Homework 7 (STAT 720)

Due: May 2, 2007

Problem 1. If $\{X_n\}_{n=1}^{\infty}$ is an arbitrary sequence of random variables on $(\Omega, \mathcal{F}, P)$ and $M_n = \sup_{k \geq n} |X_k - a|$, is it true that

$$[X_n \to a \text{ a.s.}] \Leftrightarrow [M_n \to 0 \text{ in probability}]?$$

[Prove your assertion in full.]

Problem 2.

1. If ϕ is a ch.f. corresponding to the d.f. F (and measure μ_F) prove that
$$\sum_{x \in \mathbb{R}} [\mu_F(\{x\})]^2 = \lim_{T \to \infty} \frac{1}{2T} \int_{-T}^{T} |\phi(t)|^2 \, dt.$$

2. Let $\phi(t)$ be the ch.f. of a r.v. X. Then one of the following three cases must hold
   (a) $|\phi(t)| < 1$ for all $t \neq 0$,
   (b) $|\phi(t_0)| = 1$ for some $t_0 > 0$ and $|\phi(t)| < 1$ for $0 < t < t_0$,
   (c) $|\phi(t)| = 1$ for all t.

Problem 3. Let $\{X_n\}_{n=1}^{\infty}$ be a sequence of r.v.'s such that for each n, X_n has a Poisson distribution with parameter λ_n. If $X_n \to X$ in distribution, show that X also has a Poisson distribution.

Problem 4. If $\{X_n\}$ is a sequence of r.v.'s with ch.f.'s $\{\phi_n\}$, the following are equivalent

1. $X_n \to 0$ in probability,
2. $X_n \to 0$ in distribution,
3. $\phi_n(t) \to 1$ for all t,
4. $\phi_n(t) \to 1$ for some neighborhood of $t = 0$.