Effect of Truncation on Heavy-tailed Models

Arijit Chakrabarty
Cornell University

Joint work with Gennady Samorodnitsky

June 23, 2009

Introduction

There are situations where heavy-tail has proven to be a good fit for the underlying distribution and at the same time there is a natural upper bound for possible values.

- How to resolve this apparent paradox?
  Natural model: Truncated Heavy-tail.
- How does the upper bound effect the asymptotics?
- How to decide if the upper bound is "large enough"?

Arijit Chakrabarty
Cornell University

The setup

- B separable Banach space.
- ρ α-stable probability measure on B, $\alpha \in (0, 2)$.
- $(H, H_1, H_2, \ldots)$ are i.i.d. B-valued random variables in the domain of attraction of ρ.
- $(L, L_1, L_2, \ldots)$ are i.i.d. $[0, \infty)$ valued random variables with $EL^2 < \infty$.
- The families $(H, H_1, H_2, \ldots)$ and $(L, L_1, L_2, \ldots)$ are independent.
- $X_{nj} = H_j 1_{\{\|H_j\| \le M_n\}} + \frac{H_j}{\|H_j\|}(M_n + L_j) 1_{\{\|H_j\| > M_n\}}$ where M_n is a sequence of positive numbers going to ∞.
- $S_n := \sum_{j=1}^n X_{nj}$.

Notations

- μ denotes the Lévy measure of ρ.
- σ denotes the normalized spectral measure of ρ.

Soft Truncation

Theorem (C.-Samorodnitsky)

If M_n grows fast enough so that

$$P(\|H\| > M_n) \ll n^{-1}$$

then

$$b_n^{-1}(S_n - a_n) \Longrightarrow \rho$$

as $n \longrightarrow \infty$. Here $\{a_n\}$ and $\{b_n\}$ are such that

$$b_n^{-1}\left(\sum_{j=1}^{n} H_j - a_n\right) \Longrightarrow \rho\,.$$

Hard Truncation

M_n satisfies

$$P(\|H\| > M_n) \gg n^{-1}$$

and

$$M_n \gg 1\,.$$

A one-dimensional result

Theorem (C.-Samorodnitsky)

For every $f \in B'$

$$B_n^{-1}(f(S_n) - Ef(S_n)) \Longrightarrow N\left(0, \frac{2}{2-\alpha}\int_{\mathcal{S}} f^2(s)\sigma(ds)\right),$$

where

$$B_n := [nM_n^2 P(\|H\| > M_n)]^{1/2}.$$

Corollary

If $B = \mathbb{R}^d$,

$$B_n^{-1}(S_n - ES_n) \Longrightarrow N_d(0, \Sigma)$$

where

$$\Sigma_{ij} = \frac{2}{2-\alpha}\int_{\mathcal{S}} s_i s_j \sigma(ds).$$

The genaral small ball criterion

Theorem (Ledoux-Talagrand)

$X \in L^2(B)$ satisfies CLT if and only if for every $\epsilon > 0$,

$$\text{(small ball criterion)} \quad \liminf_{n \to \infty} P(\|S_n - ES_n\| < \epsilon\sqrt{n}) > 0,$$

where

$$S_n := \sum_{i=1}^{n} X_i$$

and $X_1, X_2, \ldots$ are i.i.d. copies of X.

The small ball criterion

Theorem (C.-Samorodnitsky)

There is a Gaussian measure γ on B such that

$$B_n^{-1}(S_n - ES_n) \Longrightarrow \gamma$$

if and only if the following hold:

1. *(small ball criterion) For every $\epsilon > 0$*

$$\liminf_{n\to\infty} P(B_n^{-1}\|S_n - ES_n\| < \epsilon) > 0\,,$$

2. $\sup_{n\geq 1} B_n^{-1} E\|S_n - ES_n\| < \infty$.

In this case, the characteristic function of γ is given by

$$\hat{\gamma}(f) = \exp\left(-\frac{2}{2-\alpha}\int_S f^2(s)\sigma(ds)\right),\, f \in B'\,.$$

Type 2 spaces

Definition

B is said to be of type p if there is $C_p \in (0, \infty)$ so that for idependent zero mean $X_1, \ldots, X_N$,

$$E \left\| \sum_{j=1}^{N} X_j \right\|^p \leq C_p \sum_{j=1}^{N} E\|X_j\|^p .$$

Type 2 spaces

Definition

B is said to be of type p if there is $C_p \in (0, \infty)$ so that for idependent zero mean $X_1, \ldots, X_N$,

$$E \left\| \sum_{j=1}^{N} X_j \right\|^p \leq C_p \sum_{j=1}^{N} E \|X_j\|^p .$$

Theorem (Reference: Araujo-Giné)

Suppose B is a Banach space of type 2. Then every $X \in L^2(B)$ satisfies CLT, ie, there is a Gaussian measure γ such that

$$n^{-1/2} \sum_{i=1}^{n} [X_i - E(X)] \Longrightarrow \gamma .$$

Conversely, if B is a Banach space where every $X \in L^2(B)$ satisfies the CLT, then B is of type 2.

Type 2 spaces (contd.)

Arijit Chakrabarty
Cornell University

Theorem (C.-Samorodnitsky)

If B is of type 2, then there is a Gaussian measure γ on B such that

$$B_n^{-1}(S_n - ES_n) \Longrightarrow \gamma\,.$$

The characteristic function of γ is given by

$$\hat{\gamma}(f) = \exp\left(-\frac{2}{2-\alpha}\int_S f^2(s)\sigma(ds)\right), f \in B'\,.$$

Large Deviations in Hard Truncation

Assume:

- $B = \mathbb{R}^d$.
- If $\alpha > 1$, $EH = 0$. If $\alpha = 1$, H is symmetric.
- $Ee^{\epsilon L} < \infty$ for some $\epsilon > 0$.
- M_n positive sequence with

$$\begin{aligned} \lim M_n &= \infty \\ \text{and } \lim nP(\|H\| > M_n) &= \infty . \end{aligned}$$

Large Deviations

Theorem (C.-Samorodnitsky)

$S_n / \{nM_nP(\|H\| > M_n)\}$ follows LDP with speed $nP(\|H\| > M_n)$ and rate function Λ^* where Λ is given by

$$\Lambda(\lambda) :=$$

$$\begin{cases} \int_{\{\|x\|\le 1\}} \left(e^{\langle\lambda,x\rangle} - 1\right) \nu(dx) \\ \text{if } 0 < \alpha < 1 \\ \int_{\{\|x\|\le 1\}} \left(e^{\langle\lambda,x\rangle} - 1 - \langle\lambda,x\rangle\right) \nu(dx) \\ \text{if } \alpha = 1 \\ \int_{\{\|x\|\le 1\}} \left(e^{\langle\lambda,x\rangle} - 1 - \langle\lambda,x\rangle\right) \nu(dx) - \frac{\alpha}{\alpha-1} \int_{\mathcal{S}} \langle\lambda,s\rangle \sigma(ds) \\ \text{if } 1 < \alpha < 2 \end{cases}$$

and ν is defined by

$$\nu(A) := \frac{\mu(A \cap B_1)}{\mu(B_1^c)} + \sigma(A \cap \mathcal{S}).$$

Moderate Deviations

Theorem (C.-Samorodnitsky)

If $n^{1/2}M_nP^{1/2}(\|H\| > M_n) \ll a_n \ll nM_nP(\|H\| > M_n)$, then $a_n^{-1}(S_n - ES_n)$ follows LDP with speed $a_n^2/\{nM_n^2P(\|H\| > M_n)\}$ and rate Λ^* where

$$\Lambda(\lambda) := \frac{1}{2}\langle \lambda, D\lambda \rangle$$

and D is the $d \times d$ matrix with

$$D_{ij} := \frac{\alpha}{2-\alpha}\int_{\mathcal{S}} s_i s_j \sigma(ds)\,.$$

If, in addition, D is invertible, then Λ^* is given by

$$\Lambda^*(x) = \frac{1}{2}\langle x, D^{-1}x \rangle\,.$$

Large Deviations in Soft Truncation

- $B = \mathbb{R}^d$.
- If $\alpha > 1$, $EH = 0$. If $\alpha = 1$, H is symmetric.
- M_n positive sequence with

$$\lim nP(\|H\| > M_n) = 0 .$$

Large Deviations for the untruncated case

Theorem (Hult et al.)

Suppose $X, X_1, X_2, \ldots$ are i.i.d. $\mathbb{R}^d$-valued random variables in the domain of attraction of some α-stable distribution $(0 < \alpha < 2)$ with Lévy measure μ. Define

$$S_n := \sum_{i=1}^{n} X_i \, .$$

Then for any sequence (λ_n) increasing to ∞ such that $\lambda_n^{-1} S_n \xrightarrow{P} 0$,

$$[nP(\|X\| > \lambda_n)]^{-1} P(\lambda_n^{-1} S_n \in \cdot) \xrightarrow{v} \frac{\mu(\cdot)}{\mu(B_1^c)}.$$

Large Deviations

Theorem (C.-Samorodnitsky)

If $b_n \ll x_n \ll M_n$, then

$$\frac{P(x_n^{-1} S_n \in \cdot)}{nP(\|H\| > x_n)} \xrightarrow{v} \frac{\mu(\cdot)}{\mu(B_1^c)}$$

on $\mathbb{R}^d \setminus \{0\}$.

Large Deviations

Theorem (C.-Samorodnitsky)

Suppose $k \geq 1$ and that $P(L > x) = o(P(\|H\| > x)^{k-1})$ as $x \longrightarrow \infty$. Then, as $n \longrightarrow \infty$,

$$\frac{P(M_n^{-1} S_n \in \cdot)}{\{nP(\|H\| > M_n)\}^k} \xrightarrow{v} \frac{1}{k!}\nu^k$$

on $\mathbb{R}^d \setminus B_{k-1}$, where

$$\nu^k(A) := \int \cdots \int 1\left(\sum_{j=1}^{k} x_j \in A\right) \nu(dx_1) \ldots \nu(dx_k)$$

and

$$\nu(A) := \frac{\mu(A \cap B_1)}{\mu(B_1^c)} + \sigma(A \cap \mathcal{S}) .$$

Large Deviations

Theorem (C.-Samorodnitsky)

(The boundary case: $k = 1$) For σ-continuous $A \subset \mathcal{S}$, as $n \longrightarrow \infty$,

$$P\left(\|S_n\| > M_n, \frac{S_n}{\|S_n\|} \in A\right) \sim$$

$$nP(\|H\| > M_n) \int_A P(\langle x, \rho\rangle > 0)\sigma(dx)\,.$$

Large Deviations

Theorem (C.-Samorodnitsky)

(The boundary case: $k \geq 2$, when σ has atoms) Assume that for every s with $\sigma(\{s\}) > 0$,

$$\lim_{t\to\infty} \frac{P\left(\|H\| > t, \frac{H}{\|H\|} = s\right)}{P(\|H\| > t)} = \sigma(\{s\}).$$

Suppose $k \geq 2$ and $P(L > x) = o(P(\|H\| > x)^{k-1})$. Then, for σ-continuous $A \subset \mathcal{S}$,

$$P\left(\|S_n\| > kM_n, \frac{S_n}{\|S_n\|} \in A\right) \sim$$

$$\{nP(\|H\| > M_n)\}^k \frac{1}{k!} \sum_{s \in A} P(\langle s, \rho\rangle \geq 0)\sigma(\{s\})^k .$$

Comparison

	limit law	large deviation probability
hard truncation	Gaussian	decay "exponentially"
soft truncation	stable	decay polynomially