Spatio-temporal Gaussian models and their level crossing distributions

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Spatio-temporal models

Spatio-temporal models arise when data are collected across time as well as space. A typical example would be that of aerial laser and satellite scanning or of monitoring network (of an atmospheric pollutant or network of meteorological stations), where data are collected also over time (regular or not time intervals). Thus, the analysis has to take account of spatial dependence among the different locations but since the observations at each location are typically not independent one must take account of temporal correlations also.

Original problem: Model the non-stationary in time and space variation of the significant wave height data (a measure of the energy in carried in each sea state) complied from satellite records and buoys.

Solution -Motivation

The solution to the previous problem is the following: starting from a selected spatial covariance: $r_S(\mathbf{p}) = \sigma^2 \exp(-|\mathbf{p}|^2/(2L^2))$, we introduce temporal dependence by considering the auto-regression

$$X(\mathbf{p}, t) = \rho X(\mathbf{p} - \mathbf{v}dt, t - dt) + \sqrt{1-\rho^2}\Phi_t(\mathbf{p})$$

with $\Phi_t(\mathbf{p})$ independent in time innovations having the covariance r_S.
Motivation: at each time step the past surface is moving forward to a new location with velocity $\mathbf{v}$ and is modified by an independent innovation with prescribed (fixed) spatial covariance structure. The resulting (stationary) covariance is of the form $r(\mathbf{p}, t) = \rho^t r_S(\mathbf{p} - \mathbf{v}t)$. Then, model has been extended to account for non-stationarity in space at three different levels:

- Use of non-stationary innovations $\Phi_t(\mathbf{p})$ with $r_S(\mathbf{p}, \mathbf{p}')$
- Dependence of ρ - the autocorrelation parameter - on location
- Dependence of $\mathbf{v}$ - the velocity field - on both location and time

Extension

The work presented here extends the above model in several aspects:

- Provide with a fully continuous set-up by means of properly defined moving averages in time of independent spatial fields

$$X(\mathbf{p},t) = \int f(t-s)\Phi(\mathbf{p};\mathrm{d}s).$$

- For any covariance $r_S(\mathbf{p})$ and a general class of correlations in time $\rho(t)$ we represent Gaussian fields with covariance structure $r_S(\mathbf{p})\rho(t)$ - extension to non-stationary case
- Dynamics are included in the model by a time varying velocity field that generates a flow of diffeomorphisms ψ_t and is incorporated by means of stochastic integrals:

$$X(\mathbf{p},t) = \int f(t,s;\mathbf{p})\ \Phi(\psi_{-t-s}(\mathbf{p}),ds),$$

where $\psi_{-t-s}(\mathbf{p})$ is the position $t+s$ time units ago of the point currently at $\mathbf{p}$.

Basics on Random Fields

A random field is a stochastic process, usually taking values in a Euclidean space, and defined over a parameter space of dimensionality at least one. A real valued Gaussian random field is a random field f for which the finite dimensional distributions of $\big(f(\mathbf{t}_1), \ldots, f(\mathbf{t}_n)\big)$ are multivariate Gaussian for each $1 \leq n < \infty$ and each $(\mathbf{t}_1, \ldots \mathbf{t}_n) \in T^n$, where T is the parameter space, usually for us $T \equiv \mathbb{R}^3$ or $T \equiv \mathbb{R}^2 \times \mathbb{R}^+$.

- $\mu(\mathbf{t}) = E(f(\mathbf{t}))$ – the mean value
- $r(\mathbf{s}, \mathbf{t}) = \text{Cov}(f(\mathbf{s}), f(\mathbf{t})) = E(f(\mathbf{s}) - \mu(\mathbf{s}))(f(\mathbf{t}) - \mu(\mathbf{t}))$ – covariance function
- Field is strictly homogeneous or stationary if $\big(f(\mathbf{t}_1), \ldots, f(\mathbf{t}_k))\big) \stackrel{d}{=} \big(f(\mathbf{t}_1 + \boldsymbol{\tau}), \ldots, f(\mathbf{t}_k + \boldsymbol{\tau}))\big) \Longrightarrow \mu(\mathbf{t}) = c$ and $r(\mathbf{s}, \mathbf{t}) = r(\mathbf{s} - \mathbf{t})$
- Isotropic if $r(\mathbf{t}) = r(|\mathbf{t}|)$

Also other direction: Given any set T, a function $\mu : T \to \mathbb{R}$, and a non-negative definite $r : T \times T \to R$, there exists a Gaussian field on T with mean μ and covariance r.

Basics on Stochastic Integration I

We sketch the steps for establishing the existence of integrals

$$X(\mathbf{p}, t) = \int f(t, s)\Phi(\mathbf{p}; \mathrm{d}s)$$

for deterministic function $f \in \mathcal{L}^2$.

- $\forall t \in \mathbb{R}$ let $r_S(\mathbf{p}, \mathbf{p}'; t)$ be a spatial covariance (of independent innovations at time t)
- if $r_{(a,b]}(\mathbf{p}, \mathbf{p}') = \int_a^b r_S(\mathbf{p}, \mathbf{p}'; s)\mathrm{d}s$ is well defined as function of $\mathbf{p}$ and $\mathbf{p}'$, is a spatial covariance
- there exists Gaussian fields $\Phi(\mathbf{p}; (a, b])$ centered at zero such that

  (i) For each $a < b, c < d \in \mathbb{R}$

$$r_{(a,b]\cap(c,d]}(\mathbf{p}, \mathbf{p}') = \mathrm{Cov}(\Phi(\mathbf{p}; (a, b]), \Phi(\mathbf{p}'; (c, d]))$$

  (ii) For $(a, b] = \bigcup_{i=1}^{\infty}(a_i, b_i]$, with $(a_i, b_i]$ disjoint, with probability one

$$\Phi(\mathbf{p}; (a, b]) = \sum_{i=1}^{\infty} \Phi(\mathbf{p}; (a_i, b_i])$$

Basics on Stochastic Integration II

Thus Φ is a σ-additive measure whose values are Gaussian random fields. For a step function $f(t) = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{(a_i,b_i]}(t)$ and $(a_i, b_i]$ disjoint, we define:

$$X(\mathbf{p}) := \int f(s)\Phi(\mathbf{p}; ds) = \sum_{i=1}^{n} \alpha_i \Phi(\mathbf{p}; (a_i, b_i]).$$

It follows immediately that $X(\mathbf{p})$ is a Gaussian centered field with covariance

$$\text{Cov}(X(\mathbf{p}), X(\mathbf{p}')) = \int f^2(s) r_S(\mathbf{p}, \mathbf{p}'; s)\, ds = \sum_{i=1}^{n} \alpha_i^2 r_{(a_i,b_i]}(\mathbf{p}, \mathbf{p}').$$

Think of $X(\mathbf{p})$ as a mapping from simple functions to Gaussian random fields which needs to be extended to a full isomorphism for all complex valued functions f such that: $\int_{-\infty}^{\infty} |f|^2(s) \cdot r_S(\mathbf{p}, \mathbf{p}'; s)\, ds < \infty$, which is straightforward using standard measure theoretic arguments. General functions $f(t, s; \mathbf{p})$ can be considered also.
So existence of integrals

$$X(\mathbf{p}, t) = \int f(t, s; \mathbf{p})\Phi(\mathbf{p}, \mathrm{d}s)$$

Partial derivative fields

In an analogous way we show the existence of (mean square) partial derivative fields:

$$X^x(\mathbf{p}, t) = \int f(t,s)\ \Phi^x(\mathbf{p}; \mathrm{d}s) \text{ with governing covariance } r_S^{xx'}(\mathbf{p}, \mathbf{p}'; s)$$

$$X^y(\mathbf{p}, t) = \int f(t,s)\ \Phi^y(\mathbf{p}; \mathrm{d}s) \text{ with governing covariance } r_S^{yy'}(\mathbf{p}, \mathbf{p}'; s)$$

$$X^t(\mathbf{p}, t) = \int f^t(t,s)\ \Phi(\mathbf{p}; \mathrm{d}s) \text{ with governing covariance } r_S(\mathbf{p}, \mathbf{p}'; s)$$

Then it is easy to see that:

$$\mathrm{Cov}(X(\mathbf{p}, t), X^x(\mathbf{p}', t')) = \int f(t,s) f(t',s) \cdot r_S^{x'}(\mathbf{p}, \mathbf{p}'; s) \mathrm{d}s$$

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Examples

A wide class of spatio-temporal fields is generated using $X(\mathbf{p},t) = \int f(t,s;\mathbf{p})\Phi(\mathbf{p};\mathrm{d}s)$ for appropriate choices of r_S and f.

- Stationary moving average – $f(t,s;\mathbf{p}) = f(t-s)$ and $r_S(\mathbf{p}-\mathbf{p}')$. Then we obtain the complete stationary case with covariance $r(\mathbf{p};t) = r_S(\mathbf{p})r_T(t)$ where $r_T(t) = \int f(t-s)f(-s)\mathrm{d}s = f * \tilde{f}(t) = \hat{S}_T(t)$ since $S_T(t) = |\hat{f}|^2$, and $\tilde{f}(t) = f(-t)$.

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- Separable covariance model – $f(t, s; \mathbf{p}) = f(t, s)$ and $r_S(\mathbf{p}, \mathbf{p}')$. Then $r(\mathbf{p}, \mathbf{p}'; t, t') = r_S(\mathbf{p}, \mathbf{p}') r_T(t, t')$ with $r_T(t, t') = \int f(t, s) f(t', s) \mathrm{d}s$.

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- Separable covariance model – $f(t, s; \mathbf{p}) = f(t, s)$ and $r_S(\mathbf{p}, \mathbf{p}')$. Then $r(\mathbf{p}, \mathbf{p}'; t, t') = r_S(\mathbf{p}, \mathbf{p}') r_T(t, t')$ with $r_T(t, t') = \int f(t, s) f(t', s)\mathrm{d}s$.
- Heteroscedastic moving average – $f(t, s; \mathbf{p}) = f(t - s)$. Then $r(\mathbf{p}, \mathbf{p}'; t, t') = \int f(t - s) f(t' - s) r_S(\mathbf{p}, \mathbf{p}'; s)\mathrm{d}s$ – typically non-stationary in both space and time.

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- Separable covariance model – $f(t, s; \mathbf{p}) = f(t, s)$ and $r_S(\mathbf{p}, \mathbf{p}')$. Then $r(\mathbf{p}, \mathbf{p}'; t, t') = r_S(\mathbf{p}, \mathbf{p}')r_T(t, t')$ with $r_T(t, t') = \int f(t, s)f(t', s)\mathrm{d}s$.
- Heteroscedastic moving average – $f(t, s; \mathbf{p}) = f(t - s)$. Then $r(\mathbf{p}, \mathbf{p}'; t, t') = \int f(t - s)f(t' - s)r_S(\mathbf{p}, \mathbf{p}'; s)\mathrm{d}s$ – typically non-stationary in both space and time.
- Temporal stationary moving average – $f(t, s; \mathbf{p}) = f(t - s; \mathbf{p})$ and $r_S(\mathbf{p}, \mathbf{p}'; s) = r_S(\mathbf{p}, \mathbf{p}')$. Then $r(\mathbf{p}, \mathbf{p}'; t, t') = r_S(\mathbf{p}, \mathbf{p}') \cdot f_{\mathbf{p}} * f_{\mathbf{p}'}(-t)$. Stationary in time at any fixed location $\mathbf{p}$ while temporal models differ at various locations.

Temporal moving averages

Temporal M.A field is introduced by $f(t, s; \mathbf{p}) = f(t - s)$ and $r_S(\mathbf{p}, \mathbf{p}'; s) = r_S(\mathbf{p}, \mathbf{p}')$. Relation to M.A appearing in time series analysis clear through approximation by a sum: let $s = k\Delta t, \quad k = -M, \ldots, M$ for some large M and $t = n\Delta t$. Then,

$$X(\mathbf{p}, t) \approx \sum_{k=-M}^{M} f((n-k)\Delta t) \cdot \epsilon_k(\mathbf{p}) \cdot \sqrt{\Delta t},$$

where

- $\epsilon_k(\mathbf{p})$ - independent (in time) Gaussian fields with Cov$(\epsilon_k(\mathbf{p}), \epsilon_k(\mathbf{p}')) = r_S(\mathbf{p}, \mathbf{p}')$
- $$\epsilon_k(\mathbf{p}) = \frac{\Phi(\mathbf{p}; (k\Delta t, (k+1)\Delta t])}{\sqrt{\Delta t}}$$

Then

$$X_n(\mathbf{p}) = \lim_{M\to\infty, \Delta t \to 0} \sum_{k=-M}^{M} \alpha_k \epsilon_{n-k}(\mathbf{p}),$$

with $\alpha_k = \sqrt{\Delta t} \cdot f(k\Delta t)$.

Temporal Ornstein-Uhlenbeck field

A special case of $X(\mathbf{p},t) = \int f(t-s)\Phi(\mathbf{p};\mathrm{d}s)$ with governing covariance $r_S(\mathbf{p},\mathbf{p}')$ is for $f(t) = e^{-\lambda t}\mathbf{1}_{[0,\infty)}(t)$. Then:

$$X(\mathbf{p},t) = \int_{-\infty}^{t} e^{-\lambda(t-s)}\ \Phi(\mathbf{p};ds) \text{ with } r(\mathbf{p},\mathbf{p}';t) = r_S(\mathbf{p},\mathbf{p}')\cdot\frac{1}{2\lambda}e^{-\lambda|t|}$$

which corresponds to

$$X(\mathbf{p},t) = \rho X(\mathbf{p},t-\Delta t) + \sqrt{1-\rho^2}\ \Phi_t(\mathbf{p})$$

with $\rho = e^{-\lambda\Delta t}$.

The space dependent Ornstein-Uhlenbeck model is obtained by taking a space dependent $\lambda(\mathbf{p})$. Then

$$r(\mathbf{p},\mathbf{p}';t) = \frac{r_S(\mathbf{p},\mathbf{p}')}{\lambda(\mathbf{p})+\lambda(\mathbf{p}')}\begin{cases} e^{-\lambda(\mathbf{p}')\cdot t}\ ; & \text{if } t>0,\\ e^{-\lambda(\mathbf{p})\cdot t}\ ; & \text{if } t<0\end{cases}$$

Notice stationarity in time as in any other space dependent moving average.

Temporal Gaussian dependence

For the Gaussian kernel $f_{\mathbf{p}}(t) = f(t;\mathbf{p}) = \pi^{-1/4} \cdot e^{-t^2/L^2(\mathbf{p})}$, we have

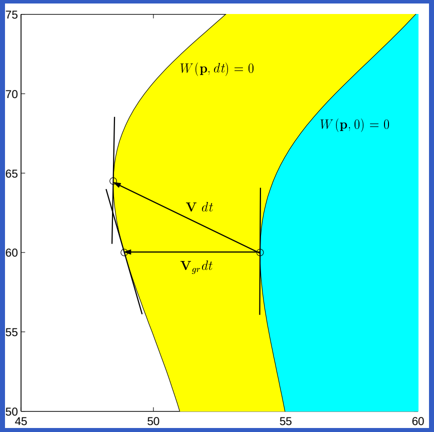
$$r(\mathbf{p},\mathbf{p}';t) = r_S(\mathbf{p},\mathbf{p}') \cdot f_{\mathbf{p}} * \tilde{f}_{\mathbf{p}'}(t) = r_S(\mathbf{p},\mathbf{p}') \cdot \left(\frac{1}{L^2(\mathbf{p})} + \frac{1}{L^2(\mathbf{p}')} \right)^{-1/2} \cdot e^{-\frac{(t-t')^2}{L^2(\mathbf{p})+L^2(\mathbf{p}')}}$$

Use of space dependent kernel $f(t,s;\mathbf{p})$ is natural for building non-stationary spatio-temporal correlation. However, one could also consider space dependent temporal spectra,

$$\int_{\mathbb{R}^{n+1}} e^{i(\mathbf{p},t)\cdot(\boldsymbol{\omega},\tau)} \sqrt{S_{\mathbf{p}}(\boldsymbol{\omega}) S^T_{\mathbf{p}}(\tau)}\, dB(\boldsymbol{\omega},\tau),$$

where $S^T_{\mathbf{p}}(\tau)$ is a location dependent temporal spectrum. The two fields are equal in distribution if $\int_{\mathbb{R}} e^{it\tau} S^T_{\mathbf{p}}(\tau)\, d\tau = f_{\mathbf{p}} * \tilde{f}_{\mathbf{p}}$, i.e. the covariances in time at a fixed point $\mathbf{p}$ are the same and the kernels $f_{\mathbf{p}}$ are symmetric with non-negative Fourier transform. For the proof is enough to show equality of covariances which is almost straightforward using Fourier transform properties. Symmetry of the kernels cannot be relaxed, for example for Ornstein-Uhlenbeck process the two approaches lead to different processes.

Velocities of a random field I



- Velocity in the direction of gradient $\mathbf{V}_{gr}$ is given by:

$$\begin{bmatrix} W_x & W_y \\ -W_y & W_x \end{bmatrix} \mathbf{V}_{gr} = - \begin{bmatrix} W_t \\ 0 \end{bmatrix}$$

- Velocity in constant gradient direction

$$\begin{aligned} W_x v_1 + W_y v_2 &= -W_t \\ (W_{xx}W_y - W_{yx}W_x)v_1 &+ (W_{xy}W_y - W_{yy}W_x)v_2 \\ &= W_{yt}W_x - W_{xt}W_y. \end{aligned}$$

Velocities for Random Surfaces II

- The speed V_{gr} in the direction of the gradient has distribution as

$$\mathbf{v}_{\max}^T \mathbf{n}_\beta + \epsilon_{gr}(\beta) T_n,$$

- The asymptotic distribution of $\mathbf{V}$ sampled on the high level contours is given by

$$\mathbf{V} = \mathbf{v}_{\max} + \frac{1}{\gamma^2 \cos^2\beta + \sin^2\beta} \frac{s_E}{s(\beta)} \frac{T_2}{\sqrt{2}} \begin{bmatrix} \gamma^2 \cos\beta \\ \sin\beta \end{bmatrix}$$

where $\mathbf{v}_{\max} = \left(-\frac{\lambda_{101}}{\lambda_{200}}, -\frac{\lambda_{011}}{\lambda_{020}} \right)$.
We say the field $X(\mathbf{p}, t)$ does not exhibit any organized motion at the point $\mathbf{p}$ and time t if the means of the distributions equal zero, i.e. if

$$\mathbb{C}ov(X^x(\mathbf{p}, t), X^t(\mathbf{p}, t)) = \mathbb{C}ov(X^y(\mathbf{p}, t), X^t(\mathbf{p}, t)) = 0$$

Special cases

For the fields we have introduced up to now

- the field

$$X(\mathbf{p}, t) = \int f(t-s)\Phi(\mathbf{p}; \mathrm{d}s)$$

  with

$$r(\mathbf{p}, \mathbf{p}'; t, t') = \int f(t-s) f(t'-s) r_s(\mathbf{p} - \mathbf{p}'; s)\mathrm{d}s$$

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- the field

$$X(\mathbf{p}, t) = \int f(t, s; \mathbf{p})\Phi(\mathbf{p}; \mathrm{d}s)$$

governed by stationary covariance, has the center of velocity at

$$-\int \left(\frac{f^x(t, s; \mathbf{p})}{A}, \frac{f^y(t, s; \mathbf{p})}{B} \right) \cdot f^t(t, s; \mathbf{p}) \cdot r_S(\mathbf{0}; s)\, ds,$$

for some A and B.

Dynamics in the spatial field

Dynamics can also be introduced by assuming the contribution to $Y(\mathbf{p})$ from the innovation $\Phi(\cdot; \mathrm{d}s)$ that occurred at time s is not evaluated at the point $\mathbf{p}$ but at the point $\psi_s(\mathbf{p})$ - the location at time s of the point currently at $\mathbf{p}$. Let $\boldsymbol{\psi}_t(\mathbf{p})$ be a group of diffeomorphisms obtained from the velocity field $\mathbf{v}(\mathbf{p}, t)$ satisfying the transport equation

$$\boldsymbol{\psi}_t(\mathbf{p}) = \mathbf{p} + \int_0^t \mathbf{v}(\boldsymbol{\psi}_u(\mathbf{p}), u)\, du,$$

i.e. for a positive t the location of $\mathbf{p}$ after t time units is $\boldsymbol{\psi}_t(\mathbf{p})$ and its location t time units ago is $\boldsymbol{\psi}_{-t}(\mathbf{p})$. Then contribution to $Y(\mathbf{p})$ at the time s is coming from $\Phi(\boldsymbol{\psi}_s(\mathbf{p}); ds)$ so the integral

$$Y(\mathbf{p}) = \int_{-\infty}^{\infty} f(s)\ \Phi(\boldsymbol{\psi}_s(\mathbf{p}); ds) := \sum_{i=1}^{n} \alpha_i \Phi(\boldsymbol{\psi}_{a_i}(\mathbf{p}); (a_i, b_i]).$$

Extension to a general f is again straightforward to give the cross-correlation formula

$$r_{X,Y}(\mathbf{p}, \mathbf{p}') = \mathrm{Cov}(X(\mathbf{p}), Y(\mathbf{p}')) = \int_{-\infty}^{\infty} f(s) \cdot g(s) \cdot r_S(\boldsymbol{\psi}_s(\mathbf{p}), \boldsymbol{\psi}_s(\mathbf{p}'); s)\ ds.$$

Spatio-temporal dynamical models

For

$$Y(\mathbf{p},t) = \int_{-\infty}^{\infty} f(t,s)\ \Phi(\boldsymbol{\psi}_{s-t}(\mathbf{p}); ds),$$

Theorem: Let the field measure $\Phi(\mathbf{p}; ds)$ be driven by stationary in space innovations, so $r_S^x(\mathbf{0}; s) = r_S^y(\mathbf{0}; s) = 0$. Then the center of the velocity is given by

$$V_1 = \frac{\displaystyle\int |f(t,s)|^2 \cdot \frac{\partial \boldsymbol{\psi}_{s-t}(\mathbf{p})}{\partial x}^T \begin{bmatrix} r_S^{xx} & r_S^{xy} \\ r_S^{yx} & r_S^{yy} \end{bmatrix} \mathbf{v}(\boldsymbol{\psi}_{s-t}(\mathbf{p}), -t)\ ds}{\displaystyle\int |f(t,s)|^2 \cdot \frac{\partial \boldsymbol{\psi}_{s-t}(\mathbf{p})}{\partial x}^T \begin{bmatrix} r_S^{xx} & r_S^{xy} \\ r_S^{yx} & r_S^{yy} \end{bmatrix} \frac{\partial \boldsymbol{\psi}_{s-t}(\mathbf{p})}{\partial x}\ ds},$$

$$V_2 = \frac{\displaystyle\int |f(t,s)|^2 \cdot \frac{\partial \boldsymbol{\psi}_{s-t}(\mathbf{p})}{\partial y}^T \begin{bmatrix} r_S^{yy} & r_S^{yx} \\ r_S^{xy} & r_S^{xx} \end{bmatrix} \mathbf{v}(\boldsymbol{\psi}_{s-t}(\mathbf{p}), -t)\ ds}{\displaystyle\int |f(t,s)|^2 \cdot \frac{\partial \boldsymbol{\psi}_{s-t}(\mathbf{p})}{\partial y}^T \begin{bmatrix} r_S^{yy} & r_S^{yx} \\ r_S^{xy} & r_S^{xx} \end{bmatrix} \frac{\partial \boldsymbol{\psi}_{s-t}(\mathbf{p})}{\partial y}\ ds},$$

where $r_S^{xx}, r_S^{yy}, r_S^{xy}, r_S^{yx}$ are all evaluated at $\boldsymbol{\psi}_{s-t}(\mathbf{p})$.

Constant velocity dynamics I

For constant velocity field $\mathbf{v}(\mathbf{p}, t) = \mathbf{v} = (v_1, v_2)$ we get

$$Y(\mathbf{p}, t) = \int_{-\infty}^{\infty} f(t, s)\ \Phi(\mathbf{p} + \mathbf{v}(s - t); ds)$$

with covariance

$$r(\mathbf{p}, \mathbf{p}'; t, t') = \int_{-\infty}^{\infty} f(t, s) \cdot f(t', s) \cdot r_S(\mathbf{p} + \mathbf{v} \cdot (s - t), \mathbf{p}' + \mathbf{v} \cdot (s - t'); s)\ ds.$$

If additionally we have spatial stationary innovation covariance r_S then

$$r(\mathbf{p} - \mathbf{p}'; t, t') = \int_{\infty}^{\infty} f(t, s) \cdot f(t', s) \cdot r_S(\mathbf{p}' - \mathbf{p} + \mathbf{v}(t' - t); s)\ ds$$

so $Y(\mathbf{p}, t) = X(\mathbf{p} + \mathbf{v} \cdot t, t)$, i.e. the dynamic field is equivalent to the static field subordinated to the deterministic dynamics.

Constant velocity dynamics II

Theorem: The center of velocities of the field Y that is driven by constant velocity $\mathbf{v} = (v_1, v_2)$ is given by

$$\left(v_1 + v_2 \frac{\int |f(t,s)|^2 r_S^{xy}(\mathbf{0};s)\ ds}{\int |f(t,s)|^2 r_S^{xx}(\mathbf{0};s)\ ds}, v_2 + v_1 \frac{\int |f(t,s)|^2 r_S^{xy}(\mathbf{0};s)\ ds}{\int |f(t,s)|^2 r_S^{yy}(\mathbf{0};s)\ ds} \right).$$

If additionally the innovations are homogeneous (isotropic and stationary), then the above velocity equals the constant flow velocity $\mathbf{v}$.

Conclusions

Thank you very much for listening!