Bayesian Model Selection for Geostatistical Regression Data

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Introduction

• Environmental data is often collected within a spatial domain (obtained through GIS layers)

• Failing to take spatial correlation into account can affect regression model selection results.

• Previous methods to account for spatial correlation in model selection
  ◦ Ver Hoef et al. (2001), Spatial stepwise selection
  ◦ Thompson (2001), Bayes factor approx.
  ◦ Hoeting et al. (2005), Spatial AICC
Benefits of a Reversible Jump MCMC approach

- Allows inclusion of expert knowledge in selection of regression covariates
  - Spatial stepwise and AICC methods treat all covariates equally

- Selection over large model spaces
  - In both the Bayes factor and AICC methods parameters in each model must be separately estimated

- A sample from the joint model and parameter posterior is obtained

- Straightforward extension to spatial generalized linear mixed models (model based geostatistics)
Model based geostatistical regression

Data Model

\[ Y(s)|Z(s) \sim \text{i.i.d. } P(\ell^{-1}\{Z(s)\}), \]

where

\[ E[Y(s)|Z(s)] = \ell^{-1}\{Z(s)\} \]

Parameter model

\[ Z = (Z(s_1), \ldots, Z(s_n))' \sim N_n(X\beta; \Sigma) \]

where \( \Sigma \) is defined by a geostatistical covariance
Covariance function

\[ \text{Cov}\{ Z(s), Z(s') \} = \sigma^2 \rho(h; \phi) \]

\[ \text{Var}\{ Z(s) \} = \sigma^2 \]

where,

- \( h = s - s' \)
- \( \sigma^2 \) is the sill \( (0 < \sigma^2 < \infty) \)
- \( \phi \) are the spatial correlation parameters
- \( \rho(h; \phi) \) is a nonnegative correlation function
  e.g. \( \rho(h; \phi) = \exp \left\{ - (h' \phi h)^{1/2} \right\} \)
Bayesian model selection

• Model incorporated as another parameter, $M$ with sample space $\mathcal{M} = \{m_0, \ldots, m_K\}$

• For each $m_k$ we have $\vartheta_k = (\beta_k, \sigma^2, \phi, Z)$

• Inference for the model can be made through the posterior model probability (PMP)

$$P(m_k|Y) \propto \int P(Y|\vartheta_k, m_k)P(\vartheta_k|m_k)P(m_k)d\vartheta_k$$

$$= P(Y|m_k)P(m_k)$$
Model prior distribution

A classic model prior is derived by treating inclusion of the $p$ coefficients as a series of independent Bernoulli trials with probability $\pi_j$. The result is the following prior

$$P(m_k) = \prod_{j=1}^{p_k} \pi_j I_{k,j} (1 - \pi_j)^{1-I_{k,j}},$$

where $I_{k,j}$ is the indicator that $\beta_{k,j} \neq 0$. 

Reversible Jump MCMC

Objective: Draw a sample from $P(\vartheta_k, m_k | Y)$

For current state $q = (\vartheta_k, m_k)$:

1. Propose move of type $i$ to $m_k^*$ from distribution $J_i(q)$

2. Draw $\vartheta_k^*$ from $G_i(q, m_k^*)$

3. Accept new state $q^*$ with probability

$$\min \left\{ 1, \frac{P(q^*|Y)J_i(q^*)G_i(q^*)}{P(q|Y)J_i(q)G_i(q)} \right\}$$
Difficulty with RJMCMC

- Low acceptance rate: Even if the appropriate model is chosen, bad parameter proposals will hinder mixing

- Conjecture: Proposals distributions $G(q)$ close to $P(\theta_k^* | m_k^*, Y)$ will produce the best results

Acceptance probability for $P(\theta_k^* | m_k^*, Y)$

$$\min \left\{ 1, \frac{P(m_k^* | Y) J(m_k^*)}{P(m_k | Y) J(m_k)} \right\}$$
Partial analytic RJMCMC (PARJ)

- Godsill (2001) for AR order selection

- Use parameter proposal
  - Propose $\beta_{k*} \sim P(\beta_{k*} | m_{k*}, \sigma^2, \phi, Z, Y)$
  - Set $(\sigma^2_{k*}, \phi_{k*}, Z_{k*}) = (\sigma^2, \phi, Z)$

- Acceptance probability

$$
\min \left\{ 1, \frac{P(m_{k*} | \sigma^2, \phi, Z, Y) J(m_{k*})}{P(m_k | \sigma^2, \phi, Z, Y) J(m_k)} \right\}
$$

No need to actually simulate $\beta_{k*}$ values
Acceptance ratio for PARJ

- Suppose $P(\beta_k | m_k) = N_{p_k}(\mu_k, V_k)$

- Since $Z = X\beta_k + \delta$,
  $$Z | m_k, \sigma^2, \phi \sim N_n(X_k \mu_k, X_k V_k X_k' + \Sigma)$$

$$P(m_k | Y, Z, \sigma^2, \phi) = P(m_k | Z, \sigma^2, \phi)$$

$$\propto \exp \left\{ -\frac{1}{2} (Z - X_k \mu_k)'(X_k V_k X_k' + \Sigma)^{-1} (Z - X_k \mu_k) \right\}$$

$$\times P(m_k)$$
Fish abundance in the Appalachian region

- In 1994 – 1995, $n = 119$ stream sites were sampled by EPA in Mid-Atlantic highlands region of U.S. (MAHA)

- $Z(s) = \text{Abundance of pollution intolerant fish} – \text{important indicators of stream health}$

- Environmental Covariates:
  - Strahler order, Elevation, Watershed area
  - Road density, % watershed disturbed
  - Habitat quality index, % fish cover
  - Dissolved O$_2$ conc., % fine sediments
Parameters and priors

• **Model:** \( \pi_j = 0.75 \) for Strahler Order, Elevation, and Watershed Area and \( \pi_j = 0.5 \) for others (a uniform prior was also used)

• \( \beta_k \sim N_{p_k}(0, 100\sigma^2(\mathbf{X}_k'\mathbf{X}_k)^{-1}) \) (\( N_{p_k}(\cdot, \cdot) \) update)

• \( \theta_1 = \log \sigma^2 \sim N(0, 10) \)

• \( \phi = \eta^{-1}\mathbf{I}, \quad \theta_2 = \log \eta \sim N(0, 1) \)
### Sampler for spatial regression

<table>
<thead>
<tr>
<th>Step</th>
<th>Update Type</th>
<th>Proposal Distribution$^a$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. Update $\log \sigma^2$</td>
<td>Metropolis</td>
<td>Gaussian</td>
</tr>
<tr>
<td>2. Update $\log \eta$</td>
<td>Metropolis</td>
<td>Gaussian</td>
</tr>
<tr>
<td>3. Update $m_k$</td>
<td>PARJ</td>
<td>Discrete random walk</td>
</tr>
<tr>
<td>4. Update $\beta_k$</td>
<td>Gibbs</td>
<td>Gaussian$^b$</td>
</tr>
<tr>
<td>5. Update $Z$</td>
<td>Langevin-Hastings</td>
<td>Gaussian</td>
</tr>
</tbody>
</table>

$^a$ Metropolis proposal distributions are centered on the current parameter value

$^b$ Full conditional distribution
### Model chain summary

<table>
<thead>
<tr>
<th>Covariate</th>
<th>PIP</th>
<th>PMP</th>
<th>0.12</th>
<th>0.06</th>
<th>0.05</th>
<th>0.05</th>
<th>0.04</th>
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<tbody>
<tr>
<td>Strahler order</td>
<td>0.88</td>
<td></td>
<td></td>
<td></td>
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<tr>
<td>Elevation</td>
<td>0.29</td>
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<tr>
<td>Area</td>
<td>0.43</td>
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<td></td>
<td></td>
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<tr>
<td>Road density</td>
<td>0.38</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>% Disturbance</td>
<td>0.79</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Habitat quality</td>
<td>0.74</td>
<td></td>
<td></td>
<td></td>
<td></td>
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<td></td>
</tr>
<tr>
<td>Dissolved O$_2$</td>
<td>0.15</td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
<td></td>
</tr>
<tr>
<td>% Fish cover</td>
<td>0.10</td>
<td></td>
<td></td>
<td></td>
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<td></td>
<td></td>
</tr>
<tr>
<td>% Fine sed.</td>
<td>0.13</td>
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</tr>
</tbody>
</table>
Marginal coefficient posterior distributions

- Strahler Order
- Watershed Area
- % Watershed Disturbed
- Habitat Quality Index
Comments / Future work

- Partial analytic RJMCMC provides a straightforward method of model update in an MCMC sampler
  - Simple addition to a standard Gibbs sampler
  - Hierarchical centering allows partial analytic approach

- Future investigations
  - Transformed Gaussian models possible
  - Covariate based model proposals
  - Robust hierarchical centering