Laplace Likelihood and LAD Estimation for Non-invertible MA(1)

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Program

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MA(1) unit root problem

MA(1):

\[ Y_t = Z_t - \theta Z_{t-1}, \quad \{Z_t\} \sim \text{IID} (0, \sigma^2) \]

Properties:

- \(|\theta| < 1 \quad \Rightarrow \quad Z_t = \sum_{j=0}^{\infty} \theta^j Y_{t-j} \quad \text{(invertible)}\)
- \(|\theta| > 1 \quad \Rightarrow \quad Z_t = -\sum_{j=1}^{\infty} \theta^j Y_{t+j} \quad \text{(non-invertible)}\)
- \(|\theta| = 1 \quad \Rightarrow \quad Z_t \in \text{sp}\{Y_t, Y_{t-1}, \ldots\} \quad \text{and} \quad Z_t \in \text{sp}\{Y_{t+1}, Y_{t+2}, \ldots,\} \quad \Rightarrow \quad P_{\text{sp}\{Y_t, s \neq 0\}} Y_0 = Y_0 \quad \text{(perfect interpolation)}\)
- \(|\theta| < 1 \quad \Rightarrow \quad \hat{\theta}_{\text{mle}} \quad \text{is} \quad \text{AN}(\theta, (1 - \theta^2) / n)\)

MLE = maximum (Gaussian) likelihood, \(n = \text{sample size}\)

What if \(\theta = 1\)?
Why study non-invertible MA(1)?

a) over-differencing

- linear trend model: \( X_t = a + bt + Z_t \).
  \[
  Y_t = X_t - X_{t-1} = b + Z_t - Z_{t-1} \sim \text{MA}(1) \text{ with } \theta = 1. 
  \]

- seasonal model: \( X_t = s_t + Z_t \), \( s_t \) seasonal component w/ period 12.
  \[
  Y_t = X_t - X_{t-12} = Z_t - Z_{t-12} \sim \text{MA}(12) \text{ with } \theta = 1. 
  \]

b) random walk + noise

\[
X_t = X_{t-1} + U_t \quad \text{(random walk signal)}
\]

\[
Y_t = X_t + V_t \quad \text{(random walk signal + noise)}
\]

Then

\[
Y_t - Y_{t-1} = U_t + V_t - V_{t-1} \sim \text{MA}(1)
\]

with \( \theta = 1 \) if and only if \( \text{Var}(U_t) = 0. \)
Identifiability and Gaussian likelihood

Identifiability

• $|\theta| > 1 \Rightarrow Y_t = \varepsilon_t - \theta^{-1} \varepsilon_{t-1}$, where $\{\varepsilon_t\} \sim \text{WN}(0, \theta^2 \sigma^2)$.

• $\{\varepsilon_t\}$ is IID if and only if $\{Z_t\}$ is Gaussian (Breidt and Davis `91)

Gaussian Likelihood

$$L_G(\theta, \sigma^2) = L_G(1/\theta, \theta^2 \sigma^2) \Rightarrow \theta \text{ is only identifiably for } |\theta| \leq 1.$$  

Notes:

i) this implies $L_G(\theta) = L_G(1/\theta)$ for the profile likelihood and $\theta = 1$ is a critical point, $L_G'(1) = 0$.

ii) a pile-up effect ensues, i.e.,

$$P(\hat{\theta} = 1) > 0$$

even if $\theta < 1.$
Gaussian likelihood examples

100 observations from $Y_t = Z_t - \theta_0 Z_{t-1}$, $\{Z_t\} \sim$ IID ($0, \sigma^2$), Laplace pdf

$\theta_0 = .8 \quad \theta_0 = 1.0 \quad \theta_0 = 1.25$
MLE (Gaussian likelihood)

Idea: build parameter normalization into the likelihood function.

Model: \[ Y_t = Z_t - \left(1 - \frac{\beta}{n}\right) Z_{t-1}, \ t = 1, \ldots, n. \]

\[ \beta = n(1 - \theta), \ \theta = 1 - \frac{\beta}{n}, \ \theta_0 = 1 - \frac{\gamma}{n} \]

Gaussian Likelihood:

\[ L_n(\beta) = \ln \left(1 - \frac{\beta}{n}\right) - \ln(1), \ \text{\(l_n(\ )\) = profile log-like}. \]

Theorem (Davis and Dunsmuir `96): Under \( \theta_0 = 1 - \frac{\gamma}{n} \),

\[ L_n(\beta) \to_d Z_\gamma(\beta) \quad \text{on} \quad C[0, \infty). \]

Results:

- \( n(1 - \hat{\theta}_{mle}) \to \hat{\beta}_{mle} = \text{argmax} \ Z_\gamma(\beta) \)
- \( n(1 - \hat{\theta}_{lm}) \to \hat{\beta}_{lm} = \text{arglocalmax} \ Z_\gamma(\beta) \)
- \( P(\hat{\theta}_{lm} = 1) \to P(\hat{\beta}_{lm} = 0) = .6518 \quad \text{if} \ \gamma = 0. \)
Extensions of MLE (Gaussian likelihood)

i) non-zero mean (Chen and Davis `00): same type of limit, except pile-up is more excessive.

\[ P(\hat{\theta}_{mle} = 1) \to 0.955 \]

This makes hypothesis testing easy!

Reject \( H_0: \theta = 1 \) if \( \hat{\theta}_{mle} < 1 \) (size of test is .045)

ii) heavy tails (Davis and Mikosch `98): \( \{Z_t\} \) symmetric alpha stable (\( S_{\alpha}S \)). Then the max Gaussian likelihood estimator has the same normalizing rate, i.e.,

\[ n(1 - \hat{\theta}_{lm}) \to_d \hat{\beta}_{lm} \]

\[ P(\hat{\theta}_{lm} = 1) \to P(\hat{\beta}_{lm} = 0) \]

The pile-up decreases with increasing tail heaviness.
Comparison of limit cdf’s for different $\alpha$’s

$\alpha = 2.0$
$\alpha = 1.5$
$\alpha = 1.0$
$\alpha = .75$
Laplace likelihood/LAD estimation

If noise distribution is non-Gaussian, the MA(1) parameter $\theta$ is identifiable for all real values.

Q1. For MLE (non-Gaussian) does one have $n$ or $n^{1/2}$ asymptotics?

Q2. Is there a pile-up effect?
Laplace likelihood – joint and exact

Model. \( Y_t = Z_t - \theta Z_{t-1} , \{Z_t\} \sim \text{IID } (0, \sigma^2) \) with median 0 and \( EZ^4 < \infty \).

Initial variable.

\[
Z_{\text{init}} = \begin{cases} 
Z_0, & \text{if } |\theta| \leq 1, \\
Z_n - \sum_{t=1}^{n} Y_t, & \text{otherwise.}
\end{cases}
\]

Joint density: Let \( Y_n = (Y_1, \ldots, Y_n) \), then

\[
f(y_n, z_{\text{init}}) = f(z_0, z_1, \ldots, z_n) \left( 1_{\{\theta \leq 1\}} + |\theta|^{-n} 1_{\{\theta > 1\}} \right),
\]

where the \( z_t \) are solved

forward by: \( z_t = Y_t + \theta z_{t-1}, \quad t = 1, \ldots, n \) for \( |\theta| \leq 1 \) with \( z_0 = z_{\text{init}} \)

backward by: \( z_{t-1} = \theta^{-1}(z_t - Y_t), \quad t = n, \ldots, 1 \) for \( |\theta| > 1 \) with \( z_n = z_{\text{init}} + Y_1 + \ldots + Y_n \)

Note: integrate out \( z_{\text{init}} \) to get exact likelihood.

\[
f(y_n) = \int_{-\infty}^{\infty} f(y_n, z_{\text{init}}) dz_{\text{init}}
\]
100 observations from $Y_t = Z_t - \theta_0 Z_{t-1}$, $\{Z_t\} \sim$ IID $(0, \sigma^2)$, Laplace pdf

$\theta_0 = 1.0$

$\theta_0 = 0.8$
Laplace likelihood examples (cont)

100 observations from \( Y_t = Z_t - \theta_0 Z_{t-1} \), \( \{Z_t\} \sim \text{IID} (0, \sigma^2) \), Laplace pdf

\[ \theta_0 = 0.8 \quad \theta_0 = 1.0 \quad \theta_0 = 1.25 \]

Exact likelihood

Joint likelihood at \( z_{\text{max}}(\theta) \)
(Joint) Laplace log-likelihood.

\[
L(\theta, z_{init}, \sigma) = -(n + 1) \log 2\sigma - \sigma^{-1} \sum_{t=0}^{n} \left| z_t \right| - n \left( \log \left| \theta \right| \right) 1_{\{\theta > 1\}}
\]

Maximizing wrt \( \sigma \), we obtain

\[
\hat{\sigma} = \frac{\sum_{t=0}^{n} \left| z_t \right|}{(n + 1)}
\]

so that maximizing \( L \) is equivalent to minimizing

\[
l_n (\theta, z_{init}) = \begin{cases} 
\sum_{t=0}^{n} \left| z_t \right|, & \text{if } |\theta| \leq 1, \\
\sum_{t=0}^{n} \left| z_t \right| \left| \theta \right|, & \text{otherwise}.
\end{cases}
\]
Joint Laplace likelihood — limit results

Theorem 1. Under the parameterizations,

\[ \theta = 1 + \beta/n \quad \text{and} \quad z_{\text{init}} = Z_0 + \alpha \sigma/n^{1/2}, \]

we have

\[ U_n(\beta, \alpha) = \sigma^{-1}(l_n(\theta, z_{\text{init}}) - l_n(1, Z_0)) \rightarrow_d U(\beta, \alpha) \]

on \( C(\mathbb{R}^2) \), where

\[ U_n(\beta, \alpha) = \int_0^1 \left( \beta \int_0^{s-} e^{\beta(s-t)} dS(t) + \alpha e^{\beta s} \right) dW(s) \]

\[ + f(0) \int_0^1 \left( \beta \int_0^{s-} e^{\beta(s-t)} dS(t) + \alpha e^{\beta s} \right)^2 ds \]

for \( \beta \leq 0 \), and

\[ U_n(\beta, \alpha) = \int_0^1 \left( -\beta \int_{s+}^{1} e^{\beta(s-t)} dS(t) + \alpha e^{-\beta(1-s)} \right) dW(s) \]

\[ + f(0) \int_0^1 \left( \beta \int_{s}^{1} e^{\beta(s-t)} dS(t) + \alpha e^{-\beta(1-s)} \right)^2 dW(s) \]

for \( \beta > 0 \), in which \( S(t) \) and \( W(t) \) are the limits of the partial sum processes.
Joint Laplace likelihood — limit results

\[
S_n(t) = \frac{1}{\sigma \sqrt{n}} \sum_{i=0}^{[nt]} Z_i \to_d S(t), \quad W_n(t) = \frac{1}{\sigma \sqrt{n}} \sum_{i=0}^{[nt]} \text{sign}(Z_i) \to_d W(t).
\]

From the limit,

\[
U_n(\beta, \alpha) \to_d U(\beta, \alpha)
\]

it follows that

\[
(n(\hat{\theta}_m - 1), \sqrt{n} \sigma^{-1}(\hat{z}^L_{init} - Z_0)) \to_d (\hat{\beta}_m, \hat{\alpha}_m)
\]

where

\[
(\hat{\beta}_m, \hat{\alpha}_m) = \text{arg(local) min } U(\beta, \alpha).
\]
Exact Laplace likelihood — limit results

Exact Laplace Likelihood:

\[ L_n(\theta, \sigma) = \int_{-\infty}^{\infty} f(y_n, z_{init}) dz_{init} \]

Theorem 2. For the MLE \( \tilde{\theta}_n, \tilde{\sigma}_n \), we have

\[ (n(\tilde{\theta}_{mle} - 1), \sqrt{n}(\tilde{\sigma}_{mle} - E[Z_0])) \rightarrow_d (\tilde{\beta}_{mle}, N), \]

where

\[ \tilde{\beta}_{mle} = \arg \min U^*(\beta), \quad N \sim N(0, \var(|Z_0|)), \]

and \( U^*(\beta) \) is a stochastic process defined in terms of \( S(t) \) and \( W(t) \).

In addition,

\[ n(\tilde{\theta}_{lm} - 1) \rightarrow_d \tilde{\beta}_{lm}, \quad \tilde{\beta}_{lm} = \arg \text{(local)} \min U^*(\beta). \]
Simulating from the limit process

Step 1. Simulate two indep sequences \((W_1, \ldots, W_m)\) and \((V_1, \ldots, V_m)\) of iid N(0,1) random variables with \(m=100000\).

Step 2. Form \(W(t)\) and \(V(t)\) by the partial sum processes,

\[
W(t) = \frac{\sum_{j=1}^{[100000 \cdot t]} W_j}{\sqrt{100000}} \quad \text{and} \quad V(t) = \frac{\sum_{j=1}^{[100000 \cdot t]} V_j}{\sqrt{100000}}.
\]

Step 3. Set \(S(t) = c_1 W(t) + c_2 V(t)\), where

\[
c_1 = \frac{E |Z_t|}{\sigma} \quad \text{and} \quad c_2 = \sqrt{\text{Var}(Z_t) / \sigma^2 - c_1^2}.
\]

Limit process depends only on \(c_1, c_2,\) and \(f(0)\).

Step 4. Compute \(U(\beta, \alpha)\) and \(U^*(\beta)\) from the definition.

Step 5. Determine the respective local and global minimizers of \(U(\beta, \alpha)\) and \(U^*(\beta)\) numerically.
2 realizations of the limit processes, $U(\beta, \alpha(\beta))$, $U^*(\beta)$.
Limit distribution

red graph = Laplace pdf for $Z_t$
blue graph = Gaussian pdf for $Z_t$

Joint Lap Likelihood

$\lim n(\hat{\theta}_{lm} - 1) \rightarrow_d \hat{\beta}_{lm}$

Exact Lap Likelihood

$\lim n(\tilde{\theta}_{lm} - 1) \rightarrow_d \tilde{\beta}_{lm}$
red graph = Laplace pdf for $Z_t$

blue graph = Gaussian pdf for $Z_t$

Joint Lap Like

Exact Lap Like
<table>
<thead>
<tr>
<th>$n$</th>
<th>Exact $\tilde{\theta}_{lm}$</th>
<th>Joint $\hat{\theta}_{lm}$</th>
<th>$\hat{\sigma}$</th>
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<td>-.0033</td>
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<td>s.d. .1438</td>
<td>.0656</td>
<td>.2430</td>
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Laplace noise

$\theta = 1, \quad \sigma = 1$

1000 reps
### Simulation results

**Exact = MLE**

**Joint = maximize over $\theta$ and $z_{init}$**

**Cond = maximize over $\theta$ conditional on $z_{init} = 0$**

**Laplace noise**

$\theta = 1, \ \sigma = 1$

1000 reps

<table>
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<tr>
<th>$n$</th>
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<th>Cond $\overline{\theta}_{ml}$</th>
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<td>cond $-0.057$</td>
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<td>rmse $0.213$</td>
<td>rmse $0.297$</td>
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**Note:**

- LM dominates ML
- joint dominates exact (rmse is half the size)
Pile-up probabilities

**Theorem 3. (joint Laplace likelihood)**

\[ P(\hat{\theta}_{lm} = 1) \rightarrow P(0 < Y < \int_0^1 dS(s) dW(s)), \]

where

\[ Y = \int_0^1 S(s) dW(s) - W(1) \int_0^1 S(s) ds + \frac{W(1)}{2 f(0)} (\int_0^1 W(s) ds - W(1)/2) \]

Idea:

\[ P(\hat{\theta}_{lm} = 1) = P(\lim_{\beta \uparrow 0} \frac{\partial}{\partial \beta} U_n(\beta, \hat{\alpha}(\beta)) < 0 \text{ and } \lim_{\beta \downarrow 0} \frac{\partial}{\partial \beta} U_n(\beta, \hat{\alpha}(\beta)) > 0) \]

\[ \rightarrow P(\lim_{\beta \uparrow 0} \frac{\partial}{\partial \beta} U(\beta, \hat{\alpha}(\beta)) < 0 \text{ and } \lim_{\beta \downarrow 0} \frac{\partial}{\partial \beta} U(\beta, \hat{\alpha}(\beta)) > 0) \]

Now,

\[ \lim_{\beta \downarrow 0} \frac{\partial}{\partial \beta} U(\beta, \hat{\alpha}(\beta)) = Y \]

\[ \lim_{\beta \uparrow 0} \frac{\partial}{\partial \beta} U(\beta, \hat{\alpha}(\beta)) = Y - \int_0^1 dS(s) dW(s) \quad \text{and the result follows.} \]
Theorem 4. (exact Laplace likelihood)

\[ P(\tilde{\theta}_{lm} = 1) \to P\left[ \frac{1}{2} < Y < \int_0^1 dS(s) dW(s) - \frac{1}{2} \right] \]

The pile-up probability is always smaller for the exact MLE than for the joint MLE (see Theorem 3).

Remark 1.

If \( Z_t \) has a Laplace density \( f(z) = \frac{1}{2\sigma} e^{-|z|/\sigma} \), then

\[ Y = \int_0^1 [W(1)s - W(s)] dV(s) + \frac{1}{2}. \]

where \( W(s) \) and \( V(s) \) are independent standard Brownian motions.
It follows that

\[ P(\hat{\theta}_{lm} = 1) \rightarrow P(0 < Y < \int_0^1 dS(s) dW(s)) \]

\[ = P(0 < \int_0^1 [W(1)s - W(s)] dV(s) + .5 < 1) \]

\[ = E \left[ P(-.5 < \int_0^1 [W(1)s - W(s)] dV(s) < .5 \mid W(t), \ t \in [0,1]) \right] \]

\[ = E \left[ 2\Phi \left( \frac{1}{2} \left\{ \int_0^1 [W(1)s - W(s)]^2 ds \right\}^{1/2} \right) - 1 \right] \]

\[ \approx 0.820 \]

\[ P(\tilde{\theta}_{lm} = 1) \rightarrow P(1/2 < Y < \int_0^1 dS(s) dW(s) - 1/2) \]

\[ = P(1/2 < Y < 1 - 1/2) \]

\[ = 0. \ \Rightarrow \text{no pile-up} \]
Remark 2.

\[ P(\hat{\theta}_{lm} = 1) \rightarrow P(0 < Y < \int_{0}^{1} dS(s)dW(s)) \]

\[ = P(0 < Y < c_1), \quad \text{where} \quad c_1 = \frac{E |Z_t|}{\sigma} > 0. \]

On the other hand

\[ P(\tilde{\theta}_{lm} = 1) \rightarrow P(1/2 < Y < c_1 - 1/2) \]

\[ > 0 \quad \text{if and only if} \quad c_1 > 1. \]

That is, there is a pile-up if and only if \( c_1 > 1 \).

Remark 3. Pile-up probability tends to be larger if the density is more concentrated around 0.
## Simulation results – pile-up probabilities

Pile-up probabilities for local maximum: \( P(\hat{\theta}_{lm} = 1) \)

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