Estimation for State-Space Models: an Approximate Likelihood Approach

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Example: Pound-Dollar Exchange Rates
(Oct 1, 1981 - June 28, 1985; Koopman website)
Motivating Examples
- Time series of counts
- Stochastic volatility

Model setup and estimation
- Exponential family
  - 2 examples
- Estimation
  - GLM
    - Davis, Dunsmuir, Wang
  - Importance sampling
    - Durbin and Koopman
  - Approximation to the likelihood
    - Davis, Dunsmuir, and Wang

Simulation and Application
- Time series of counts
- Stochastic volatility

How good is the posterior approximation?
- posterior mode vs posterior mean
Exponential Family Setup for Parameter-Driven Model

Time series data: $Y_1, \ldots, Y_n$

Regression (explanatory) variable: $x_t$

Observation equation:

$$p(y_t | \alpha_t) = \exp \{ (\alpha_t + \beta^T x_t) y_t - b(\alpha_t + \beta^T x_t) + c(y_t) \}.$$

State equation: \{\alpha_t\} follows an autoregressive process satisfying the recursions

$$\alpha_t = \gamma + \phi_1 \alpha_{t-1} + \phi_2 \alpha_{t-2} + \ldots + \phi_p \alpha_{t-p} + \varepsilon_t,$$

where \{\varepsilon_t\} ~ IID N(0,\sigma^2).

Note: $\alpha_t = 0$ corresponds to standard generalized linear model.

Original primary objective: Inference about $\beta$. 
Examples of parameter driven models

Poisson model for time series of counts

Observation equation:

\[ p(y_t | \alpha_t) = \frac{e^{(\beta^T x_t + \alpha_t)y_t} e^{-e^{(\beta^T x_t + \alpha_t)}}}{y_t!}, \quad y_t = 0, 1, ..., \]

State equation: State variables follow a Gaussian AR(1) process

\[ \alpha_t = \phi \alpha_{t-1} + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2) \]

The resulting transition density of the state variables is

\[ p(\alpha_{t+1} | \alpha_t) = n(\alpha_{t+1} ; \phi \alpha_t , \sigma^2 ) \]

Remark: The case \( \sigma^2 = 0 \) corresponds to a log-linear model with Poisson noise.
Examples of parameter driven models-cont

A stochastic volatility model for financial data (Taylor `86):

Model:

\[ Y_t = \sigma_t Z_t, \{Z_t\} \sim \text{IID } N(0,1) \]

\[ \alpha_t = \phi \alpha_{t-1} + \epsilon_t, \{\epsilon_t\} \sim \text{IID } N(0,\sigma^2), \]

where \( \alpha_t = 2 \log \sigma_t \).

The resulting observation and state transition densities are

\[ p(y_t | \alpha_t) = n(y_t ; 0, \exp(2\alpha_t)) \]

\[ p(\alpha_{t+1} | \alpha_t) = n(\alpha_{t+1} ; \phi \alpha_t, \sigma^2) \]

Properties:

- Martingale difference sequence.
- Stationary.
- Strongly mixing at a geometric rate.
Estimating equations (Zeger ‘88): Let \( \hat{\beta} \) be the solution to the equation

\[
\frac{\partial \mu}{\partial \beta} \Gamma_n (y_n - \mu) = 0,
\]

where \( \mu = \exp(\mathbf{X} \beta) \) and \( \Gamma_n = \text{var}(Y_n) \).

- Monte Carlo EM (Chan and Ledolter ‘95)
- GLM (ignores the presence of the latent process, i.e., \( \alpha_t = 0 \)).
- Importance sampling (Durbin & Koopman ‘01, Kuk ‘99, Kuk & Chen ‘97):
- Approximate likelihood (Davis, Dunsmuir & Wang ‘98)
Estimation Methods — Importance Sampling (Durbin and Koopman)

Model:

\[ Y_t \mid \alpha_t, x_t \sim Pois(\exp(x_t^T \beta + \alpha_t)) \]
\[ \alpha_t = \phi \alpha_{t-1} + \varepsilon_t, \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2) \]

Relative Likelihood: Let \( \psi = (\beta, \phi, \sigma^2) \) and suppose \( g(y_n, \alpha_n; \psi_0) \) is an approximating joint density for \( Y_n = (Y_1, \ldots, Y_n)' \) and \( \alpha_n = (\alpha_1, \ldots, \alpha_n)' \).

\[
L(\psi) = \int p(y_n \mid \alpha_n) p(\alpha_n) d\alpha_n \\
= \int \frac{p(y_n \mid \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(y_n, \alpha_n; \psi_0) d\alpha_n \\
= \int \frac{p(y_n \mid \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(\alpha_n \mid y_n; \psi_0) g(y_n; \psi_0) d\alpha_n \\
\frac{L(\psi)}{L_g(\psi_0)} = \int \frac{p(y_n \mid \alpha_n) p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(\alpha_n \mid y_n; \psi_0) d\alpha_n
\]
Importance Sampling (cont)

\[
\frac{L(\psi)}{L_g(\psi_0)} = \int \frac{p(y_n | \alpha_n)p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} g(\alpha_n | y_n; \psi_0) d\alpha_n
\]

\[
= E_g \left[ \frac{p(y_n | \alpha_n)p(\alpha_n)}{g(y_n, \alpha_n; \psi_0)} | y_n; \psi_0 \right]
\]

\[
\sim \frac{1}{N} \sum_{j=1}^{N} \frac{p(y_n | \alpha_n^{(j)})p(\alpha_n^{(j)})}{g(y_n, \alpha_n^{(j)}; \psi_0)}
\]

where \( \{\alpha_n^{(j)}; j = 1,...,N\} \sim \text{iid } g(\alpha_n | y_n; \psi_0) \).

Notes:

• This is a “one-sample” approximation to the relative likelihood. That is, for one realization of the \( \alpha \)'s, we have, in principle, an approximation to the whole likelihood function.

• Approximation is only good in a neighborhood of \( \psi_0 \). Geyer suggests maximizing ratio wrt \( \psi \) and iterate replacing \( \psi_0 \) with \( \hat{\psi} \).
Importance Sampling — example

Simulation example: \( Y_t \mid \alpha_t \sim Pois(\exp(0.7 + \alpha_t)) \),

\[ \alpha_t = 0.5 \alpha_{t-1} + \epsilon_t, \quad \{\epsilon_t\} \sim \text{IID N}(0, 0.3), \quad n = 200, \quad N = 1000 \]
Simulation example:  \( Y_t \mid \alpha_t \sim Pois(\exp(0.7 + \alpha_t)) \),  \( \phi = 0.5, \sigma^2 = 0.3, n = 200, N = 1000 \)
Importance Sampling — example

Simulation example: $\beta = .7$, $\phi = .5$, $\sigma^2 = .3$, $n = 200$, $N = 1000$, 50 realizations plotted
Importance Sampling (cont)

Choice of importance density $g$:

Durbin and Koopman suggest a linear state-space approximating model

$$Y_t = \mu_t + x_t^T \beta + \alpha_t + Z_t, \quad Z_t \sim N(0,H_t),$$

with

$$\mu_t = y_t - \hat{\alpha}_t - x_t' y_t e^{-(\hat{\alpha}_t + x_t' \beta)} + 1,$$

$$H_t = e^{-(\hat{\alpha}_t + x_t' \beta)},$$

where the $\hat{\alpha}_t = E_g(\alpha_t \mid y_n)$ are calculated recursively under the approximating model until convergence.

With this choice of approximating model, it turns out that

$$g(\alpha_n \mid y_n; \psi_0) \sim N(\Gamma_n^{-1} \tilde{y}_n, \Gamma_n^{-1}),$$

where

$$\tilde{y}_n = y_n - e^{X\beta + \hat{\alpha}_n} + e^{X\beta + \hat{\alpha}_n} \hat{\alpha}_n,$$

$$\Gamma_n = \text{diag}(e^{X\beta + \hat{\alpha}_n}) + (E(\alpha_n \alpha_n'))^{-1}.$$
Components required in the calculation.

- \( g(y_n, \alpha_n) \)
  - \( \tilde{y}'_n \Gamma_n^{-1} \tilde{y}_n \)
  - \( \det(\Gamma_n) \)
- simulate from \( N(\Gamma_n^{-1} \tilde{y}_n, \Gamma_n^{-1}) \)
  - compute \( \Gamma_n^{-1} \tilde{y}_n \)
  - simulate from \( N(0, \Gamma_n^{-1}) \)

Remark: These quantities can be computed quickly using a version of the innovations algorithm or the Kalman smoothing recursions.
Consider a Gaussian approximation \( p_a(\alpha_n | y_n) = \phi(\alpha_n ; \mu_0 , \Sigma_0) \) to the posterior

\[
p(\alpha_n | y_n) \propto p(\alpha_n | y_n) p(\alpha_n)
\]

where

\[
G_n^{-1} = E(\alpha_n - \mu)^T (\alpha_n - \mu)
\]

Likelihood:

\[
L(\psi) = \int p(y_n | \alpha_n) p(\alpha_n) d\alpha_n
\]

Consider a Gaussian approximation \( p_a(\alpha_n | y_n) = \phi(\alpha_n ; \mu_0 , \Sigma_0) \) to the posterior

\[
p(\alpha_n | y_n) \propto p(\alpha_n | y_n) p(\alpha_n)
\]

Setting equal the respective posterior modes \( \alpha_a^* \) and \( \alpha^* \) of \( p_a(\alpha_n | y_n) \) and \( p(\alpha_n | y_n) \), we have \( \mu_0 = \alpha^* \), where \( \alpha^* \) is the solution of the equation

\[
\frac{\partial}{\partial \alpha_n} \log p(y_n | \alpha_n, \psi) - G_n (\alpha_n - \mu) = 0
\]
Matching Fisher information matrices:

\[
\Sigma_0 = \left( -\frac{\partial^2}{\partial \alpha \partial \alpha^T} \log p(y_n | \alpha_n, \psi) \bigg|_{\hat{\alpha}} + G_n \right)^{-1}
\]

Approximating posterior:

\[
p_a(\alpha_n | y_n, \psi) = \phi(\alpha_n; \alpha^*, \left( -\frac{\partial^2}{\partial \alpha \partial \alpha^T} \log p(y_n | \alpha_n, \psi) \bigg|_{\hat{\alpha}} + G_n \right)^{-1})
\]

Notes:

1. This approximating posterior is identical to the importance sampling density used by Durbin and Koopman.
2. In traditional Bayesian setting, posterior is approximately \( p_a \) for large \( n \) (see Bernardo and Smith, 1994).
3. Obtain same result if one applies a Taylor series expansion to the joint likelihood and ignore terms of order > 2.
Approximate likelihood: Note that

\[ p(\alpha_n \mid y_n) = \frac{p(y_n \mid \alpha_n) p(\alpha_n)}{L(\psi; y_n)}, \]

which by solving for L in the expression,

\[ p_a(\alpha_n^* \mid y_n, \psi) = p(\alpha_n^* \mid y_n, \psi), \]

we obtain

\[
L_a(\psi; y_n) = p(y_n \mid \alpha^*, \psi)p(\alpha^*, \psi)/p_a(\alpha^* \mid y_n, \psi)
\]

\[
= |G_n|^{1/2} p(y_n \mid \alpha^*, \psi) \exp\{- (\alpha^* - \mu)^T G_n (\alpha^* - \mu)/2\}
\]

\[
det\left( - \frac{\partial^2}{\partial \alpha \partial \alpha^T} \log p(y_n \mid \alpha_n, \psi) \bigg|_{\alpha^*} + G_n \right)^{1/2}
\]
Case of exponential family:

\[ L_a(\psi; y_n) = \frac{|G_n|^{1/2}}{(K + G_n)^{1/2}} \exp \{ y_n^T \alpha^* - 1^T \{ b(\alpha^*) - c(y_n) \} - (\alpha^* - \mu)^T G_n (\alpha^* - \mu) / 2 \}, \]

where

\[ K = \text{diag} \left\{ \frac{\partial^2}{\partial \alpha_t^2} b_t(\alpha_t) \bigg|_{\alpha_t^*} \right\}, \]

and \( \alpha^* \) is the solution to the equation

\[ y_n - \frac{\partial}{\partial \alpha_n} b(\alpha_n) - G_n (\alpha_n - \mu) = 0. \]

Using a Taylor expansion, the latter equation can be solved iteratively.
Estimation Methods — Approximation to the likelihood

Implementation:

1. Let \( \alpha^* = \alpha^*(\psi) \) be the converged value of \( \alpha^{(j)}(\psi) \), where

\[
\alpha^{(j+1)}(\psi) = (\ddot{b}^j + G_n)^{-1}\tilde{y}_n^j(\psi),
\]

and

\[
\tilde{y}_n^j = y_n - \dot{b}^j + \ddot{b}^j \alpha^{(j)} + G_n \mu.
\]

2. Maximize \( L_\alpha(\psi ; y_n) \) with respect to \( \psi \).
Model: \( Y_t | \alpha_t \sim Pois(\exp(0.7 + \alpha_t)) \), \( \alpha_t = 0.5 \alpha_{t-1} + \varepsilon_t \), \( \{\varepsilon_t\} \sim IID \ N(0, 0.3) \), \( n = 200 \)

Estimation methods:

- Importance sampling (\( N=1000, \psi_0 \) updated a maximum of 10 times)

  \[
  \begin{array}{ccc}
  beta & phi & sigma2 \\
  \text{mean} & 0.6982 & 0.4718 & 0.3008 \\
  \text{std} & 0.1059 & 0.1476 & 0.0899 \\
  \end{array}
  \]

- Approximation to likelihood

  \[
  \begin{array}{ccc}
  beta & phi & sigma2 \\
  \text{mean} & 0.7036 & 0.4579 & 0.2962 \\
  \text{std} & 0.0951 & 0.1365 & 0.0784 \\
  \end{array}
  \]
Model: $Y_t \mid \alpha_t \sim Pois(\exp(0.7 + \alpha_t))$, $\alpha_t = 0.5 \alpha_{t-1} + \varepsilon_t$, $\{\varepsilon_t\} \sim \text{IID } N(0, 0.3)$, $n = 200$

Approx likelihood

Importance Sampling
Application to Model Fitting for the Polio Data

Model for \( \{\alpha_t\} \):
\[
\alpha_t = \phi \alpha_{t-1} + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IID } N(0, \sigma^2).
\]

- Importance sampling (\( \psi_0 \) updated 5 times for each \( N=100, 500, 1000, \))
- Simulation based on 1000 replications and the fitted AL model.

<table>
<thead>
<tr>
<th></th>
<th>Import Sampling</th>
<th>Approx Like</th>
<th>GLM</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( \hat{\beta}_{\text{IS}} )</td>
<td>( \hat{\beta}_{\text{AL}} )</td>
<td>( \hat{\beta}_{\text{GLM}} )</td>
</tr>
<tr>
<td><strong>Mean</strong></td>
<td><strong>SD</strong></td>
<td><strong>Mean</strong></td>
<td><strong>SD</strong></td>
</tr>
<tr>
<td><strong>Simulation</strong></td>
<td><strong>Simulation</strong></td>
<td><strong>Simulation</strong></td>
<td></td>
</tr>
<tr>
<td>Intercept</td>
<td>0.203</td>
<td>0.202</td>
<td>0.207</td>
</tr>
<tr>
<td>Trend((\times 10^{-3}))</td>
<td>-2.675</td>
<td>-2.690</td>
<td>-4.18</td>
</tr>
<tr>
<td>cos(2(\pi t/12))</td>
<td>0.110</td>
<td>0.113</td>
<td>-0.152</td>
</tr>
<tr>
<td>sin(2(\pi t/12))</td>
<td>-0.456</td>
<td>-0.454</td>
<td>-.532</td>
</tr>
<tr>
<td>cos(2(\pi t/6))</td>
<td>0.399</td>
<td>0.396</td>
<td>.169</td>
</tr>
<tr>
<td>sin(2(\pi t/6))</td>
<td>0.015</td>
<td>0.016</td>
<td>-.432</td>
</tr>
<tr>
<td>(\phi)</td>
<td>0.865</td>
<td>0.845</td>
<td></td>
</tr>
<tr>
<td>(\sigma^2)</td>
<td>0.088</td>
<td>0.104</td>
<td></td>
</tr>
</tbody>
</table>
Simulation Results

Stochastic volatility model:

\[ Y_t = \sigma_t Z_t, \{Z_t\} \sim \text{IID N}(0,1) \]

\[ \alpha_t = \gamma + \phi \alpha_{t-1} + \varepsilon_t, \{\varepsilon_t\} \sim \text{IID N}(0,\sigma^2), \text{ where } \alpha_t = 2 \log \sigma_t; \ n=1000, \ NR=500 \]

<table>
<thead>
<tr>
<th>CV=10</th>
<th>True</th>
<th>AL</th>
<th>RMSE</th>
<th>IS</th>
<th>RMSE</th>
</tr>
</thead>
<tbody>
<tr>
<td>(\gamma)</td>
<td>-.411</td>
<td>-.491</td>
<td>.210</td>
<td>-.490</td>
<td>.216</td>
</tr>
<tr>
<td>(\phi)</td>
<td>0.950</td>
<td>0.940</td>
<td>.025</td>
<td>0.940</td>
<td>.026</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>0.484</td>
<td>0.478</td>
<td>.065</td>
<td>0.481</td>
<td>.073</td>
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<table>
<thead>
<tr>
<th>CV=1</th>
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<th>AL</th>
<th>RMSE</th>
<th>IS</th>
<th>RMSE</th>
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</thead>
<tbody>
<tr>
<td>(\gamma)</td>
<td>-.368</td>
<td>-.499</td>
<td>.341</td>
<td>-.485</td>
<td>.324</td>
</tr>
<tr>
<td>(\phi)</td>
<td>0.950</td>
<td>0.932</td>
<td>.046</td>
<td>0.934</td>
<td>.043</td>
</tr>
<tr>
<td>(\sigma)</td>
<td>0.260</td>
<td>0.270</td>
<td>.068</td>
<td>0.268</td>
<td>.068</td>
</tr>
</tbody>
</table>
Application to Sydney Asthma Count Data

Data: \(Y_1, \ldots, Y_{1461}\) daily asthma presentations in a Campbelltown hospital.

Preliminary analysis identified.

- no upward or downward trend
- annual cycle modeled by \(\cos(2\pi t/365), \sin(2\pi t/365)\)
- seasonal effect modeled by
  \[
  P_{ij}(t) = \frac{1}{B(2.5,5)} \left( \frac{t - T_{ij}}{100} \right)^{2.5} \left( 1 - \frac{t - T_{ij}}{100} \right)^5
  \]
  where \(B(2.5,5)\) is the beta function and \(T_{ij}\) is the start of the \(j^{th}\) school term in year \(i\).
- day of the week effect modeled by separate indicator variables for Sunday and Monday (increase in admittance on these days compared to Tues-Sat).
- Of the meteorological variables (max/min temp, humidity) and pollution variables (ozone, NO, NO\(_2\)), only humidity at lags of 12-20 days and NO\(_2\)(max) appear to have an association.
## Results for Asthma Data—(IS & AL)

<table>
<thead>
<tr>
<th>Term</th>
<th>IS</th>
<th>AL</th>
<th>Mean</th>
<th>SD</th>
</tr>
</thead>
<tbody>
<tr>
<td>Intercept</td>
<td>0.590</td>
<td>0.591</td>
<td>0.593</td>
<td>0.0658</td>
</tr>
<tr>
<td>Sunday effect</td>
<td>0.138</td>
<td>0.138</td>
<td>0.139</td>
<td>0.0531</td>
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<tr>
<td>Monday effect</td>
<td>0.229</td>
<td>0.231</td>
<td>0.230</td>
<td>0.0495</td>
</tr>
<tr>
<td>$\cos(2\pi t/365)$</td>
<td>-0.218</td>
<td>-0.218</td>
<td>-0.217</td>
<td>0.0415</td>
</tr>
<tr>
<td>$\sin(2\pi t/365)$</td>
<td>0.200</td>
<td>0.179</td>
<td>0.181</td>
<td>0.0437</td>
</tr>
<tr>
<td>Term 1, 1990</td>
<td>0.188</td>
<td>0.198</td>
<td>0.194</td>
<td>0.0638</td>
</tr>
<tr>
<td>Term 2, 1990</td>
<td>0.183</td>
<td>0.130</td>
<td>0.129</td>
<td>0.0664</td>
</tr>
<tr>
<td>Term 1, 1991</td>
<td>0.080</td>
<td>0.075</td>
<td>0.070</td>
<td>0.0733</td>
</tr>
<tr>
<td>Term 2, 1991</td>
<td>0.177</td>
<td>0.164</td>
<td>0.157</td>
<td>0.0665</td>
</tr>
<tr>
<td>Term 1, 1992</td>
<td>0.223</td>
<td>0.221</td>
<td>0.214</td>
<td>0.0667</td>
</tr>
<tr>
<td>Term 2, 1992</td>
<td>0.243</td>
<td>0.239</td>
<td>0.237</td>
<td>0.0620</td>
</tr>
<tr>
<td>Term 1, 1993</td>
<td>0.379</td>
<td>0.397</td>
<td>0.394</td>
<td>0.0625</td>
</tr>
<tr>
<td>Term 2, 1993</td>
<td>0.127</td>
<td>0.111</td>
<td>0.108</td>
<td>0.0682</td>
</tr>
<tr>
<td>Humidity $H_t/20$</td>
<td>0.009</td>
<td>0.010</td>
<td>0.007</td>
<td>0.0032</td>
</tr>
<tr>
<td>$\text{NO}_2$ max</td>
<td>-0.125</td>
<td>-0.107</td>
<td>-0.108</td>
<td>0.0347</td>
</tr>
<tr>
<td>AR(1), $\phi$</td>
<td>0.385</td>
<td>0.788</td>
<td>0.468</td>
<td>0.3790</td>
</tr>
<tr>
<td>$\sigma^2$</td>
<td>0.053</td>
<td>0.010</td>
<td>0.018</td>
<td>0.0153</td>
</tr>
</tbody>
</table>
Is the posterior distribution close to normal?

Compare posterior mean with posterior mode: Can compute the posterior mean using SIR (sampling importance-resampling)

Posterior mode: The mode of $p(\alpha_n \mid y_n)$ is $\alpha^*$ found at the last iteration.

Posterior mean: The mean of $p(\alpha_n \mid y_n)$ can be found using SIR.

Let $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)}$ be independent draws from the multivariate distr $p_a(\alpha_n \mid y_n)$. For $N$ large, an approximate iid sample from $p(\alpha_n \mid y_n)$ can be obtained by drawing a random sample from $\alpha^{(1)}, \alpha^{(2)}, \ldots, \alpha^{(N)}$ with probabilities

$$p_i = \frac{w_i}{\sum_{i=1}^{N} w_i}, \quad w_i = \frac{p(\alpha^{(i)} \mid y_n)}{p_a(\alpha^{(i)} \mid y_n)} \propto \frac{L(\psi; y_n, \alpha^{(i)})}{p_a(\alpha^{(i)} \mid y_n)}, \quad i = 1, \ldots, N.$$
Posterior mean vs posterior mode?

Polio data: blue = mean, red = mode
Posterior mean vs posterior mode?

Pound/US exchange rate data: blue = mean, red = mode
Summary Remarks

1. Importance sampling offers a nice clean method for estimation in parameter driven models.

2. Relative likelihood approach is a one-sample based procedure, but may have convergence problems.

3. Approximation to the likelihood is a non-simulation based procedure which may have great potential especially with large sample sizes and/or large number of explanatory variables.

5. Approximation likelihood approach is amenable to bootstrapping procedures for bias correction.

6. Posterior mode matches posterior mean reasonably well.

7. Extension to more general latent process models (e.g., long memory) is in progress.