Large Deviation Principle for a Class of Long Range Dependent Infinitely Divisible Process

Souvik Ghosh

Columbia University

June 25, 2009.

Infinitely Divisible Random Variables

- A random variable X is said to be ID if for every $n \geqslant 1$ there exists i.i.d $Y_1, \ldots Y_n$ such that

$$X \stackrel{d}{=} Y_1 + \cdots + Y_n$$

- Examples: Normal, Gamma, Stable, compound Poisson, Pareto, t, χ^2, negative binomial, Gumbel, F, logistic and more.

Infinitely Divisible Random Variables

- A random variable X is said to be ID if for every $n \geqslant 1$ there exists i.i.d $Y_1, \dots Y_n$ such that

$$X \stackrel{d}{=} Y_1 + \cdots + Y_n$$

- Examples: Normal, Gamma, Stable, compound Poisson, Pareto, t, χ^2, negative binomial, Gumbel, F, logistic and more.
- Lévy Khintchine representation:

$$E\left[e^{itX}\right] = \exp\left\{it\mu - \frac{1}{2}t^2\sigma^2 + \int_{\mathbb{R}} \left(e^{itz} - 1 - it[\![z]\!]\right)\pi(dz)\right\}$$

where

$$[\![z]\!] := \frac{z}{|z| \vee 1} \quad \text{and} \quad \int_{\mathbb{R}} [\![z]\!]^2 \pi(dz) < \infty.$$

ID Processes

- $(X_n, n \in \mathbb{Z})$ is said to be Infinitely Divisible if $(X_{t_1}, \ldots, X_{t_k})$ is ID for all $t_1 < t_2 < \cdots < t_k$.
- (X_n) is the independent sum of a Gaussian process and a Poissonian process.

ID Processes

- $(X_n, n \in \mathbb{Z})$ is said to be Infinitely Divisible if $(X_{t_1}, \dots, X_{t_k})$ is ID for all $t_1 < t_2 < \cdots < t_k$.
- (X_n) is the independent sum of a Gaussian process and a Poissonian process.
- Maruyama '70: There exists a Lévy measure π on $\mathbb{R}^{\mathbb{Z}}$ which characterizes a Poissonian ID process.

ID Processes

- $(X_n, n \in \mathbb{Z})$ is said to be Infinitely Divisible if $(X_{t_1}, \ldots, X_{t_k})$ is ID for all $t_1 < t_2 < \cdots < t_k$.
- (X_n) is the independent sum of a Gaussian process and a Poissonian process.
- Maruyama '70: There exists a Lévy measure π on $\mathbb{R}^{\mathbb{Z}}$ which characterizes a Poissonian ID process.
- (X_n) stationary $\Rightarrow$ π is shift-invariant.

ID Processes

- $(X_n, n \in \mathbb{Z})$ is said to be Infinitely Divisible if $(X_{t_1}, \dots, X_{t_k})$ is ID for all $t_1 < t_2 < \cdots < t_k$.
- (X_n) is the independent sum of a Gaussian process and a Poissonian process.
- Maruyama '70: There exists a Lévy measure π on $\mathbb{R}^{\mathbb{Z}}$ which characterizes a Poissonian ID process.
- (X_n) stationary $\Rightarrow$ π is shift-invariant.

- We consider stationary, mean-zero, ergodic ID processes without a Gaussian component.

Long Range Dependence

- (X_n) is iid $\Rightarrow \pi$ is supported on $z = (\ldots, z_{-1}, z_0, z_1, \ldots)$ such that $z_i = 0$ for all but one $i \in \mathbb{Z}$.

Long Range Dependence

- (X_n) is iid $\Rightarrow \pi$ is supported on $z = (\ldots, z_{-1}, z_0, z_1, \ldots)$ such that $z_i = 0$ for all but one $i \in \mathbb{Z}$.
- (X_n) will be p-independent if π is supported on points z for which atmost p consecutive co-ordinates are non-zero.
- Generalize: short memory if π is supported on z such that $z_i \to 0$ if $|i| \to \infty$.

Long Range Dependence

- (X_n) is iid $\Rightarrow \pi$ is supported on $z = (\ldots, z_{-1}, z_0, z_1, \ldots)$ such that $z_i = 0$ for all but one $i \in \mathbb{Z}$.
- (X_n) will be p-independent if π is supported on points z for which atmost p consecutive co-ordinates are non-zero.
- Generalize: short memory if π is supported on z such that $z_i \to 0$ if $|i| \to \infty$.

- (X_n) is Long Range Dependent if there exists $\theta > 0$ such that $\pi\{z : |z_i| > \theta, \text{i.o.}\} > 0$

A Model

- Suppose there exists a recurrent Markov chain with kernel $P(x, \cdot)$ and invariant measure μ such that

$$\begin{aligned} & \pi\big(z \in \mathbb{R}^{\mathbb{Z}} : (z_0, \ldots, z_n) \in A_0 \times \cdots \times A_n\big) \\ = & \int_{A_0} \cdots \int_{A_n} \mu(dz_0) P(z_0, dz_1) \cdots P(z_{n-1}, dz_{n-1}). \end{aligned}$$

A Model

- Suppose there exists a recurrent Markov chain with kernel $P(x,\cdot)$ and invariant measure μ such that

$$\begin{aligned} &\pi\big(z \in \mathbb{R}^{\mathbb{Z}} : (z_0, \dots, z_n) \in A_0 \times \cdots \times A_n\big) \\ = \;& \int_{A_0} \cdots \int_{A_n} \mu(dz_0) P(z_0, dz_1) \cdots P(z_{n-1}, dz_{n-1}). \end{aligned}$$

- The MC will have to be null recurrent.
- Then (X_n) is long range dependent.

The Space S

- Suppose (Z_n) is a null-recurrent Markov chain on $\mathbb{Z}_+$ with transition kernel P and invariant measure μ

- Let $S = \mathbb{Z}_+^{\mathbb{Z}}$ equipped with the cylindrical σ-field.

The Space S

- Suppose (Z_n) is a null-recurrent Markov chain on $\mathbb{Z}_+$ with transition kernel P and invariant measure μ
- Let $S = \mathbb{Z}_+^{\mathbb{Z}}$ equipped with the cylindrical σ-field.
- Define the control measure m as

$$m\big(s : (s_n, \dots, s_{n+k}) \in A_0 \times \cdots \times A_k\big)$$

$$= \int_{A_0} \cdots \int_{A_k} \mu(ds_0) P(s_0, ds_1) \cdots P(s_{k-1}, ds_k)$$

The Model

- M is an IDRM on $(S, \mathcal{S})$ such that for all $\lambda \in \mathbb{R}$

$$E\left[e^{\lambda M(A)}\right] = \exp\left\{m(A)g(\lambda)\right\} \quad m(A) < \infty$$

- $f : \mathbb{Z}_+ \to \mathbb{R}$ is a “suitable” function.

- Define a stationary ID process

$$X_n = \int_S f(s_n) M(ds)$$

- $S_n = X_1 + \cdots + X_n$

Structure of (Z_n)

- Return times: $T = \inf\{n \geqslant 1 : Z_n = 0\}$

$$P_0[T > n] \in RV_{-\alpha}$$

Structure of (Z_n)

- Return times: $T = \inf\{n \geqslant 1 : Z_n = 0\}$

$$P_0[T > n] \in RV_{-\alpha}$$

- There exists $\zeta > -1$ such that

$$\frac{\mu([rn])}{\mu(n)} \to r^{\zeta} \qquad \text{for all } r > 0$$

Structure of (Z_n)

- Return times: $T = \inf\{n \geqslant 1 : Z_n = 0\}$

$$P_0\big[T > n\big] \in RV_{-\alpha}$$

- There exists $\zeta > -1$ such that

$$\frac{\mu([rn])}{\mu(n)} \to r^{\zeta} \qquad \text{for all } r > 0$$

- There exists $\psi \in RV_\beta$ for some $\beta > 0$ and a random variable V such that

$$P_n\left[\frac{T}{\psi(n)} \in \cdot\right] \Longrightarrow P\big[V \in \cdot\big]$$

Large Deviation Principle

- Assume $E(e^{\lambda X_0}) < \infty$ for λ in nbhd of 0.
- $S_n = X_1 + \cdots + X_n$
- (S_n/a_n) satisfies LDP with speed (b_n) and rate function $J(\cdot)$ if

$$\frac{1}{b_n} \log P\left[\frac{S_n}{a_n} \in \cdot\right] \approx J(\cdot)$$

- (a_n): normalizing sequence, (b_n): speed sequence.

Large Deviation Principle

- Assume $E(e^{\lambda X_0}) < \infty$ for λ in nbhd of 0.
- $S_n = X_1 + \cdots + X_n$
- (S_n/a_n) satisfies LDP with speed (b_n) and rate function $J(\cdot)$ if
$$\frac{1}{b_n} \log P\left[\frac{S_n}{a_n} \in \cdot\right] \approx J(\cdot)$$
- (a_n): normalizing sequence, (b_n): speed sequence.
- Cramer's Theorem: If (X_n) are iid then (S_n/n) satisfies LDP with speed n.

Large Deviation Principle

- Assume $E(e^{\lambda X_0}) < \infty$ for λ in nbhd of 0.
- $S_n = X_1 + \cdots + X_n$
- (S_n/a_n) satisfies LDP with speed (b_n) and rate function $J(\cdot)$ if

$$\frac{1}{b_n} \log P\left[\frac{S_n}{a_n} \in \cdot\right] \approx J(\cdot)$$

- (a_n): normalizing sequence, (b_n): speed sequence.
- Cramer's Theorem: If (X_n) are iid then (S_n/n) satisfies LDP with speed n.
- Same holds for some short memory process as well but not for long memory processes.

Large Deviation Principle

$$Y_n(t) = \frac{1}{a_n} S_{[nt]} \qquad \text{for all } t \in [0,1]$$

where

$$a_n = \mu([\psi^{-1}(n)])\psi^{-1}(n)\gamma(n) \qquad a_n \in RV_{(\zeta+1+\alpha\beta)/\beta}$$

Large Deviation Principle

$$Y_n(t) = \frac{1}{a_n} S_{[nt]} \qquad \text{for all } t \in [0,1]$$

where

$$a_n = \mu([\psi^{-1}(n)])\psi^{-1}(n)\gamma(n) \qquad a_n \in RV_{(\zeta+1+\alpha\beta)/\beta}$$

Theorem

"Under some conditions" (Y_n) satisfies LDP in $\mathcal{BV}$ with sup-nom topology, a good rate function and speed (b_n)

$$b_n = a_n/\gamma(n) \qquad b_n \in RV_{(\zeta+1)/\beta}$$

Ruin Probability

$$R(n) = P\left[S_k - a_k\mu > n \text{ for some } k \geqslant 1\right]$$

Theorem

Under the assumptions made before there exists $C > 0$ such that

$$\lim_{n\to\infty} \frac{1}{b_n} \log R(n) = -C$$

Behind the Scenes

Gartner-Ellis:

$$\Lambda_n(\lambda) = \frac{1}{b_n} \log E\left[\exp\left(\lambda \frac{b_n}{a_n} S_n\right)\right] \to \Lambda(\lambda)$$

Behind the Scenes

Gartner-Ellis:

$$\Lambda_n(\lambda) = \frac{1}{b_n} \log \mathsf{E}\Big[\exp\Big(\lambda \frac{b_n}{a_n} S_n\Big)\Big] \to \Lambda(\lambda)$$

How to approximate $\Lambda_n(\lambda)$

$$\begin{aligned} \Lambda_n(\lambda) &= \frac{1}{b_n} \int_S g\Big(\frac{\lambda}{\gamma(n)}\big(f(z_1) + \cdots f(z_n)\big)\Big) m(dz) \\ &= \frac{1}{b_n} \sum_{k=0}^{\infty} \mu(k) \int_{[z_0=k]} g\Big(\frac{\lambda}{\gamma(n)}\big(f(z_1) + \cdots f(z_n)\big)\Big) \frac{m(dz)}{\mu(k)} \end{aligned}$$

Proof Continued...

One term:

$$\int_{[z_0=0]} g\Big(\frac{\lambda}{\gamma(n)}\big(f(z_1)+\cdots f(z_n)\big)\Big)\frac{m(dz)}{\mu(0)}$$

$$= \quad E_0\Big[g\Big(\frac{\lambda}{\gamma(n)}\big(f(Z_1)+\cdots f(Z_n)\big)\Big)\Big]$$

Proof Continued...

One term:

$$\int_{[z_0=0]} g\Big(\frac{\lambda}{\gamma(n)}\big(f(z_1)+\cdots f(z_n)\big)\Big)\frac{m(dz)}{\mu(0)}$$
$$= \quad E_0\Big[g\Big(\frac{\lambda}{\gamma(n)}\big(f(Z_1)+\cdots f(Z_n)\big)\Big)\Big]$$

CLT for MC: Chen '99

$$\frac{1}{\gamma(n)}\sum_{k=1}^{n} f(Z_k) \Rightarrow S_\alpha^{-\alpha} \quad \text{for any initial distribution.}$$

Hence the assumption on g.

Proof Continued...

Another term: $k = [\psi^{-1}(n)]$

$$\begin{aligned} & \int_{[z_0=k]} g\Big(\frac{\lambda}{\gamma(n)}\big(f(z_1)+\cdots f(z_n)\big)\Big)\frac{m(dz)}{\mu(0)} \\ = \quad & E_k\Big[g\Big(\frac{\lambda}{\gamma(n)}\big(f(Z_1)+\cdots f(Z_n)\big)\Big)\Big] \\ \approx \quad & E_k\Big[g\Big(\frac{\lambda}{\gamma(n)}\big(f(Z_{T+1})+\cdots f(Z_n)\big)\Big)\Big] \end{aligned}$$

Proof Continued...

Another term: $k = [\psi^{-1}(n)]$

$$\int_{[z_0=k]} g\Big(\frac{\lambda}{\gamma(n)}\big(f(z_1)+\cdots f(z_n)\big)\Big)\frac{m(dz)}{\mu(0)}$$

$$= \quad E_k\Big[g\Big(\frac{\lambda}{\gamma(n)}\big(f(Z_1)+\cdots f(Z_n)\big)\Big)\Big]$$

$$\approx \quad E_k\Big[g\Big(\frac{\lambda}{\gamma(n)}\big(f(Z_{T+1})+\cdots f(Z_n)\big)\Big)\Big]$$

What about T?

$$T/n \Rightarrow V$$

There are about $n(1-V)$ terms left in the above sum given $V \leqslant 1$.

Example: Simple Symmetric RW on $\mathbb{Z}$

- Suppose (Z_n) is the simple symmetric RW on $\mathbb{Z}$
- $P_0[T > x] \sim cx^{-1/2}$. Therefore $\alpha = 1/2, \quad \gamma(n) = \sqrt{n}$

Example: Simple Symmetric RW on $\mathbb{Z}$

- Suppose (Z_n) is the simple symmetric RW on $\mathbb{Z}$
- $P_0[T > x] \sim cx^{-1/2}$. Therefore $\alpha = 1/2, \quad \gamma(n) = \sqrt{n}$
- μ is the counting measure. So $\zeta = 0$. Also

$$P_n \left[\frac{T}{n^2} \in \cdot \right] \Longrightarrow P\big(S_{1/2} \in \cdot\big) \quad \beta = 2$$

Example: Simple Symmetric RW on $\mathbb{Z}$

- Suppose (Z_n) is the simple symmetric RW on $\mathbb{Z}$
- $P_0[T > x] \sim cx^{-1/2}$. Therefore $\alpha = 1/2$, $\gamma(n) = \sqrt{n}$
- μ is the counting measure. So $\zeta = 0$. Also

$$P_n\left[\frac{T}{n^2} \in \cdot\right] \Longrightarrow P\big(S_{1/2} \in \cdot\big) \qquad \beta = 2$$

- $a_n = n$, $b_n = \sqrt{n}$. $Y_n(t) = n^{-1}S_{[nt]}$ satisfies LDP in $\mathcal{BV}$ with speed $\sqrt{n}$.

Another Example

- (Z_n) is a MC on $\mathbb{Z}_+$ with transition kernel

$$P(i,j) = \begin{cases} p_i q_i & \text{if } i \neq 0, j = i+1 \\ p_i(1-q_i) & \text{if } i \neq 0, j = 0 \\ 1 - p_i & \text{if } j = i \\ 1 & \text{if } i = 0, j = 1 \\ 0 & \text{otherwise} \end{cases}$$

Another Example

- (Z_n) is a MC on $\mathbb{Z}_+$ with transition kernel

$$P(i,j) = \begin{cases} p_i q_i & \text{if } i \neq 0, j = i+1 \\ p_i(1-q_i) & \text{if } i \neq 0, j = 0 \\ 1 - p_i & \text{if } j = i \\ 1 & \text{if } i = 0, j = 1 \\ 0 & \text{otherwise} \end{cases}$$

- For some $s, t > 0$, $1/2 < t - s < 1$

$$p_n = \frac{1}{(n+1)^s} \text{ and } q_n = \left(\frac{n}{n+1}\right)^t \quad \text{for every } n \geqslant 1.$$

Example Continued

- $P_0[T > x] \sim cx^{-t/(s+1)}$ as $x \to \infty$. Therefore $\alpha = t/(s+1)$.

Example Continued

- $P_0\big[T > x\big] \sim cx^{-t/(s+1)}$ as $x \to \infty$. Therefore $\alpha = t/(s+1)$.
- $\mu\big(\{n\}\big) \sim n^{s-t}$. Therefore $\zeta = s - t$

Example Continued

- $P_0\left[T > x\right] \sim cx^{-t/(s+1)}$ as $x \to \infty$. Therefore $\alpha = t/(s+1)$.
- $\mu\left(\{n\}\right) \sim n^{s-t}$. Therefore $\zeta = s - t$
- $\beta = s + 1$ since

$$P_n\left[\frac{T}{n^{s+1}} > x\right] \to \left(1 + (1+s)x\right)^{-t/(s+1)}$$

Example Continued

- $P_0[T > x] \sim cx^{-t/(s+1)}$ as $x \to \infty$. Therefore $\alpha = t/(s+1)$.
- $\mu(\{n\}) \sim n^{s-t}$. Therefore $\zeta = s - t$
- $\beta = s + 1$ since

$$P_n\left[\frac{T}{n^{s+1}} > x\right] \to \big(1 + (1+s)x\big)^{-t/(s+1)}$$

- $a_n = n$ and $b_n = n^{\frac{s-t+1}{s+1}}$. $Y_n(t) = n^{-1}S_{[nt]}$ satisfies LDP with speed b_n

Thank You!