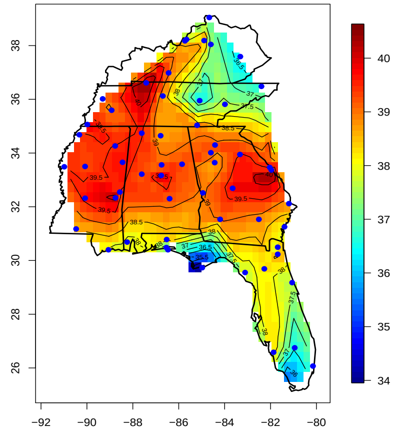

Nonparametric Spatial Models for Extreme Temperatures

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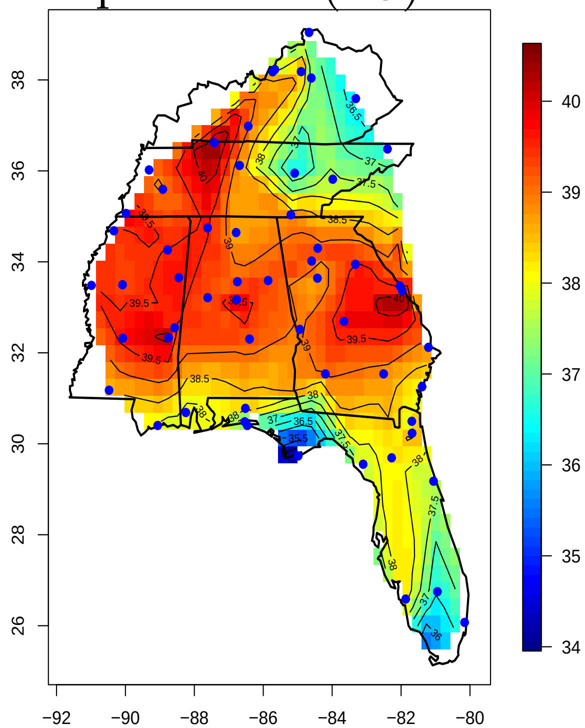
Motivation: return levels for temperatures

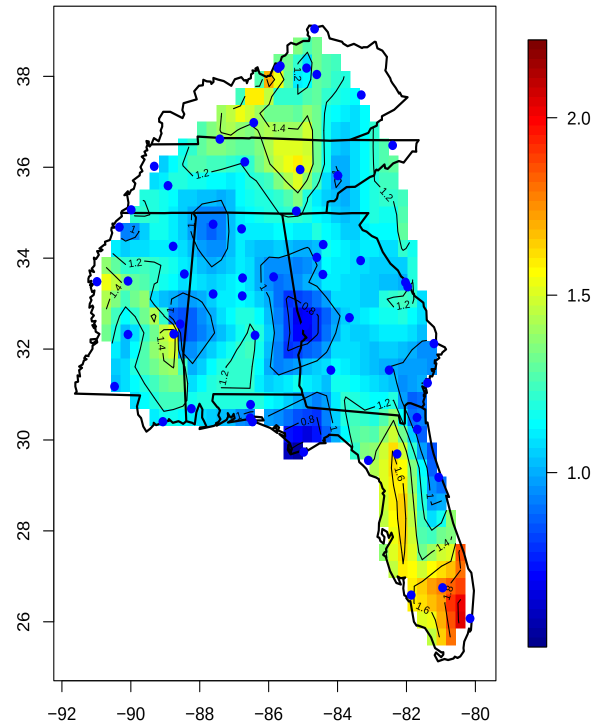
- Some of the impacts of climate change on society and wildlife will be experienced through changes in extreme weather events as global temperatures increase.
- The IPCC First Assessment Report showed that a warmer mean temperature increases the probability of extreme warm days and decreases the probability of extreme cold days.
- In North America the greatest increase in the 20-year return values of daily maximum temperature (IPCC 3rd report), is found in central and southeast North America where there is a decrease in soil moisture content.

n-year return level for annual maximum daily temperatures: the quantile which has probability $1/n$ of being exceeded in a particular year.

Data

We study air daily max temperatures in the east-south-central U.S. from 1978 to 2007. We present the mean and SD of the yearly maximum temperatures (°C). The circles are the observation locations.





Statistical challenges

- Introduce flexible measures of spatial dependence for spatial extremes.
- Study and characterize potential nonstationarity in the dependence of the extreme temperatures.
- Introduce spatial models for extreme temperatures, that allow for residual correlation (after accounting for spatially varying parameters).
- Provide uncertainty measures in the estimated return levels.
- Provide computational efficient methods.

We introduce two new Bayesian statistical methods to characterize and model spatial nonstationary dependence in extreme temperature values:

- *a flexible mixture-based copula framework, and*
- *a spatial nonparametric approach that allows for residual correlation and has univariate GEV marginals.*

We also present measures to characterize complex spatial dependence structures for extreme temperatures.

Background GEV

The generalised Extreme Value GEV distribution at each site $s \in D$, is given by

$$F_s(x; \mu, \sigma, \psi) = \exp\left[- \left\{ 1 - \frac{\xi(s)(x - \mu(s))}{\sigma(s)} \right\}_+^{-1/\xi(s)} \right], \qquad (1)$$

where μ is the location parameter, σ is the scale, and ξ is the shape.

The probability integral transformation

$$y = \left\{ 1 - \frac{\xi(s)(x - \mu(s))}{\sigma(s)} \right\}^{1/\xi(s)}, \qquad (2)$$

has a standard Fréchet distribution function ($F_s(y) = e^{-1/y}$), so the transformation $z = \frac{1}{y}$ have an exponential with mean 1 distribution function, $F_s(z) = 1 - e^{-z}$.

Measuring spatial dependence

Extremal coefficient. $(Y(s_1), \ldots, Y(s_m))$ follow an $m-$variate extreme value distribution with iid univariate margins. The extremal coefficient (Smith, 1990), θ, between sites $s_1, \ldots, s_m$ is

$$P(\max(Y(s_1), \ldots, Y(s_m)) < z) = (P(Y(s_1) < z))^{\theta}$$

for all $z \in \mathcal{R}$, where θ is independent of the value of z.

Nonstationary extremal coefficient. We have

$$P(\max(Y(s_1), Y(s_2)) < 1) = (P(Y(s_1) < 1))^{\theta(s_1, s_2)}.$$

A extremal function $\theta(s_1, s_2)$ that is a function of locations s_1 and s_2, we call it a *nonstationary extremal function.*

Threshold-specific extremal coefficient. Consider a extremal coefficient that satisfies

$$P(\max(Y(s_1), Y(s_2)) < u) = (P(Y(s_1) < u))^{\theta(u)},$$

and there is a function θ_0, such that,

$$\theta(u) = \theta_0(1),$$

for all u. Then, we name it a threshold-independent extremal coefficient.

A extremal coefficient $\theta(u)$ that depends on u is called here a *threshold-specific extremal coefficient.*

Theorem. Max-stable processes can not have threshold-specific dependence structure.

Spatial Gaussian copula

The spatial copula introduces a latent Gaussian process $R(s)$ with mean 0, variance 1, and covariance $\text{cov}(R(s_1), R(s_2)) = \rho_R(s_1, s_2)$.

Then, $T(s) = \Phi(R(s)) \sim \text{Unif(0,1)}$, where Φ is the normal CDF. To relate the latent and data processes, let G be the CDF of the standard Fréchet distribution. Then,

$$Y(s) = G^{-1}(T(s)) \sim G. \tag{3}$$

$T(s)$ determines the $Y(s)$'s percentile, and since the $T(s)$ have spatial correlation (via $R(s)$), the outcomes also have spatial correlation. The Gaussian copula C_R for the distribution function of R

$$C_R(u_1, \ldots, u_m) = F_R(\Phi^{-1}(u_1), \ldots, \Phi^{-1}(u_m)),$$

where $(u_1, \ldots, u_m) \in [0, 1]^m$, F_R is the distribution of R.

Let F_Y be the distribution of Y, for $(y_1, \ldots, y_m) \in \mathcal{R}^m$,

$$F_Y(y_1, \ldots, y_m) = C_R(G(y_1), \ldots, G(y_m)) = F_R(\Phi^{-1}G(y_1), \ldots, \Phi^{-1}G(y_m)).$$

If ρ_R is stationary, then the extremal function is stationary. Since,

$$\theta(s_1, s_2) = \theta_0(s_1 - s_2) = -\log\left(F_R(\Phi^{-1}G_{s_1}(1), \Phi^{-1}G_{s_2}(1)\right),$$

only depends on s_1 and s_2 through its vector distance, because F_R has a stationary covariance. If ρ_R is nonstationary, then the extremal function is nonstationary.

The extremal function could be also made threshold-specific,

$$\theta(s_1, s_2; u) = -u\log\left(F_R(\Phi^{-1}G_{s_1}(u), \Phi^{-1}G_{s_2}(u)\right)),$$

The multivariate (spatial) Gaussian copula may not be able to characterize complex tail spatial dependence structures, since asymptotically it does not allow for tail dependence.

Review of the non-spatial stick-breaking prior.

- To account for uncertainty in the parametric form of a distribution, F, we use a Bayesian np model and put a **prior** on the unknown distribution F.
- The stick-breaking prior for F is the potentially infinite mixture $F \stackrel{d}{=} \sum_{i=1}^{m} p_i \delta(\theta_i)$, where p_i are the mixture probabilities and $\delta(\theta_i)$, is the Dirac distribution with point mass at θ_i.
- The mixture probabilities "break the stick" into m pieces, so the sum of the pieces is one, $\sum_{i=1}^{m} p_i = 1$.
- $p_1 = V_1$, and the subsequent probabilities are $p_i = V_i \prod_{j=1}^{i-1}(1 - V_j)$, where $1 - \sum_{j=1}^{i-1} p_j = \prod_{j=1}^{i-1}(1 - V_j)$ is the probability not accounted by the first $i - 1$ components.

- $V_i \sim Beta(a, b)$ control the distribution of the probability mixture.
- the locations $\theta_i \sim F_0$, where F_0 is a known density, say Gaussian.
- A special case of this prior is the Dirichlet process prior $DP(\nu, F_0)$ with infinite m and $V_i \overset{iid}{\sim} \text{Beta}(1, \nu)$ (Ferguson, 1973).
- The SB can be extended to the spatial setting by incorporating spatial information into the locations θ_i or the p_i.

A spatial Dirichlet process (DP) copula model

The spatial DP copula introduces a latent process Z_t, s.t. in year t, for $t = 1, \ldots, T$, the joint density of $\mathbf{Z}_t = (Z_t(s_1), \ldots, Z_t(s_m))$ *given* H^m, the m-random prob. measure of the spatial part and τ^2 (the nugget component), $f(\mathbf{Z}|H^m, \tau^2)$, is a.s.

$$f_{Z_t} = \sum_{i=1}^{\infty} p_i N_m(\mathbf{Z}|\boldsymbol{\theta}_i, \tau^2 I_m), \tag{4}$$

where $\boldsymbol{\theta}_i = (\theta_i(s_1), \ldots, \theta_i(s_m))$, $p_i = V_i \prod_{j<i}(1 - V_j)$, $V_i \sim Beta(1, \nu)$,

$$\theta_{t,i}|H^m \sim_{ind} H^m, \; t = 1, ...T,$$

and $H^m = DP(\nu, H_0^m)$, $H_0^m = N_m(.|\mathbf{0}_m, \Sigma)$.

We denote F_Z the distribution of Z associated to the density in (4).

Then, $T_t(s) = H(Z_t(s)) \sim \text{Unif}(0,1)$, where H_s is the CDF for $Z_t(s)$,

$$H_s = \sum_{i=1}^{\infty} p_i \Phi(\theta_i(s)).$$

The copula C_Z for the distribution function of $Z_t(s_1), \ldots, Z_t(s_m)$ is (conditioning on the $\boldsymbol{\theta}_i$ components),

$$C_{Z_t}(u_1, \ldots, u_m) = F_{Z_t}(H_{s_1}^{-1}(u_1), \ldots, H_{s_m}^{-1}(u_m)),$$

where $u_1, \ldots u_m \in [0,1]^m$.

Then,

$$Y_t(s) \sim G^{-1}(T_t(s)) \sim G.$$

G is the CDF of the standard Fréchet distribution. The multivariate distribution of Y_t is

$$F_{Y_t}(y_1, \ldots, y_m) = C_{Z_t}(G(y_1), \ldots, G(y_m)).$$

Spatial dependence for spatial extremes using the DP copula. If the spatial covariance Σ in f_Z is nonstationary, then the resulting extremal function is nonstationary

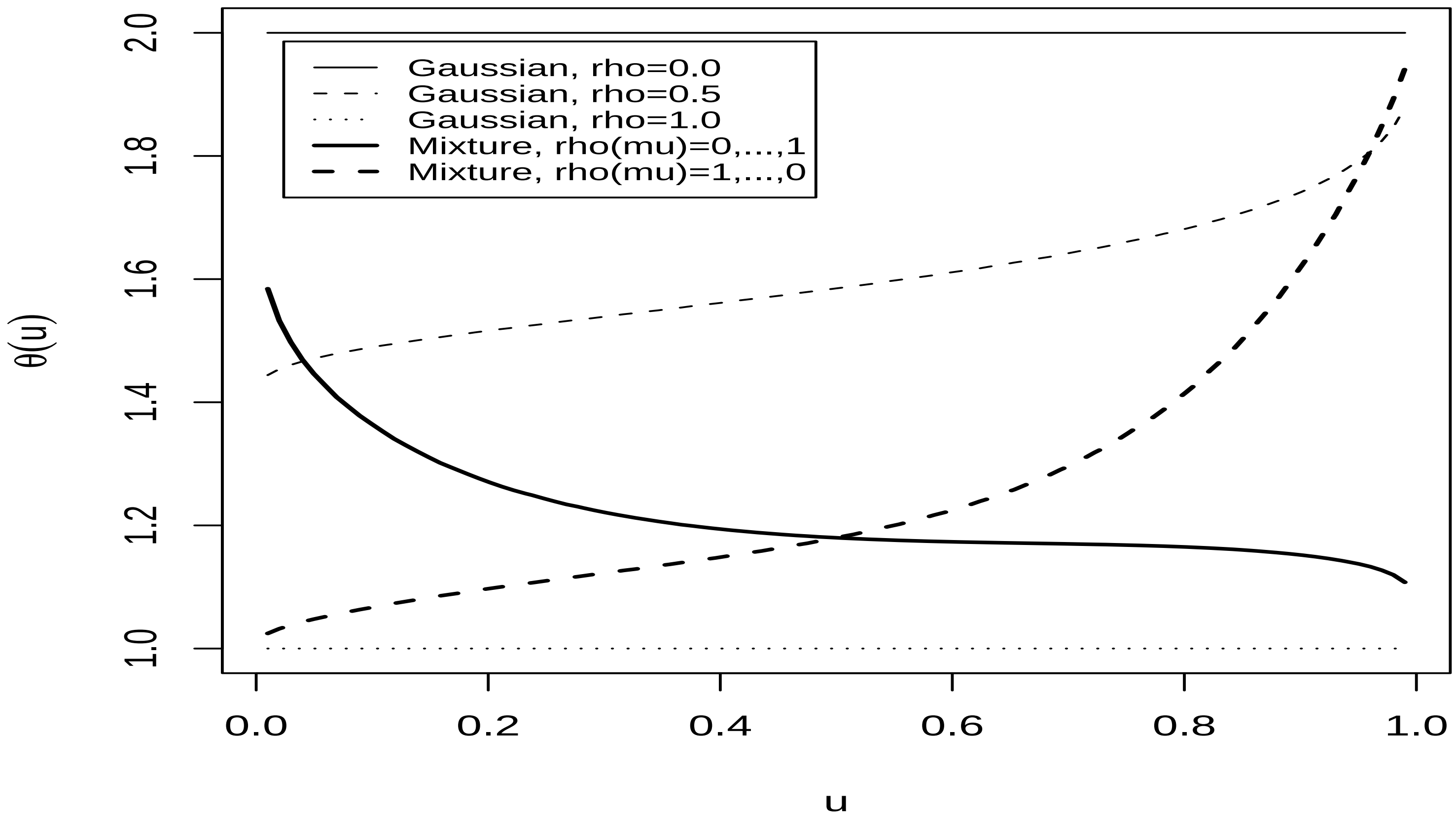
$$\theta(s_1, s_2) = -\log\big(F_{Z_t}(H_{s_1}^{-1}G(1), H_{s_2}^{-1}G(1))\big),$$

since the covariance Σ in F_{Z_t} is nonstationary. The extremal function could be also made threshold-specific,

$$\theta(s_1, s_2; u) = -u\log\big(F_{Z_t}(H_{s_1}^{-1}G(u), H_{s_2}^{-1}G(u))\big).$$

Since F_Z is a multivariate distribution, the results above can be extended and calculated simultaneously for any number of sites $\{s_1, \ldots, s_m\}$.

Figure 2: Extremal coefficient function for a Gaussian copula and for a copula that is a mixture of normal distributions, evaluated for different values of the correlation parameter ρ, and for different quantiles u (of the Fréchet).



Nonparametric approach

We introduce a new method to account for spatial correlation, while the data's marginal distribution is assumed to be a common extreme model, such as the GEV. We make two important modifications to analyze extreme temperature events:

- We allow the locations, scale, and shape parameters of the GEV distribution to vary with space and time.
- Second, we model the spatial residual associations at nearby observations. Accounting for residual correlation is crucial to obtaining reasonable measures of uncertainty for estimates of return levels.

We assume $Y_t(s)$ has a standard Fréchet distribution, for all $s \in D$.

We introduce the transformation $z = 1/y$, then $Z_t(s)$ has an Expo(1) distribution.

We model $Z_t(s) = U_t(s) + W_t(s)$, where W_t is a pure error term with a Gamma$(\gamma_W, 1)$ distribution.

We assign to U_t a version of the stick breaking prior,

$$F_{U_t(s)} = \sum_{i=1}^{M} p_i(s)\delta(\tau_i),$$

where τ_i has a Gamma$(\gamma_U, 1)$ with $\gamma_U + \gamma_W = 1$, $p_i(s)$ is the probability mass at location s modelled using kernel functions w_i.

We have $p_i(s) = V_i(s)\prod_{j=1}^{i-1}(1 - V_j(s))$, and $V_i(s) = w_i(s)V_i$. w_i(s) is centered at knot $\psi_i = (\psi_{1i}, \psi_{2i})$ and has bandwidth λ_i. The knots and the bandwidths are modelled as unknown parameters.

The distributions $F_{U_t(s)}$ are related through their dependence on the V_i and τ_i, which are given the priors $V_i \sim \text{Beta}(1, \nu)$ and $\tau_i \sim \text{Gamma}(\gamma_U, 1)$, each independent across i.

By allowing the kernel function to be space-dependent (bandwith that depends on space) we introduce nonstationarity in the extremal coefficient.

In our model we also allow the extremal function to be threshold specific.

Application

The annual maximum temperature at location s for year t, $Y_t(s)$, follows a GEV distribution with location parameter $\mu_t(s)$, scale parameter $\sigma_t(s)$ and shape parameter $\xi_t(s)$.

First, we use a copula framework to characterize the residual spatial dependence in the extreme temperatures, after accounting for the spatial structure of the GEV parameters.

To characterize the lack of stationarity in the copula, we use a covariance function for the latent process R that is a mixture of local stationary covariance functions, as in Fuentes (2001).

To allow for potential lack of stationarity in the GEV parameters, we also model them at the state level, and we allow the location's time trend to be a spatial process. We have,

$$\mu_t(s) = \alpha_\mu(S(s)) + \beta_\mu(s)t + \gamma_\mu(S(s)) * \text{elevation}(s) \quad (5)$$

$$\log[\sigma_t(s)] = \alpha_\sigma(S(s)) + \beta_\sigma(S(s))t + \gamma_\sigma(S(s)) * \text{elevation}(s) \quad (6)$$

$$\xi(s) = \alpha_\xi(S(s)) \quad (7)$$

where $S(\text{s}) \in \{1, ..., K\}$ is the state of location s, $t = 1$ corresponds to the first year of data collection, 1978, and elevation(s) is the elevation at location s.

Figure 3: Elevation in meters above the sea level.

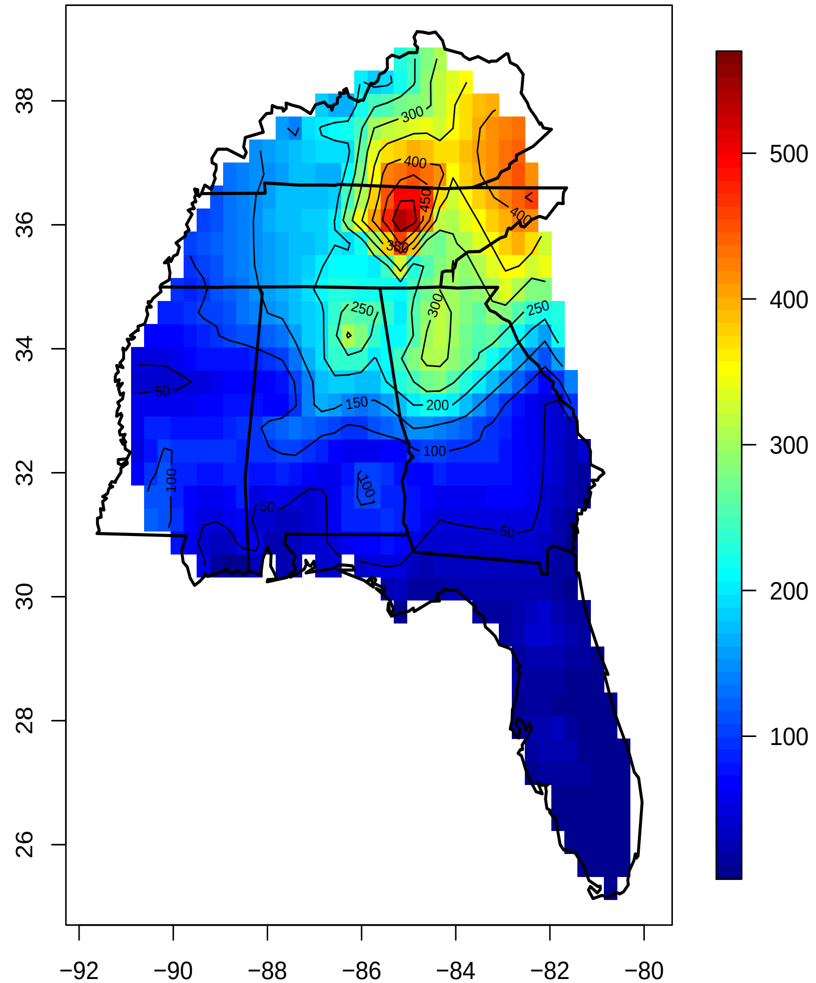


Figure 4: Posterior mean and standard deviation of the spatially-varying coefficient in the location parameter that multiplies the temporal trend, using the nonstationary Gaussian copula framework.

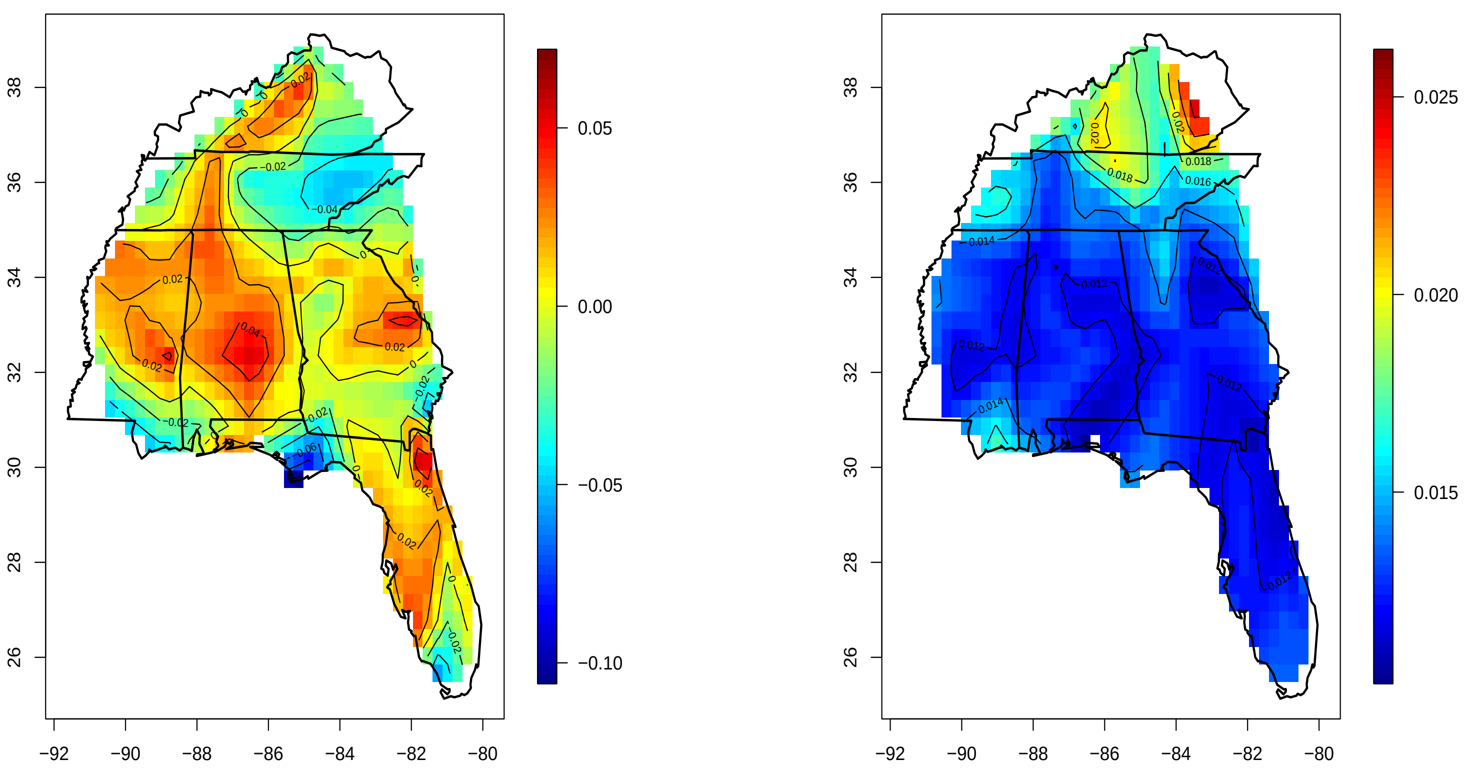


Figure 5: Extremal coefficient functions for the maximum temperatures in GA (black line) and TN (blue line) using a nonstationary Gaussian copula. In this graph we present the median of the posterior distribution for the extremal coefficient (thick lines), as well as 95% posterior bands (thin lines).

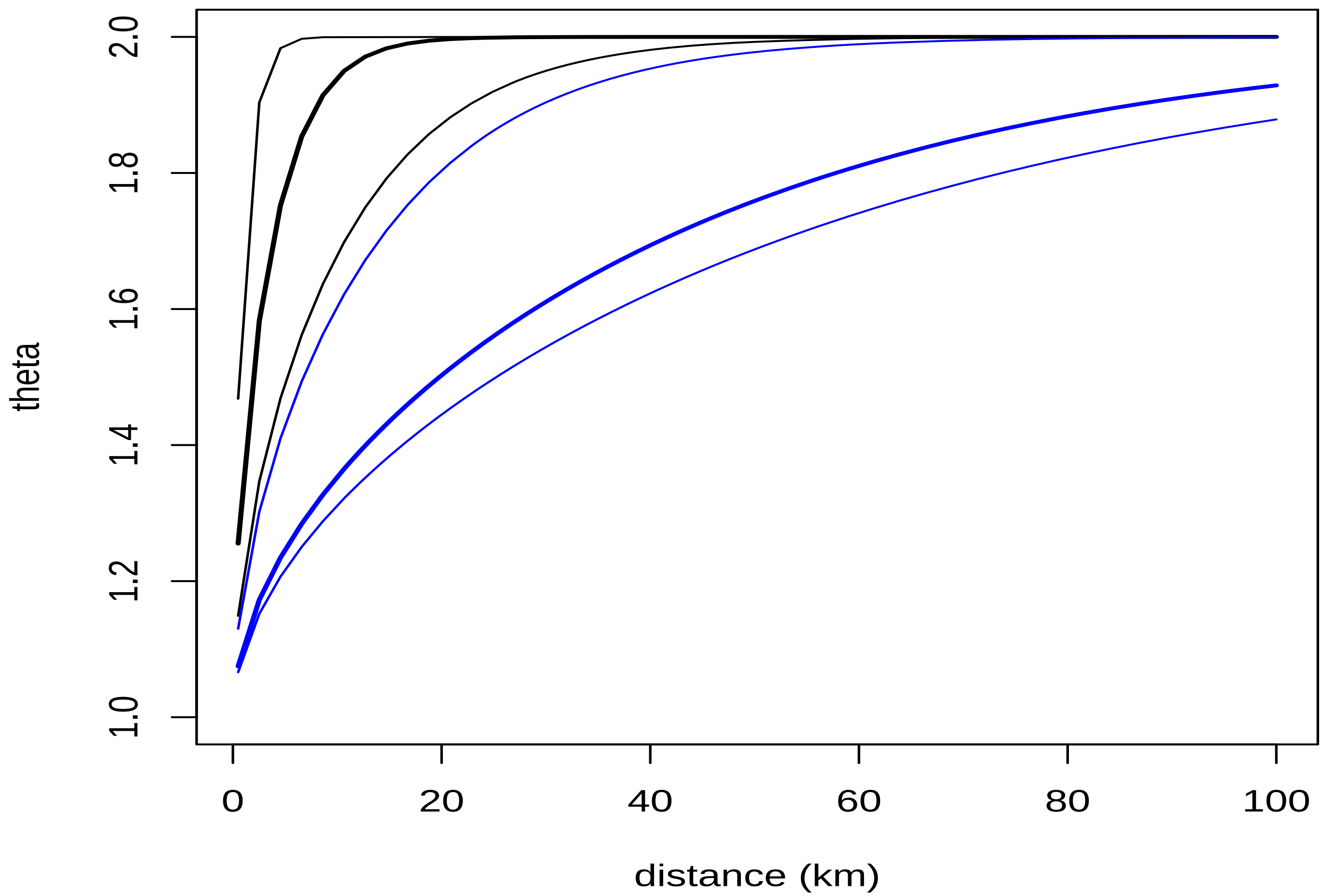
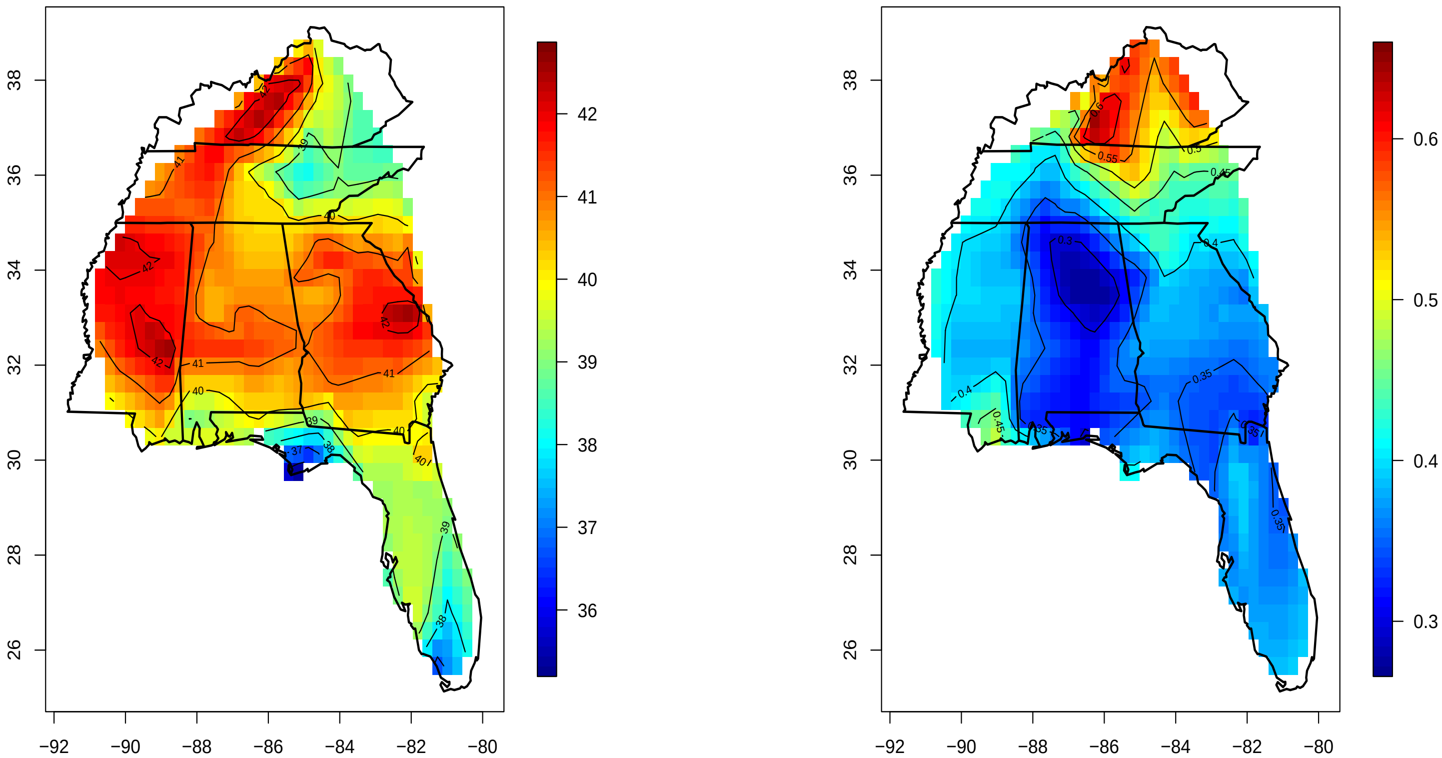
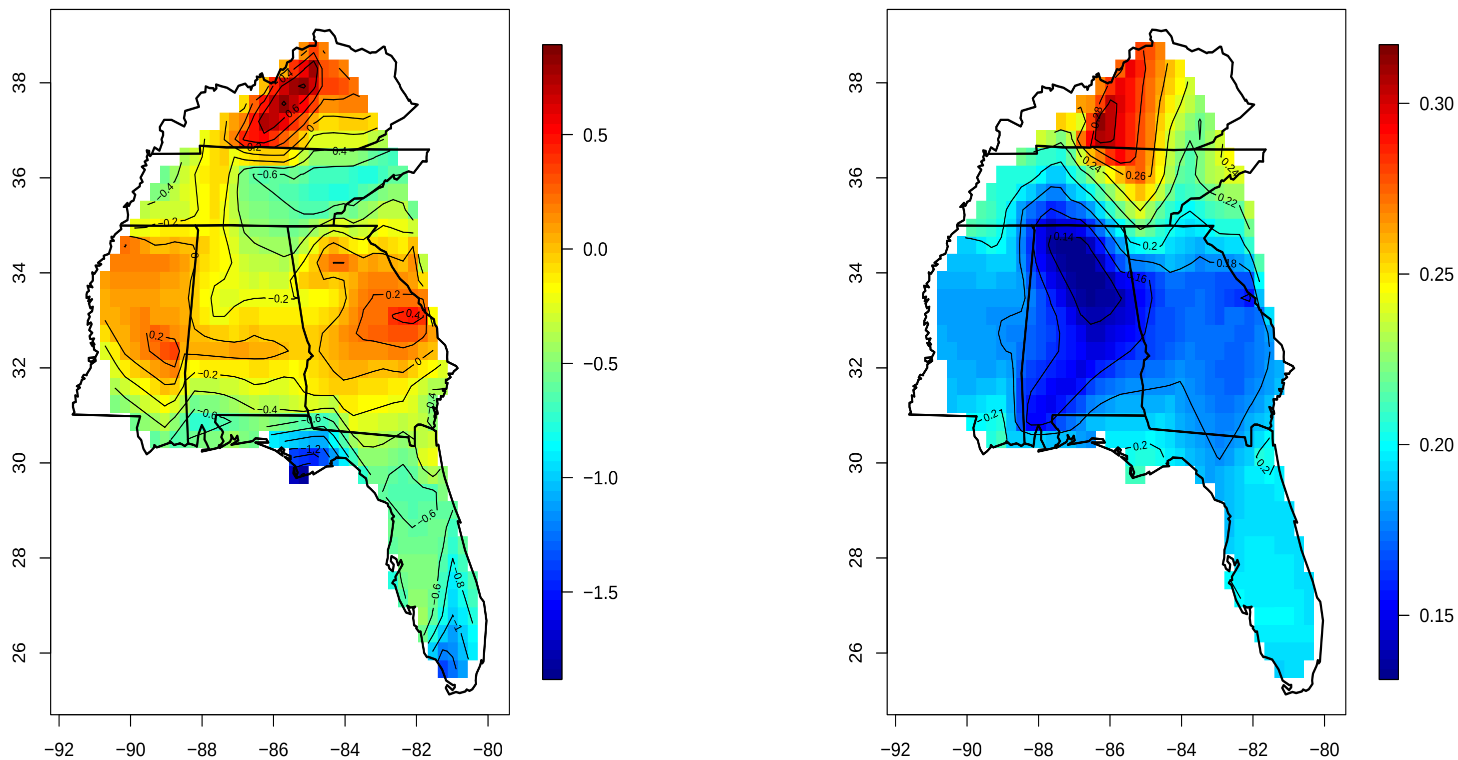


Figure 6: Mean and SD of the posterior distribution for the 20 year-return levels for surface air temperature (°C), respectively, using a nonstationary Gaussian copula.



(a) 20-year return levels (b) 20-year return levels (SD)

Figure 8: Mean and SD of the posterior distribution for the difference in the 20-year return levels for surface air temperature (°C), using a nonstationary Gaussian copula. The differences are obtained by calculating the return levels using data from 1978 - 2007, versus data from 1978-1997.



(a) Difference in 20-year return levels (b) Difference in 20-year return levels (SD)

We fit two models using the np spatial approach by varying the form of the GEV parameters. "GEV Model 1": allows all GEV parameters and the location's time trend to be spatial processes:

$$\begin{aligned} \mu_t(\mathrm{s}) &= \alpha_\mu(\mathrm{s}) + \beta_\mu(\mathrm{s})t + \gamma_\mu * \mathrm{elevation}(\mathrm{s}) \\ \log\left[\sigma_t(\mathrm{s})\right] &= \alpha_\sigma(\mathrm{s}) \\ \xi_t(\mathrm{s}) &= \alpha_\xi(\mathrm{s}) \end{aligned} \tag{8}$$

The spatial processes $\alpha_\mu()$, $\alpha_\sigma()$, $\alpha_\xi()$ have ind. spatial GP priors with mean $\bar{\alpha}_j$ and covariance $\tau_j^2 \exp(-||\mathrm{s}-\mathrm{s}'||/\rho_j)$, for $j = 1, 2, 3$.

"GEV Model 2": replaces the processes $\alpha(\mathrm{s})$ with the scalers $\bar{\alpha}_j$ for $j \in \{1, 2, 3\}$ so that only the parameter of primary interest, the location's time trend, is allowed to vary spatially.

The kernel bandwidth is space-dependent to allow for nonstationarity in the dependence structure of the extreme temperatures.

Table 1: Posterior medians (95% intervals) for the yearly maximum temperature data. The first model has spatially varying coefficients for all GEV parameters, the second model allows only the location's time trend to vary spatially.

Parameter	GEV Model 1	GEV Model 2
Stick-breaking parameter (γ_U)	0.03 (0.02, 0.07)	0.60 (0.51, 0.67)
Mean location, intercept ($\bar{\alpha}_1$)	36.8 (34.1, 39.0)	36.9 (36.9, 37.2)
Mean location, time trend ($\bar{\beta}_\mu$)	0.01 (-0.14, 0.13)	-0.01 (-0.19, 0.16)
Mean log scale ($\bar{\alpha}_2$)	-0.10 (-0.49, 0.27)	0.29 (0.22, 0.37)
Mean shape ($\bar{\alpha}_3$)	-0.20 (-0.39, -0.03)	-0.06 (-0.09, -0.02)
Mean location, elevation (γ_μ)	-0.63 (-1.00, -0.28)	-0.03 (-0.21, 0.18)
SD location, intercept (τ_1)	2.02 (1.37, 2.98)	–
SD location, time trend (τ_β)	0.10 (0.08, 0.13)	0.13 (0.10, 0.17)
SD log scale (τ_2)	0.28 (0.18, 0.50)	–
SD shape (τ_3)	0.28 (0.18, 0.44)	–

Figure 10: Posterior mean of the average bandwidth $\sum_{i=1}^{M} p_i(\mathrm{s})\lambda_i$ using two nonparametric models. Model 1 allows the other GEV parameters to vary spatially, Model 2 does not.

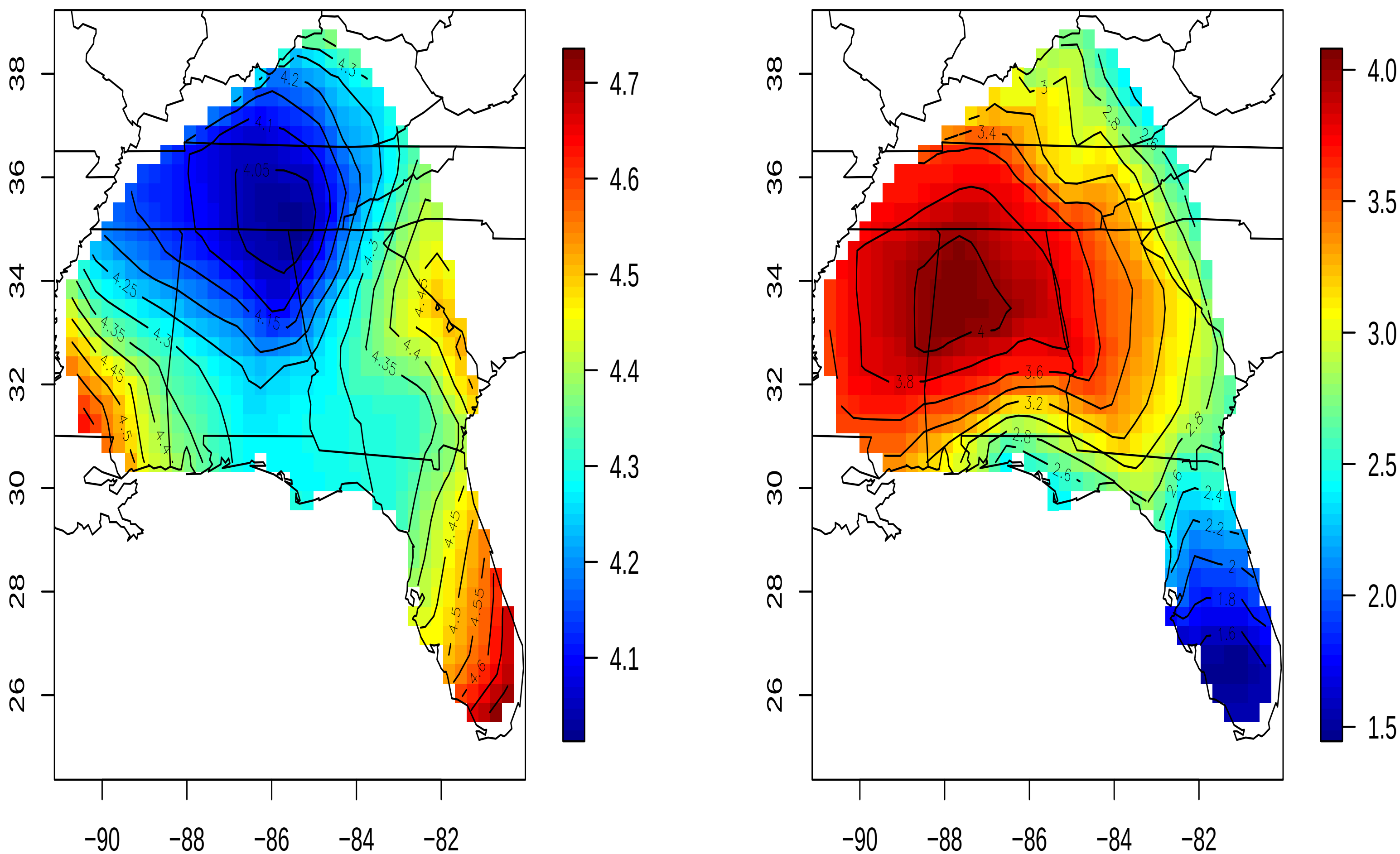
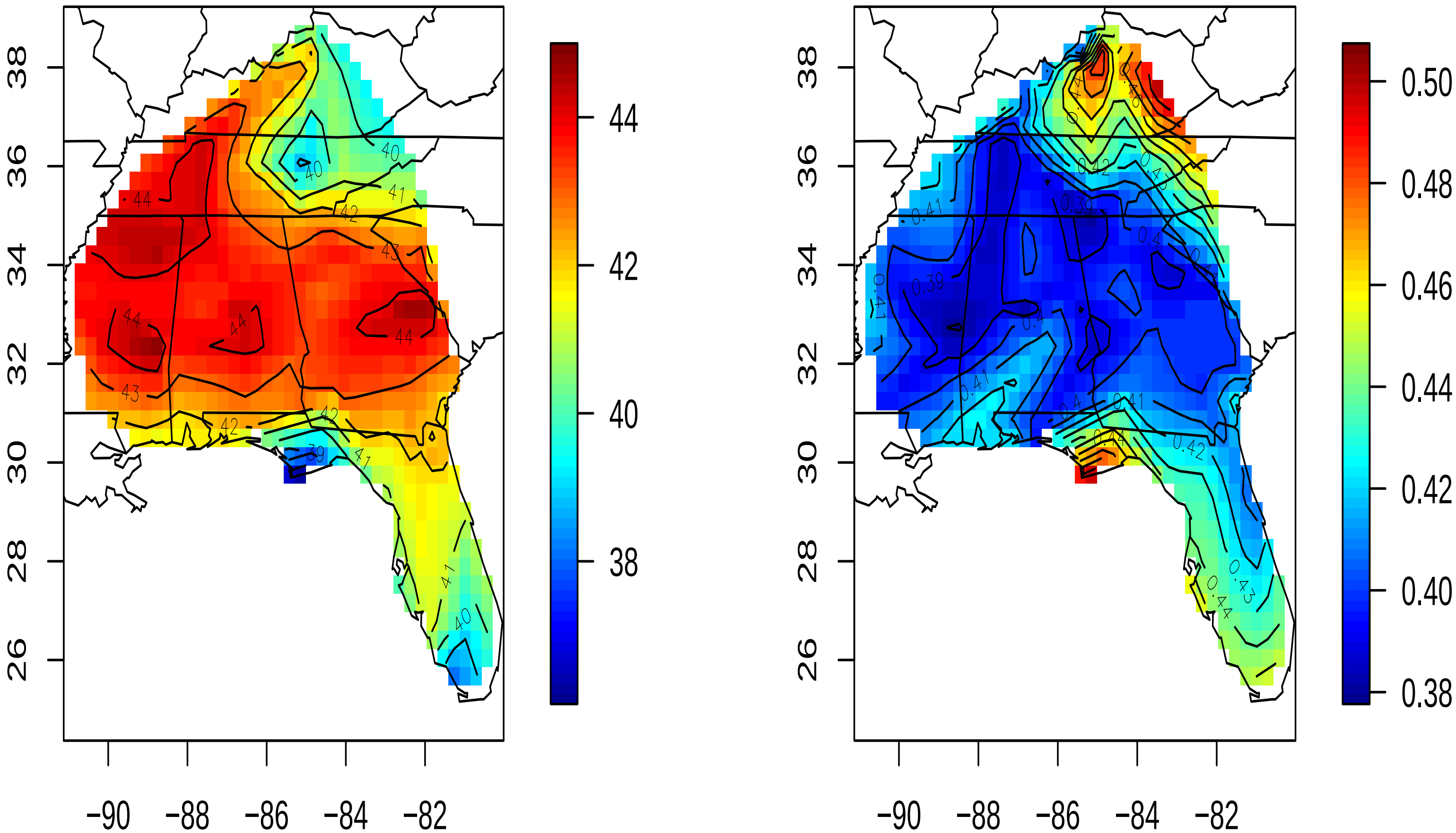


Figure 14: Mean and the SD of the posterior distribution for the 20 year-return levels, with Model 2 (no spatially varying parameters, except for location parameter).



Conclusions

- We study the spatial structure of extreme temperature values.
- We introduce modelling frameworks that offer flexible approaches to characterize complex spatial patterns and explain potential nonstationarity in the extremes.
- We present extensions of copula frameworks using Dirichlet type of mixtures.
- We present a novel nonparametric approach that has GEV marginal distributions, but also allows for residual correlation, avoiding the matrix inversion that is needed in the spatial copula methods. This offers a lot of flexibility, since complex spatial models for the GEV parameters can be avoided by putting some relatively simple structure in the residual component.