

where \( C_0 \) is a stationary spatial-temporal covariance function simply transformed from \( f_0 \), and

\[
\tilde{h} = \left\{ \tilde{h}_i \right\}_{i=1}^{d} = \left\{ \left( 1 - \sum_{j \neq i} v_{1j}v_{2j} \right) h_i + v_{2i} \sum_{j \neq i} v_{1j}h_j \right\}_{i=1}^{d}
\]
A New Class of Asymmetric Covariance Models

- Closed Form of Asymmetric Covariance Function:

\[
C_v(h; u) = \frac{1}{1 - v_1'v_2} \times \frac{\gamma \pi^{(d+1)/2} \alpha^{-2\nu + d} \beta^{-2\nu + 1}}{2^{\nu-(d+1)/2-1} \Gamma(\nu)}
\times M_{\nu - \frac{d+1}{2}} \left( \sqrt{\frac{\alpha \|\tilde{h} - u v_2\|}{1 - v_1'v_2}} \right)^2 + \left( \frac{\beta (u - h'v_1)}{1 - v_1'v_2} \right)^2, \]

\[
- M_\nu(r) \equiv r^\nu K_\nu(r)
\]
A New Class of Asymmetric Covariance Models

- $\mathbf{v} = (\mathbf{v}_1', \mathbf{v}_2')'$ controls (A)symmetry.
  - Axial Symmetry in Time if $\mathbf{v}_1 = \mathbf{v}_2 = 0$,
  - Axial Symmetry in Space if $v_{11} \neq 0$ or $v_{21} \neq 0$ and $v_{12} = v_{22} = 0$,
  - Diagonal Symmetry in Space if $v_{11} = v_{12} = v_{10}$, $v_{21} = v_{22} = v_{20}$, and at least one of $v_{10}$ and $v_{20}$ is nonzero,
  - Asymmetry in Space and Time otherwise.
A New Class of Asymmetric Covariance Models

\[ C_v(h; u) = \frac{1}{1 - v_1'v_2} C_0 \left( \frac{\tilde{h} - uv_2}{1 - v_1'v_2}; \frac{u - h'v_1}{1 - v_1'v_2} \right) \]

- Temporal component \((u - h'v_1)\)
  - the units of \(v_1\) are time divided by distances
  - reciprocals of speed

- Spatial component \((\tilde{h} - u v_2)\)
  - the units of \(v_2\) are distances divided by time
  - velocities

Symmetry and Separability In Spatial-Temporal Processes
A New Class of Asymmetric Covariance Models

- Subclass of Asymmetric Covariance Models
  - $v_2 = 0$
    
    $$C_{v_1}(h; u) \propto \mathcal{M}_{\nu - \frac{d+1}{2}} \left( \alpha \sqrt{ \left\{ \frac{\beta(u - h'v_1)}{\alpha} \right\}^2 + \|h\|^2 } \right)$$
  
  - $v_1 = 0$
    
    $$C_{v_2}(h; u) \propto \mathcal{M}_{\nu - \frac{d+1}{2}} \left( \alpha \sqrt{ \left( \frac{\beta u}{\alpha} \right)^2 + \|h - uv_2\|^2 } \right)$$
**Axial Symmetry in Time**

\[ C(h_1, h_2; u) = C(h_1, h_2; -u) \]

(a) for any \( u \)

(b) for any \( h_2 \) (or \( h_1 \))

- the covariance function is invariant to the change of any lags.
**Axial Symmetry in Space** \( C(h_1, h_2; u) = C(h_1, -h_2; u) \)

(c) \( u = -10, 0, 10 \)    (d) \( h_1 = -200, 0, 200 \)

- the covariance function is invariant to the change of \( h_2 \).
- the centroids lie on the axis of \( h_2 = 0 \).
Diagonal Symmetry in Space $C(h_1, h_2; u) = C(h_2, h_1; u)$

- the covariance function depends on the spatial lags, $h_1$ and $h_2$.
- the centroids lie on the line with slope of $v_{12}/v_{11} = 1$. 

(e) $u = -10, 0, 10$

(f) $h_2(h_1) = -200, 0, 200$
Asymmetry in Space and Time: $v_{11} \neq 0$, $v_{12} \neq 0$, $v_{11} \neq v_{12}$

- the covariance function depends on the spatial lags, $h_1$ and $h_2$.
- the centroids lie on the line with slope of $v_{12}/v_{11}$.

\[(g) \ u = -10, 0, 10\]
Asymmetry in Space and Time: $v_{11} \neq 0$, $v_{12} \neq 0$, $v_{11} \neq v_{12}$

(h) $h_2 = -400, 0, 400$

(i) $h_1 = -400, 0, 400$

- the centroids lie on the axes of the spatial lags.
- how much the centroids are shifted is related to $|v_{11}|$ and $|v_{12}|$. 
• Both consider spatial-temporal dependency (interaction)
  – **Separability (Fuentes et al. (2005))**
    \[ f_\epsilon(\omega, \tau) = \gamma \left( \alpha^2 \beta^2 + \beta^2 \|\omega\|^2 + \alpha^2 \tau^2 + \epsilon \|\omega\|^2 \tau^2 \right)^{-\nu} \]
  – **Symmetry (Our Model)**
    \[ f_\nu(\omega; \tau) = \gamma \left( \alpha^2 \beta^2 + \beta^2 \|\omega + \tau v_1\|^2 + \alpha^2 (\tau + v'_2 \omega)^2 \right)^{-\nu} \]
• Every symmetric covariance is always “nonseparable”
Figure 5: PM$_{2.5}$ daily concentrations obtained from FRM network (18 stations; 08/01/03–08/31/03)
Table 1: Parameter Estimates based on Gaussian Asymmetric Spatial-Temporal Covariance from WLS and ML methods. Note that \((-4) \equiv 10^{-4}\).

<table>
<thead>
<tr>
<th>Θ</th>
<th>M.1 WLS</th>
<th>M.1 ML</th>
<th>M.2 WLS</th>
<th>M.2 ML</th>
<th>M.3 WLS</th>
<th>M.3 ML</th>
<th>M.4 WLS</th>
<th>M.4 ML</th>
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<td>v_{11}</td>
<td>6.(-4)</td>
<td>1.(-4)</td>
<td></td>
<td></td>
<td>1.(-3)</td>
<td>2.(-3)</td>
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<tr>
<td>v_{12}</td>
<td>-6.(-4)</td>
<td>-3.(-4)</td>
<td></td>
<td></td>
<td>-1.(-3)</td>
<td>1.(-3)</td>
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<td>v_{21}</td>
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<td>20.065</td>
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<td>v_{22}</td>
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Discussion

• Introduce new concepts of symmetry in spatial-temporal processes

• Propose new classe of asymmetric stationary spatial-temporal covariance models

• The environmental data influenced by some external meteorological conditions
1. Axial Symmetry in Time: $C(h_1, h_2; u) = C(h_1, h_2; -u)$

- Cross-spectrum between $Z(a; t)$ and $Z(b; t)$ as

$$f_{ab}(\tau) \equiv (2\pi)^{-1} \int_{\mathbb{R}} \exp\{-i u \tau\} C(a - b; u) \, du = f_{ba}^{c}(\tau)$$

- Under $\mathcal{H}_0$: $C(h; u) = C(h; -u)$

$$C(h; u) = C(h; -u) \iff f_{ab}(\tau) = f_{ba}(\tau) \iff \text{Im.} f_{ab}(\tau) = 0 \iff \phi_{ab}(\tau) \equiv \tan^{-1}\left\{\frac{\text{Im.} f_{ab}(\tau)}{\text{Re.} f_{ab}(\tau)}\right\} = 0$$
1. Axial Symmetry in Time: \( C(h_1, h_2; u) = C(h_1, h_2; -u) \)

Let \( \hat{\phi}_{ab}^*(\tau) = \frac{\hat{\phi}_{ab}(\tau)}{\left[|\hat{R}_{ab}(\tau)|^{-2} - 1\right]^{1/2}} \)

- Asymptotic Distribution of \( \sqrt{B_T T} \left[ \hat{\phi}_{ab}^*(\tau) - \hat{\phi}_{ab}(\tau) \right] \)

\[
\sim \mathcal{N} \left( 0, \pi \left( \int_{\mathbb{R}} W^2(\alpha) \, d\alpha \right) [1 - \eta\{2\tau\}] \right), \quad (\text{Brillinger (2001))}
\]
Fuentes (2006)

- Asymptotic normality of cross-spectral density function, $\hat{f}_{ab}(\tau)$
- $\hat{f}_{a_i b_i}(\tau_k)$ is approximately independent of $\hat{f}_{a_j b_j}(\tau_l)$ if either
  
  **C.1** $\|\tau_k + \tau_l\|$ is sufficiently large so that
  
  $$\int_{\mathbb{R}} |W(\alpha + \tau_k)|^2 |W(\alpha + \tau_l)|^2 \, d\alpha = 0,$$
  
  i.e. $\|\tau_k + \tau_l\| \gg$ bandwidth of $|W(\alpha)|^2$ or
  
  **C.2** the distance between pairs $(a_i, b_i)$ and $(a_j, b_j)$ is greater than the bandwidth of the function $\{g_\rho(s)\}$. 
1. **Axial Symmetry in Time:** \( C(h_1, h_2; u) = C(h_1, h_2; -u) \)

Let \( \hat{\phi}_{ik}^* \equiv \hat{\phi}_{a_i b_i}(\tau_k) / \left[ |\hat{R}_{a_i b_i}(\tau_k)|^{-2} - 1 \right]^{1/2} \)

Under \( \mathcal{H}_0 \): \( \phi_{a_i b_i}(\tau_k) = 0 \) and \( \phi_{a_j b_j}(\tau_l) = 0 \)

- Asymptotic Distribution of \( \hat{\Phi}^* \equiv (\hat{\phi}_{ik}^*, \hat{\phi}_{jl}^*)' \)

\[ \sqrt{B_T T} \left( \hat{\Phi}^* - \Phi^* \right) \sim \mathcal{N}_2(0, A' \Sigma A), \]

- \( A = \partial \Phi^*/\partial \theta' \), where \( \theta = (\phi_{ik}, R_{ik}, \phi_{jl}, R_{jl})' \)

- \( \Sigma \) is a variance-covariance matrix of \( \sqrt{B_T T} \left( \hat{\theta} - \theta \right) \)

- \( \hat{f}_{a_i b_i}(\tau_k) \overset{ind}{\sim} \hat{f}_{a_j b_j}(\tau_l) \iff \hat{\phi}_{a_i b_i}(\tau_k) \overset{ind}{\sim} \hat{\phi}_{a_j b_j}(\tau_l) \)
1. Axial Symmetry in Time: \( C(h_1, h_2; u) = C(h_1, h_2; -u) \)

- Two-way ANOVA procedure:
  \[
  \hat{\phi}_{ik}^* = \alpha_i + \beta_k + \epsilon_{ik},
  \]
  \( \epsilon_{ik} \sim N \left( 0, \sigma_{\epsilon}^2 \right), \forall i, k \)
  \( \text{Cov} \left( \epsilon_{ik}, \epsilon_{jl} \right) = 0, \forall i, j, k, l \) satisfying C.1 or C.2

- Check axial symmetry in time by studying if \( \beta_k = 0, \forall k \)
- Check stationarity by studying if \( \alpha_i = 0, \forall i \)
2. Axial Symmetry in Space: \( C(h_1, h_2; u) = C(-h_1, h_2; u) \)

- Cross-spectrum between \( Z(s, a_2 + h_2; t + u) \) and \( Z(s, a_2; t) \) as
  \[
  k(\omega; h_2, u) = (2\pi)^{-1} \int_{\mathbb{R}} \exp\{-ih_1\omega\} C(h_1, h_2; u) \, dh_1
  \]
  \[
  = k^c(-\omega; h_2, u)
  \]

- Under \( \mathcal{H}_0 \): \( C(h_1, h_2; u) = C(-h_1, h_2; u) \)

\[
C(h_1, h_2; u) = C(-h_1, h_2; u) \iff k(\omega; h_2, u) = k(-\omega; h_2, u)
\]
\[
\iff \text{Im.} k(\omega; h_2, u) = 0
\]
\[
\iff \psi(\omega; h_2, u) \equiv \tan^{-1} \left\{ \frac{\text{Im.} k(\omega; h_2, u)}{\text{Re.} k(\omega; h_2, u)} \right\} = 0
\]
2. Axial Symmetry in Space: \( C(h_1, h_2; u) = C(-h_1, h_2; u) \)

Suppose \( Z \) is observed at \( N(N_1 \cdot N_2) \) spatial sites on regular grids with unit spacing \( \Delta = (\Delta_1, \Delta_2)' \) and at same measuring times \( T \).

Let \( \hat{\psi}^*_{\Delta_1}(\omega; h_2, u) = \hat{\psi}_{\Delta_1}(\omega; h_2, u) \left/ \left| \hat{Q}_{\Delta_1}(\omega; h_2, u) \right|^{-2} - 1 \right. \)\(^{1/2} \)

- Asymptotic Distn of \( \sqrt{B_{N_1} N_1} \left[ \hat{\psi}^*_{\Delta_1}(\omega; h_2, u) - \psi^*_{\Delta_1}(\omega; h_2, u) \right] \)

\[ \sim N \left( 0, \pi \left( \int_{\mathbb{R}} W^2(\alpha) \, d\alpha \right) [1 - \eta\{2\omega\}] \right), \quad \text{(Brillinger (2001))} \]
2. Axial Symmetry in Space: \( C(h_1, h_2; u) = C(-h_1, h_2; u) \)

Let \( \hat{\psi}^*_m \equiv \hat{\psi}^*_1 (\omega_m; a_i^2 - b_i^2, t_i^a - t_i^b) \)

Under \( H_0: \psi_{im} = 0 \) and \( \psi_{jn} = 0 \)

- Asymptotic Distribution of \( \hat{\Psi}^* \equiv (\hat{\psi}^*_m, \hat{\psi}^*_j)' \)

\[
\sqrt{B_{N_1} N_1} \left( \hat{\Psi}^* - \Psi^* \right) \sim N_2 \left( 0, B' \Omega B \right),
\]

- \( B = \partial \Psi^*/\partial \theta' \), where \( \theta = (\psi_{im}, Q_{im}, \psi_{jn}, Q_{jn})' \)

- \( \Omega \) is a variance-covariance matrix of \( \sqrt{B_{N_1} N_1} \left( \hat{\theta} - \theta \right) \)

- \( \hat{k}_{\Delta_1} (\omega_m; a_i^2 - b_i^2, t_i^a - t_i^b) \) \( \overset{\text{ind}}{\sim} \) \( \hat{k}_{\Delta_1} (\omega_n; a_j^2 - b_j^2, t_j^a - t_j^b) \)

\( \iff \) \( \hat{\psi}_{\Delta_1} (\omega_m; a_i^2 - b_i^2, t_i^a - t_i^b) \) \( \overset{\text{ind}}{\sim} \) \( \hat{\psi}_{\Delta_1} (\omega_n; a_j^2 - b_j^2, t_j^a - t_j^b) \)
2. Axial Symmetry in Space: \( C(h_1, h_2; u) = C(-h_1, h_2; u) \)

- Two-way ANOVA procedure:
  \[
  \widehat{\psi}_{im} = \gamma_i + \delta_m + e_{im},
  \]
  \(- e_{im} \sim \mathcal{N}(0, \sigma^2_e), \ \forall i, m \)
  \(- \text{Cov}(e_{im}, e_{jn}) = 0 \)

- Check axial symmetry in space by studying if \( \delta_m = 0, \ \forall m \)
- Check spatial-temporal interaction by studying if \( \gamma_i = 0, \ \forall i \)
Description: Air-Polution Data

- Models-3/Community Multiscale Air Quality (CMAQ) model
  - the hourly concentrations of air pollutants with the resolution of $36km \times 36km$
  - the Particulate Matter with diameter less than $2.5\mu m$ (PM$_{2.5}$)
  - one of the critical factors for public health problem
Figure 6: PM$_{2.5}$ daily concentrations obtained from CMAQ output 
$(61(N_1) \times 61(N_2) \times 363(T))$
1. Testing Lack of Axial Symmetry in Time

Figure 7: The plot of the locations of 16 pairs; ENE direction (D1), NNE direction (D2), NNW direction (D3), and WNW (D4).
1. Testing Lack of Axial Symmetry in Time

Table 2: Analysis of variance

<table>
<thead>
<tr>
<th>Item</th>
<th>DF</th>
<th>Sum of squares</th>
<th>$F$ value</th>
<th>Pr($F$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Directions</td>
<td>4</td>
<td>1.6028</td>
<td>15.9463</td>
<td>0.0000</td>
</tr>
<tr>
<td>Locations</td>
<td>12</td>
<td>0.9811</td>
<td>3.2537</td>
<td>0.0003</td>
</tr>
<tr>
<td>Frequencies</td>
<td>12</td>
<td>0.1681</td>
<td>0.5576</td>
<td>0.8736</td>
</tr>
<tr>
<td>Residuals</td>
<td>180</td>
<td>4.5229</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Apparent evidence of nonstationarity exists.
- Nonzero $\hat{\phi}_{ijk}^*$ mainly comes from the “Directions”.
- We can not reject “Axial Symmetry in Time”.
1. Testing Lack of Axial Symmetry in Time

Table 3: Analysis of variance

<table>
<thead>
<tr>
<th>Direction</th>
<th>Item</th>
<th>DF</th>
<th>SS</th>
<th>$F$ value</th>
<th>Pr($F$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>D1</td>
<td>Locations</td>
<td>4</td>
<td>1.3933</td>
<td>25.8807</td>
<td>0.0000</td>
</tr>
<tr>
<td></td>
<td>Frequencies</td>
<td>12</td>
<td>0.4283</td>
<td>2.6519</td>
<td>0.0118</td>
</tr>
<tr>
<td></td>
<td>Residuals</td>
<td>36</td>
<td>0.4845</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D2</td>
<td>Locations</td>
<td>4</td>
<td>0.4026</td>
<td>4.0371</td>
<td>0.0083</td>
</tr>
<tr>
<td></td>
<td>Frequencies</td>
<td>12</td>
<td>0.1201</td>
<td>0.4016</td>
<td>0.9536</td>
</tr>
<tr>
<td></td>
<td>Residuals</td>
<td>36</td>
<td>0.8974</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D3</td>
<td>Locations</td>
<td>4</td>
<td>0.3358</td>
<td>2.5233</td>
<td>0.0578</td>
</tr>
<tr>
<td></td>
<td>Frequencies</td>
<td>12</td>
<td>0.2819</td>
<td>0.7059</td>
<td>0.7354</td>
</tr>
<tr>
<td></td>
<td>Residuals</td>
<td>36</td>
<td>1.1979</td>
<td></td>
<td></td>
</tr>
<tr>
<td>D4</td>
<td>Locations</td>
<td>4</td>
<td>0.4522</td>
<td>5.3382</td>
<td>0.0018</td>
</tr>
<tr>
<td></td>
<td>Frequencies</td>
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<td>0.5186</td>
<td>2.0406</td>
<td>0.0491</td>
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<tr>
<td></td>
<td>Residuals</td>
<td>36</td>
<td>0.7624</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

• “Axial Symmetry in Time” can be rejected for some directions.
1. Testing Lack of Axial Symmetry in Time

![QQplots of the residuals](image)

(a) for D1.  
(b) for D4.

**Figure 8: The QQplots of the residuals**

- The normality condition for error term is reasonable.
2. Testing Lack of Axial Symmetry in Space

Figure 9: The plot of the locations of 16 pairs; D5, D6.
2. Testing Lack of Axial Symmetry in Space

Table 4: Analysis of variance

<table>
<thead>
<tr>
<th>Item</th>
<th>DF</th>
<th>Sum of squares</th>
<th>$F$ value</th>
<th>Pr($F$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Directions</td>
<td>2</td>
<td>0.0263</td>
<td>0.0273</td>
<td>0.9730</td>
</tr>
<tr>
<td>Interactions</td>
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<td>4.8976</td>
<td>0.7262</td>
<td>0.7400</td>
</tr>
<tr>
<td>Frequencies</td>
<td>4</td>
<td>6.6600</td>
<td>3.4564</td>
<td>0.0132</td>
</tr>
<tr>
<td>Residuals</td>
<td>60</td>
<td>28.9032</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Lack of “Axial Symmetry in Space” exists regardless of the direction.
2. Testing Lack of Axial Symmetry in Space

Table 5: Analysis of variance

<table>
<thead>
<tr>
<th>Direction</th>
<th>Item</th>
<th>DF</th>
<th>SS</th>
<th>( F ) value</th>
<th>Pr((F))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Interactions</td>
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<td>1.5599</td>
<td>0.3106</td>
<td>0.9555</td>
</tr>
<tr>
<td>D5</td>
<td>Frequencies</td>
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<td>3.7268</td>
<td>1.4842</td>
<td>0.2336</td>
</tr>
<tr>
<td></td>
<td>Residuals</td>
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<td>17.5772</td>
<td></td>
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</tr>
<tr>
<td>D6</td>
<td>Interactions</td>
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<td>3.3641</td>
<td>1.2229</td>
<td>0.3222</td>
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<td></td>
<td>Frequencies</td>
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</tr>
<tr>
<td></td>
<td>Residuals</td>
<td>28</td>
<td>9.6284</td>
<td></td>
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</tr>
</tbody>
</table>

- Lack of “Axial Symmetry in Space” comes from the direction D6.
Conclusions & Further Study

+ Concept of Symmetry in Spatial-Temporal Processes
+ Formal Tests for Lack of Symmetry
+ Classes of Asymmetric Spatial-Temporal Covariance Models
− Bayesian Estimation Approach
Thank you very much !!