Homework 4 (STAT 720)

Due: March 20, 2007

Problem 1. Let Ω be the unit interval, $\mathcal{F}$ its Borel subsets, and P Lebesgue measure on $\mathcal{F}$. Let $X_1(\omega) = \omega$, $X_2(\omega) = 1 - \omega$. Show that X_1 and X_2 have the same distribution, but are not identical.

Problem 2. Let X be a r.v. on $(\Omega, \mathcal{F}, P)$ and define $E_n = \{\omega : |X(\omega)| \geq n\}$. Show that

$$\sum_{n=1}^{\infty} P(E_n) \leq E\,|X| \leq 1 + \sum_{n=1}^{\infty} P(E_n)$$

and hence that $E\,|X| < \infty$ if and only if $\sum_{n=1}^{\infty} P(E_n) < \infty$. If X takes only positive integer values, show that $EX = \sum_{n=1}^{\infty} P(E_n)$.

Problem 3. If X is a non-negative r.v. with distribution function F, show that

$$EX = \int_0^{\infty} [1 - F(x)]dx.$$

If X is a real valued r.v. with distribution function F, show that

$$E\,|X| = \int_{-\infty}^{0} F(x)dx + \int_0^{\infty} [1 - F(x)]dx$$

and thus $E\,|X| < \infty$ if and only if $\int_{-\infty}^{0} F(x)dx < \infty$ and $\int_0^{\infty} [1 - F(x)]dx < \infty$, in which case

$$EX = -\int_{-\infty}^{0} F(x)dx + \int_0^{\infty} [1 - F(x)]dx.$$

Problem 4. Let X_1, X_2 be r.v.'s with $E\,|X_1|^p < \infty$, $E\,|X_2|^p < \infty$. Show that for $p > 0$,

$$E\,|X_1 + X_2|^p \leq c_p\{E\,|X_1|^p + E\,|X_2|^p\},$$

where $c_p = 1$ if $0 < p \leq 1$, $c_p = 2^{p-1}$ if $p > 1$.