

Problem 1.

1. Prove that $X_n \rightarrow X$ in probability if and only if

$$\lim_{n \rightarrow \infty} E\left(\frac{|X_n - X|}{1 + |X_n - X|}\right) = 0.$$

2. If F, G are d.f.'s the "Lévy distance" $d(F, G)$ is defined by

$$d(F, G) = \inf\{\varepsilon > 0 : G(x - \varepsilon) - \varepsilon \leq F(x) \leq G(x + \varepsilon) + \varepsilon \text{ for all real } x\}$$

d has the metric properties, and $F_n \Rightarrow F$ if and only if $d(F_n, F) \rightarrow 0$.

Problem 2. Show that for each probability measure μ on the line there exist probability measure μ_n with finite support such that $\mu_n \Rightarrow \mu$. Show further that $\mu_n\{x\}$ can be taken rational and that each point in the support can be taken rational.

Problem 3. Let $\{X_n\}$ be a sequence of r.v.'s and let $S_n = \sum_1^n X_i$. Then $\sum_1^n X_n$ converges a.s. if and only if

$$\lim_{n \rightarrow \infty} \lim_{k \rightarrow \infty} P\left\{\max_{n \leq r \leq k} |S_r - S_n| > \varepsilon\right\} = 0$$

for each $\varepsilon > 0$.

Problem 4. Let $\{X_n\}_{n=1}^\infty$ be a sequence of independent random variables such that $X_n = \pm 1$ each with probability $\frac{1}{2}$ and let $\{a_n\}_{n=1}^\infty$ be a sequence of real numbers.

1. Find a necessary and sufficient condition for the series $\sum_{n=1}^\infty a_n X_n$ to converge a.s.
2. If $a_n = \frac{1}{2^n}$ prove that $\sum_{n=1}^\infty a_n X_n$ has the uniform distribution over $[-1, 1]$.