Testing for spatial asymptotic independence of extremes using the madogram function

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Motivation

Extremes in environmental and climate fields

Spatiotemporal models

with special attention to

spatial extremal behaviour

Aim:

Test for asymptotic independence from spatial data
Extremal coefficient

\((X, Y)\) Bivariate Extreme Value distribution with same margin \(F\)

\[ \chi = \lim_{u \to u^*} \frac{P(X > u, Y > u)}{P(Y > u)} = \lim_{u \to u^*} \frac{P(X > u | Y > u)}{P(Y > u)} \]

Asymptotic independence \(\Leftrightarrow \chi = 0\)

If \(F\) Fréchet unity, \(F(x) \equiv GEV_{1,1,1}(x) = \exp(-1/x)\) (no loss of generality)

\[ P(X < x, Y < x) = F(x)^\theta \]

\[ \chi = 2 - \theta \]

\(\theta = 2\) : Asymptotic Independence
\(1 \leq \theta < 2\) : non Asymptotic Independence
Madogram test

$(X, Y)$ vector with bivariate extreme distribution $G$, margins $F_X$ and $F_Y$.

\[ Z = \frac{1}{2} |F_X(X) - F_Y(Y)| \equiv \frac{1}{2} |U - V| \]

\[ E(Z) = \frac{\theta - 1}{2(\theta + 1)} \]

\[ V(Z) = \frac{1}{6} - \left[ \frac{\theta - 1}{2(\theta + 1)} \right]^2 - \int_0^1 \frac{1}{(1 + A(1 - t))^2} dt \]

If asymptotic independence

- $E(Z) = \frac{1}{6}$
- $V(Z) = \frac{1}{72}$
- $f_Z(z) = 4 - 8z$, $z \in [0, \frac{1}{2}]$
Bivariate logistic distribution

Z distributions

pdf

independence
Madogram test

Asymptotic independence test : madogram $\nu = E(Z)$

$$H_0 : \nu = \frac{1}{6} \quad H_1 : \nu < \frac{1}{6}$$

$(X_1, Y_1), \ldots, (X_n, Y_n)$ i.i.d.

$\nu$ estimated by

$$\hat{\nu} = \frac{1}{2n} \sum_{i=1}^{n} |\hat{F}_X(X_i) - \hat{F}_Y(Y_i)|$$

Central limit theorem

$$\sqrt{n} \frac{\hat{\nu} - 1/6}{\hat{\sigma}_\nu} \to \mathcal{N}(0, 1)$$
Madogram test

Logistic model

Gaussian model

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Testing Asymptotic Independence

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Spatial case: multiple realisations

\( X \) a stationary maxima random field, with Fréchet margins \( X(s_1), X(s_2), \ldots, X(s_n) \) observations.

**Case of multiple independent realisations** \( (X^\ell(s_i))_{\ell=1,L} \)

- for each pair of points \( (s_j, s_k) \) estimate \( \hat{\nu}_F(X(s_j), X(s_k)) \)
- test: \( H_0 : \nu_{jk} = 1/6 \quad H_1 : \nu_{jk} < 1/6 \)
- derive \( p \)-values \( p_{jk} \)

Multiple hypothesis testing procedure (FDR, Positive Regression Dependency on Subset):

\[ \leftarrow \text{give pairs of points significatively non asymptotically independent.} \]
Spatial case: multiple realisations

Asymptotic independence at distance $h$ or for a class of distances $C_i = ]h_i; h_{i+1}]$?

$R$: number of rejected hypothesis under the null

Resampling procedure

1. fix a site $s_j$ and resample the vector $\mathbf{X} = t(X^\ell(s_j)), \ell = 1, \ldots, n$
2. compute the madogram with all other sites in $C_i$
3. compute the associated $p$-values
4. repeat (1) to (3) for pair of points in the class $C_i$
5. consider the obtained sample of $p$-values and use the FDR approach to count the rejected hypotheses $R^*$
6. repeat (1) to (5) a number $k$ times.

Empirical law of $R^*$ number of rejected hypothesis under null.

Conclude with

$$P(R^* \geq R)$$
Simulations : models

1. Gaussian field with autocorrelation function $\rho(\cdot)$

   \[ \theta(h) \equiv 2 \]

2. Storm process (Smith (1991), Schlather (2002))

   \[ Z(s) = \sup_{j=1,2,...} \zeta_j g(x_j - s), \quad (\zeta_j, x_j)_j \text{ Poisson on } \mathbb{R}^+ \times \mathbb{R}^2 \]

   \[ \theta(h) = 2 \Phi(\sqrt{t h M^{-1} h} / 2) \]
Simulations: Gaussian process, multiple realisations

Gaussian model: repetitions case

[0 ; 0.025]
\( pv = 0 \)

[0.025 ; 0.05]
\( pv = 0.01 \)

[0.05 ; 0.075]
\( pv = 0.09 \)

[0.075 ; 0.1]
\( pv = 0.01 \)

[0.1 ; 0.2]
\( pv = 0.12 \)

[0.2 ; 0.3]
\( pv = 0.13 \)
Simulations: Storm process, multiple realisations

Storm model: repetitions case

[0 ; 1[
pv = 0

[1 ; 2[
pv = 0

[2 ; 2.5[
pv = 0

[2.5 ; 3[
pv = 0.15

[3 ; 4[
pv = 0.17

[4 ; 5[
pv = 0.17
Spatial case: single maxima realisation

Case of a single realisation of maxima

- compute the madogram function on classes of distances $C_i = ]h_i; h_{i+1}]$

$$
\hat{\nu}(h) = \frac{1}{2n_i} \sum_{|s_j - s_k| \in C_i} \left| \hat{F}(X(s_j)) - \hat{F}(X(s_k)) \right|
$$

- compute the associated $p$-values

- FDR multiple-test procedure to determine which are significant
Simulations: gaussian process, single realisation

Gaussian model, spatial case

Dark gray: significant p-values, light gray: non significant p-values

Number of p-values

[0 ; 0.025] [0.025 ; 0.05] [0.075 ; 0.1] [0.1 ; 0.2] [0.2 ; 0.3]

h
Simulations: storm process, single realisation

Storm model, spatial case

Dark gray: significant p-values, light gray: non significant p-values

Number of p-values

0 20 40 60 80 100

[0 ; 1] [1 ; 2] [2 ; 2.5] [2.5 ; 3] [3 ; 4] [4 ; 5]

h

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Examples

**maxima daily temperatures in 29 French towns for 100 years**

**Burgundy maxima precipitations for 30 years**

*Highest residual temperatures, year 1997*

*Burgundy maxima precipitations*
Example: French maxima temperatures

Repetitions case

French maxima temperatures

[4 ; 10]  
pv = 0

[10 ; 15]  
pv = 0

[15 ; 20]  
pv = 0

[20 ; 30]  
pv = 0

[30 ; 40]  
pv = 0

[40 ; 50]  
pv = 0
Example: Burgundy maxima precipitations

Single maxima realisation

Burgundy maxima precipitations

- significant p-values
- non significant p-values