Nonparametric Model-Assisted Estimation of Distribution Functions from Survey Data

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Specific versus Generic

- **Specific (brand-name?):** not a black box; expensive; very good for its purpose
- **Generic:** pretty much a black box; cheap; good for many purposes
Finite Population CDF Estimation

- Some notation:
  \[ F(t) = \frac{1}{N} \sum_{i \in U} I\{y_i \leq t\} \]
  where \( U = \{1, 2, \ldots, N\} \) ("landscape")
  - \( y_i \) observed for sample \( s \subset U \) of size \( n \)
  - auxiliary information \( x_i \) available for all of \( U \)

- **Idea:** Use auxiliary information to improve finite population distribution function estimation
  - model \( (x_i, y_i) \) relationship and use to predict non-sampled \( y_i, i \in U - s \)
Modeling Contexts

• **Specific:** few study variables, few population parameters
  – lots of modeling resources to specify, estimate, and diagnose a model
  – willingness to defend the model

• **Generic:** many study variables, many population parameters
  – no resources to model every variable
  – no single model is adequate/defensible
Generic Inferences in Natural Resources Monitoring

• **Example:** conduct a survey and prepare a report
  – analyze large numbers of chemical, biological, and physical variables
  – estimate means, quantiles, and distribution functions
  – break down both by political classifications and by various ecological classifications

• Generic inference is a common problem for federal, state, and tribal agencies
Very Generic: Horvitz-Thompson Estimator

- Let $\pi_i = \Pr \{i \in s\}$, $\pi_{ij} = \Pr \{i, j \in s\}$ and $\Delta_{ij} = \pi_{ij} - \pi_i \pi_j$

- Then

$$\hat{F}_{HT}(t) = \frac{1}{N} \sum_{i \in s} \frac{I\{y_i \leq t\}}{\pi_i}$$

and

$$\overline{\text{Var}}(\hat{F}_{HT}(t)) = \frac{1}{N^2} \sum_{i,j \in s} \frac{\Delta_{ij} I\{y_i \leq t\} I\{y_j \leq t\}}{\pi_i \pi_j \pi_i \pi_j}$$

are design unbiased and consistent
- no dependence on any model
- does not incorporate auxiliary information $x_i$
Estimation with Auxiliary Information

- **Superpopulation model:**
  \[ y_i = m(x_i) + v^{1/2}(x_i)\epsilon_i \]
  where \( \epsilon_i \sim G \) with \( E(\epsilon_i) = 0, \ Var(\epsilon_i) = \sigma^2 \)

- **Model-based:** biased if model is wrong
  \[ \sum_{i \in U-s} (\text{model-based prediction}) + \sum_{i \in s} (\text{sampled values}) \]

- **Model-assisted:** design-unbiased even if model is wrong
  \[ \sum_{i \in U} (\text{model-based prediction}) + (\text{design bias adjustment}) \]
Model-Based Parametric: CD Estimator

- Chambers and Dunstan (1986)
  - model-based
    
    \[
    \hat{F}_{CD}(t) = \frac{1}{N} \sum_{i \in \bar{U} - s} \hat{G}_i + \frac{1}{N} \sum_{i \in s} I\{y_i \leq t\}
    \]

    - \(\hat{G}_i\) estimates \(G\left(\frac{t - m(x_i)}{v^{1/2}(x_i)}\right) = E_m \left(I\{y_i \leq t\}\right)\)
    - asymptotically unbiased when \(m(x_i)\) and \(v(x_i)\) correctly specified
Model-Assisted Parametric: RKM Estimator

- Rao, Kovar, Mantel (1990)
  - model-assisted

\[
\hat{F}_{RKM}(t) = \frac{1}{N} \sum_{i \in U} \tilde{G}_i + \sum_{i \in S} \frac{I\{y_i \leq t\}}{N\pi_i} \tilde{G}_{ic}
\]

where \(\tilde{G}_{ic}\) is \(\tilde{G}_i\) weighted with conditional probabilities

- asymptotically design and model unbiased
Motivation for Nonparametric Methods

\[ E_m I_{y_i \leq t} = \Pr \{ y_i \leq t \} = G\left(v^{-1/2}(x_i)(t - m(x_i))\right) \]

• **Mean function misspecification**
  - CD will be biased
  - RKM will be inefficient
  - nonparametric methods only assume \( m(x_i) \) is smooth

• **Variance misspecification**
  - CD and RKM assume \( v(x_i) \) is known
  - nonparametric assumes \( v(x_i) \) smooth, positive
Possible Smoothing Strategies

- **Specific:** use original response
  - smooth $y_i$ versus $x_i$ to get $\hat{m}(x_i)$
  - smooth $(y_i - \hat{m}(x_i))^2$ versus $x_i$ to get $\hat{v}(x_i)$
  - plug in to CD- or RKM-like estimator

- **Generic:** use indicators
  - smooth $I\{y_i \leq t\}$ versus $x_i$ to get $\hat{G}(v^{-1/2}(x_i)(t-m(x_i)))$
  - plug in to model-based or model-assisted estimator
Possible Smoothing Strategies

- Smooth response $y_i$ or indicator $I\{y_i \leq t\}$ versus $x_i$.

![Bump vs. x](image)

![Indicator mean at quartile vs. x](image)
A Nonparametric Method: Local Polynomial Regression

- Smooth function locally approximated by $q$th-order polynomial

- Sample design matrix $(n \times (q + 1))$:

$$X_{si} = \begin{bmatrix} 1 & x_j - x_i & \cdots & (x_j - x_i)^q \end{bmatrix}_{j \in s}$$

- Sample weighting matrix $(n \times n)$:

$$W_{si} = \text{diag}\left\{ \frac{1}{\pi_j h} K \left( \frac{x_j - x_i}{h} \right) \right\}_{j \in s}$$

- Sample smoother vector at $x_i$:

$$s'_{si} = [1 \ 0 \ \cdots \ 0] (X'_{si} W_{si} X_{si})^{-1} X'_{si} W_{si}$$
LPR in Survey Sampling Estimation

- Finite population total: \( T_y = \sum_{i \in U} y_i \)
- Breidt and Opsomer (2000):

\[
\hat{T}_{LPR} = \sum_{i \in U} \hat{m}_i + \sum_{i \in s} \frac{y_i - \hat{m}_i}{\pi_i}
\]

where \( \hat{m}_i = \mathbf{s}'_i[y_i]_{i \in s} \), and

\[
\text{Var}(\hat{T}_{LPR}) = \sum_{i,j \in s} \frac{\Delta_{ij}(y_i - \hat{m}_i)(y_j - \hat{m}_j)}{\pi_{ij}\pi_i\pi_j}
\]

are asymptotically design unbiased and consistent

- comparable efficiency to REG under linear model
- more efficient than REG otherwise
Local Polynomial Regression CDF Estimator

- Model-assisted approach
- Define $I_s = [I_{y_i \leq t}]_{i \in s}$
- Estimate $\hat{G}(v^{-1/2}(x_i)(t - m(x_i)))$ by $\hat{g}_i = s'_{si}I_s$
- Then
  $$\hat{F}_{LPR}(t) = \frac{1}{N} \sum_{i \in U} \hat{g}_i + \frac{1}{N} \sum_{i \in s} \frac{I_{y_i \leq t} - \hat{g}_i}{\pi_i}$$
- Can construct model-based nonparametric estimator analogously
Properties of LPR CDF Estimator

• From Breidt and Opsomer (2000), \( \hat{F}_{LPR}(t) \) and

\[
\text{Var}(\hat{F}_{LPR}(t)) = \frac{1}{N^2} \sum_{i,j \in s} \frac{\Delta_{ij} (I\{y_i \leq t\} - \hat{g}_i)(I\{y_j \leq t\} - \hat{g}_j)}{\pi_i \pi_j}
\]

are asymptotically design unbiased and consistent

• Weighted form for generic inference:

\[
\hat{F}_{LPR}(t) = \sum_{i \in s} \omega_{is} I\{y_i \leq t\},
\]

where the weights \( \{\omega_{is}\} \) do not depend on \( y \) or \( t \) – can be applied to any response at any quantile
Internal Consistency

- LPR weights guarantee internal consistency:

\[
\sum_{i \in s} \omega_{is}(y_i + z_i) = \sum_{i \in s} \omega_{is}y_i + \sum_{i \in s} \omega_{is}z_i
\]

- Mean of LPR-estimated cdf is LPR-estimated mean

\[
\int y \, d\hat{F}_{LPR}(y) = \sum_{i \in s} \omega_{is}y_i = \frac{\hat{T}_{LPR}}{N}
\]

- assuming weights are non-negative
### Range of Estimators

- **From specific to generic:**

<table>
<thead>
<tr>
<th>Specific</th>
<th>Generic</th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>Model-based parametric</td>
<td>Design-based</td>
<td>CD</td>
</tr>
<tr>
<td>Model-assisted parametric</td>
<td></td>
<td>RKM</td>
</tr>
<tr>
<td>Model-based nonparametric</td>
<td></td>
<td>DORF</td>
</tr>
<tr>
<td>Model-assisted nonparametric</td>
<td></td>
<td>LPR</td>
</tr>
<tr>
<td></td>
<td></td>
<td>HT</td>
</tr>
</tbody>
</table>
CDF Simulation Study Design

- Compare seven estimators via simulation:

<table>
<thead>
<tr>
<th>Type</th>
<th>Estimator</th>
<th>Model</th>
</tr>
</thead>
<tbody>
<tr>
<td>design-based</td>
<td>HT</td>
<td>no model</td>
</tr>
<tr>
<td>model-based</td>
<td>CD0</td>
<td>$\beta_1 x + x^{1/2}\epsilon$</td>
</tr>
<tr>
<td></td>
<td>CD1</td>
<td>$\beta_0 + \beta_1 x + \epsilon$</td>
</tr>
<tr>
<td></td>
<td>DORF</td>
<td>$m(x) + v^{1/2}(x)\epsilon$</td>
</tr>
<tr>
<td>model-assisted</td>
<td>RKM0</td>
<td>$\beta_1 x + x^{1/2}\epsilon$</td>
</tr>
<tr>
<td></td>
<td>RKM1</td>
<td>$\beta_0 + \beta_1 x + \epsilon$</td>
</tr>
<tr>
<td></td>
<td>LPR</td>
<td>$m(x) + v^{1/2}(x)\epsilon$</td>
</tr>
</tbody>
</table>
Simulated Response Variables

- 7 response variables with $x_i \sim \text{Unif}(0, 1), \epsilon_i \sim \text{N}(0, \sigma^2)$
CDF Simulation Study Design, Continued

- $N = 1000$, $n = 100$, $\pi_i = n/N$
- 1000 reps
- $\hat{F}_{LPR}$ calculated using Epanechnikov kernel:
  \[
  K(x) = \frac{3}{4}(1 - x^2)I\{|x|<1\}
  \]
- Bandwidth $h = 0.1$ or $0.25$
  - single choice of $h$ is not optimal
  - single choice means one generic set of weights, $\{\omega_{is}\}$
- $\sigma = 0.1$ or $0.4$
- CDF estimated at median and first quartile
CDF Simulation Study Output

• Return MSE ratios: ( > 1 favors LPR)

\[
\frac{MSE(\hat{F}_*(t))}{MSE(\hat{F}_{LPR}(t))}
\]

• Return percent relative biases:

\[
\left(\frac{(\hat{F}(t) - F(t))/F(t)}{}\right) 100\%
\]
Smoothing to Estimate the Linear CDF

Linear mean versus x

Indicator mean at quartile vs. x

Indicator mean at median vs. x

Linear CDF
Smoothing to Estimate the Bump CDF
Smoothing to Estimate the Jump CDF

Jump mean versus $x$

Jump CDF

Indicator mean at quartile vs. $x$

Indicator mean at median vs. $x$
CDF Simulation Results: Bias

- \((2\ \sigma)(2\ \text{bandwidths})(7\ \text{responses})(2\ \text{quantiles})=56\ \text{cases}\)

- RB = Relative Bias

<table>
<thead>
<tr>
<th>Method</th>
<th>HT</th>
<th>&lt; 0.5% RB</th>
<th>&lt; 2% RB</th>
<th>usually &gt; 2% RB</th>
<th>several 2%–6% RB</th>
</tr>
</thead>
<tbody>
<tr>
<td>Design-based</td>
<td>RKM0, RKM1, LPR</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model-assisted</td>
<td>CD0, CD1</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model-based par.</td>
<td>DORF</td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>Model-based nonpar.</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

- Design bias adjustment works!
CDF Simulation Numerical Results

- MSE ratios for CDF estimation at the median, $h = 0.25, \sigma = 0.4$

<table>
<thead>
<tr>
<th>Response</th>
<th>HT</th>
<th>CD0</th>
<th>CD1</th>
<th>RKM0</th>
<th>RKM1</th>
<th>DORF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio</td>
<td>1.24</td>
<td>0.71</td>
<td>1.94</td>
<td>0.95</td>
<td>0.97</td>
<td>1.22</td>
</tr>
<tr>
<td>Linear</td>
<td>2.16</td>
<td>2.86</td>
<td>0.56</td>
<td>0.97</td>
<td>0.97</td>
<td>1.40</td>
</tr>
<tr>
<td>Expo</td>
<td>1.06</td>
<td>1.02</td>
<td>0.83</td>
<td>1.20</td>
<td>0.99</td>
<td>1.17</td>
</tr>
<tr>
<td>Bump</td>
<td>2.26</td>
<td>6.36</td>
<td>2.62</td>
<td>1.08</td>
<td>1.14</td>
<td>1.39</td>
</tr>
<tr>
<td>Jump</td>
<td>1.26</td>
<td>1.26</td>
<td>0.95</td>
<td>1.13</td>
<td>1.18</td>
<td>1.24</td>
</tr>
<tr>
<td>Quad</td>
<td>1.04</td>
<td>0.51</td>
<td>0.97</td>
<td>1.34</td>
<td>1.05</td>
<td>1.20</td>
</tr>
<tr>
<td>Cycle</td>
<td>2.79</td>
<td>3.11</td>
<td>1.19</td>
<td>4.29</td>
<td>1.38</td>
<td>1.57</td>
</tr>
</tbody>
</table>

- $m(x)$ not misspecified
- $m(x)$ misspecified
MSE Comparisons to Generic Estimators

- LPR dominates HT and DORF in nearly all cases

<table>
<thead>
<tr>
<th></th>
<th>Min</th>
<th>Q1</th>
<th>Med</th>
<th>Q3</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>HT</td>
<td>0.91</td>
<td>1.22</td>
<td>2.01</td>
<td>3.16</td>
<td>10.25</td>
</tr>
<tr>
<td>DORF</td>
<td>0.98</td>
<td>1.13</td>
<td>1.38</td>
<td>1.63</td>
<td>3.29</td>
</tr>
</tbody>
</table>

- LPR is competitive with RKM0, RKM1

<table>
<thead>
<tr>
<th></th>
<th>Min</th>
<th>Q1</th>
<th>Med</th>
<th>Q3</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>RKM0</td>
<td>0.69</td>
<td>0.94</td>
<td>1.16</td>
<td>1.85</td>
<td>16.55</td>
</tr>
<tr>
<td>RKM1</td>
<td>0.69</td>
<td>0.96</td>
<td>1.03</td>
<td>1.37</td>
<td>4.65</td>
</tr>
</tbody>
</table>
MSE Comparisons for Specific Estimators

- **Mean correct, variance correct**

<table>
<thead>
<tr>
<th></th>
<th>$\sigma = 0.1$</th>
<th>$\sigma = 0.4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CD0</td>
<td>0.14, 0.14, 0.39, 0.39</td>
<td>0.56, 0.60, 0.67, 0.71</td>
</tr>
<tr>
<td>CD1</td>
<td>0.17, 0.19, 0.19, 0.23</td>
<td>0.54, 0.56, 0.58, 0.60</td>
</tr>
</tbody>
</table>

- **Mean correct, variance incorrect**

<table>
<thead>
<tr>
<th></th>
<th>$\sigma = 0.1$</th>
<th>$\sigma = 0.4$</th>
</tr>
</thead>
<tbody>
<tr>
<td>CD0</td>
<td>0.07, 0.08, 0.55, 0.67</td>
<td>1.15, 1.21, 2.74, 2.86</td>
</tr>
<tr>
<td>CD1</td>
<td>0.27, 0.27, 0.74, 0.75</td>
<td>0.71, 0.77, 1.85, 1.94</td>
</tr>
</tbody>
</table>

- **Mean incorrect**

<table>
<thead>
<tr>
<th></th>
<th>Min</th>
<th>Q1</th>
<th>Med</th>
<th>Q3</th>
<th>Max</th>
</tr>
</thead>
<tbody>
<tr>
<td>CD0</td>
<td>0.48</td>
<td>1.01</td>
<td>2.71</td>
<td>14.98</td>
<td>84.39</td>
</tr>
<tr>
<td>CD1</td>
<td>0.64</td>
<td>0.91</td>
<td>1.44</td>
<td>3.68</td>
<td>18.36</td>
</tr>
</tbody>
</table>
Quantile Estimation Simulation

- Quantile is $\theta(\alpha) = \min\{t : F(t) \geq \alpha\}$
- Estimate by $\hat{\theta}(\alpha) = \min\{t : \hat{F}(t) \geq \alpha\}$
- Simulation study design identical to CDF simulation
Results for Estimation of Median

- MSE ratios for median estimation, $h = 0.25, \sigma = 0.4$

<table>
<thead>
<tr>
<th>Population</th>
<th>HT</th>
<th>CD0</th>
<th>CD1</th>
<th>RKM0</th>
<th>RKM1</th>
<th>DORF</th>
</tr>
</thead>
<tbody>
<tr>
<td>Ratio</td>
<td>1.26</td>
<td>0.64</td>
<td>1.90</td>
<td>0.97</td>
<td>0.99</td>
<td>1.03</td>
</tr>
<tr>
<td>Linear</td>
<td>2.57</td>
<td>3.77</td>
<td>0.61</td>
<td>1.08</td>
<td>1.08</td>
<td>1.18</td>
</tr>
<tr>
<td>Expo</td>
<td>1.06</td>
<td>0.94</td>
<td>0.97</td>
<td>1.21</td>
<td>1.01</td>
<td>1.02</td>
</tr>
<tr>
<td>Bump</td>
<td>2.37</td>
<td>6.22</td>
<td>1.99</td>
<td>1.12</td>
<td>1.17</td>
<td>1.16</td>
</tr>
<tr>
<td>Jump</td>
<td>1.26</td>
<td>1.85</td>
<td>0.88</td>
<td>1.14</td>
<td>1.18</td>
<td>1.07</td>
</tr>
<tr>
<td>Quad</td>
<td>1.02</td>
<td>2.71</td>
<td>0.92</td>
<td>1.33</td>
<td>1.02</td>
<td>1.02</td>
</tr>
<tr>
<td>Cycle</td>
<td>3.52</td>
<td>16.68</td>
<td>0.78</td>
<td>5.51</td>
<td>1.51</td>
<td>1.55</td>
</tr>
</tbody>
</table>

- Results very similar to CDF simulation results for estimation at the median
Summary

- Finite population CDF estimation
  - incorporation of auxiliary information
  - comparison of generic vs. specific inference

- In generic context, nonparametric model-based (LPR)
  - dominates design-based (HT) and model-based nonparametric (DORF)
  - is competitive with model-assisted parametric (RKM)
  - loses to model-based parametric (CD) for correct mean, correct variance
  - beats CD for incorrect mean or “noticeably” incorrect variance

- Similar results for quantile estimation