The maximum increment of a heavy-tailed random walk[1]

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The problem

- X_i, $i = 1, 2, \dots$ iid real-valued random variables with *partial sums*

$$S_0 = 0\,, \quad S_n = X_1 + \cdots + X_n\,, \quad n \geq 1\,.$$

- We intend to investigate the asymptotic behavior of the quantities

$$T_n = \max_{1 \leq \ell \leq n} \max_{0 \leq k \leq n-\ell} (\ell\,(1 - \ell/n))^{-1/2} \left(S_{k+\ell} - S_k - \ell\,\overline{X}_n\right), \quad n \geq 1\,.$$

- The normalization in T_n is motivated by the fact that, under the assumption of iid finite variance X_i, $\mathrm{var}(S_{k+\ell} - S_k - \ell\overline{X}_n)$ is proportional to $\ell(1 - \ell/n)$.

- **Csörgő and Horvath (1997): nothing seems to be known about the (asymptotic) distributional properties of T_n.**

- **Approaches to the problem:**

  (a) Restrict the range $\ell_n \leq \ell \leq n - \ell_n$, $\ell_n \to \infty$, $\ell_n = o(n)$; e.g. Qiwei Yao (1993).

  (b) Choose different normalizing constants $\sqrt{\ell(1 - \ell/n)}$; e.g. Račkauskas and Suquet (2004).

- Statistics of type T_n appear in the context of tests for *change points in the mean under epidemic alternatives.* Given that $X_1, \dots, X_n$ are independent random variables, test
- the null hypothesis of constant mean

  H_0: $EX_1 = EX_2 = \cdots = EX_n = \mu$
- against the *epidemic alternative*

  H_A: There exist integers $1 \le k^* < m^* < n$ such that

$$EX_1 = \cdots = EX_{k^*} = EX_{m^*+1} = \cdots = EX_n = \mu\,,$$

$$EX_{k^*+1} = \cdots = EX_{m^*} = \nu \quad \text{and} \quad \mu \neq \nu\,.$$

- For iid normal X_i, under H_0 with alternative $\mu < \nu$, T_n is asymptotically equivalent to the square root of a generalized log-likelihood ratio statistic.

Related quantities

$$\widetilde{T}_n = \max_{1 \le \ell \le n} (\ell(1-\ell/n))^{-1/2} \max_{0 \le k \le n-\ell} |S_{k+\ell} - S_k - \ell\,\overline{X}_n| \,.$$

and if $\mu = EX_1$ is assumed to be known:

$$M_n = \max_{1 \le \ell \le n} \ell^{-1/2} \max_{0 \le k \le n-\ell} (S_{k+\ell} - S_k - \ell\,\mu)\,,$$

$$\widetilde{M}_n = \max_{1 \le \ell \le n} \ell^{-1/2} \max_{0 \le k \le n-\ell} |S_{k+\ell} - S_k - \ell\,\mu|\,.$$

Related results

- **Darling-Erdős (1956)** for iid standard normal X_i, **U. Einmahl (1989)** (NASC): there are $a_n > 0, b_n \in \mathbb{R}$

$$P\left(a_n^{-1}(\max_{\ell=1,\dots,n} \ell^{-1/2} S_\ell - b_n) \le x\right) \to \Lambda(x) = \mathrm{e}^{-\mathrm{e}^{-x}}, \quad x \in \mathbb{R}.$$

- **Siegmund and Venkatraman (1995)** for iid standard normal X_i: there are $a_n > 0, b_n \in \mathbb{R}$

$$P(a_n^{-1}(M_n - b_n) \le x) \to \Lambda(x), \quad x \in \mathbb{R}.$$

- **Erdős-Renýi laws**: Limits of maxima of $(S_{k+\ell_n} - S_k)_{k=1,\dots,n}$ for $\ell_n \to \infty$, $\ell_n = o(n)$, e.g. **Deheuvels and Devroye (1987)**.

Regular variation in a Banach space

- $(\mathcal{B}, \|\cdot\|)$ separable Banach space.
- $\mathcal{B}$-valued random element X is *regularly varying with index* $\alpha > 0$ if there exists a boundedly finite non-null measure μ on $\mathcal{B}_0 = \mathcal{B}\backslash\{0\}$ such that

$$\mu_n = n\, P(a_n^{-1} X \in \cdot) \xrightarrow{\widehat{w}} \mu\,,$$

where $\xrightarrow{\widehat{w}}$ is convergence in the sense that $\int_{\mathcal{B}_0} f\, d\mu_n \to \int_{\mathcal{B}_0} f\, d\mu$ for any bounded and continuous function f on $\mathcal{B}_0$ with bounded support and $P(\|X\| > a_n) \sim n^{-1}$.

- The measure μ necessarily satisfies the relation $\mu(t\cdot) = t^{-\alpha}\mu(\cdot)$, $t > 0$.

- **Equivalently, as $x \to \infty$,**

$$\frac{P(\|X\| > tx)}{P(\|X\| > x)} \to t^{-\alpha}, \quad t > 0,$$

and

$$P\left(x^{-1}\frac{X}{\|X\|} \in \cdot \mid \|X\| > x\right) \xrightarrow{w} P_\Theta$$

for some probability measure P_Θ on $\{x : \|x\| = 1\}$.

Examples

- $Y \in \mathcal{B}$ with $E\|Y\|^{\alpha+\delta} < \infty$. $\eta \in \mathbb{R}$ regularly varying with index α. Then $X = \eta Y$ is regularly varying with index α.
- (Y_i) iid $\mathbb{C}([0,1]^d)$-valued, $0 < E(Y_1(s))_+^\alpha < \infty$, $s \in [0,1]^d$ and $E\|Y_1\|^\alpha < \infty$. $\Gamma_1 < \Gamma_2 < \cdots$ points of unit rate Poisson process, independent of (Y_i). Then

$$X(s) = \sup_{j\geq 1} \Gamma_j^{-1/\alpha} Y_j(s)\,, \quad s \in [0,1]^d\,,$$

  is a *max-stable field* and regularly varying.
- (Y_i) iid $\mathcal{B}$-valued with $0 < E\|Y\|^\alpha < \infty$, $\alpha \in (0,2)$, (r_i) Rademacher, (Γ_i) mutually independent. Then $X = \sum_{i=1}^\infty \Gamma_i^{-1/\alpha}\, r_i\, Y_i$ is α*-stable* and regularly varying.

- $X \in \mathcal{B}$ regularly varying with index α. Then the *sample covariance operator* $X \otimes X$ is regularly varying with index $\alpha/2$.

Results on the maximum increment of random walks

- $X_i \in \mathcal{B}$, $i = 1, 2, \ldots$, iid regularly varying with index $\alpha > 0$.
- If $\alpha > 1$, $E\|X\| < \infty$ and then its expectation $\mu = EX$ exists. Assume $\mu = 0$.
- For $\gamma \geq 0$, we introduce the following quantities:

$$\widetilde{M}_n^{(\gamma)} = \max_{1 \leq \ell \leq n} \ell^{-\gamma} \max_{0 \leq k \leq n-\ell} \|S_{k+\ell} - S_k\|, \quad n \geq 1,$$

$$\widetilde{T}_n^{(\gamma)} = \max_{1 \leq \ell < n} (\ell(1 - \ell/n))^{-\gamma} \max_{0 \leq k \leq n-\ell} \|S_{k+\ell} - S_k - \ell \overline{X}_n\|, \quad n \geq 1.$$

- Observe that $\widetilde{M}_n = \widetilde{M}_n^{(0.5)}$ and $\widetilde{T}_n = \widetilde{T}_n^{(0.5)}$.

Theorem. Let (X_i) be a sequence of iid regularly varying random elements with values in a separable Banach space $\mathcal{B}$ and X regularly varying with index $\alpha > 0$. In addition, assume that $EX = 0$ if $\alpha > 1$ and

$$(1) \quad \sup_{n\ge 1} E\|n^{-1/\beta} S_n\| < \infty \quad \begin{cases} \beta = 2 & \text{if } \alpha > 2, \\ \text{every } \beta < \alpha & \text{if } 1 < \alpha \le 2. \end{cases}$$

Then for $\gamma > \max(0, 0.5 - \alpha^{-1})$, $P(\|X\| > a_n) \sim n^{-1}$, $x > 0$,

$$P(a_n^{-1} \widetilde{M}_n^{(\gamma)} \le x) \to \Phi_\alpha(x) = \mathrm{e}^{-x^{-\alpha}},$$

$$P(a_n^{-1} \widetilde{T}_n^{(\gamma)} \le x) \to \Phi_\alpha(x).$$

Why does this result hold?

Lemma. Assume that (X_n) is an iid sequence of regularly varying random elements with values in a separable Banach space $\mathcal{B}$ and index $\alpha > 0$. Then the following statements hold.

(1) For any $\gamma \geq 0$ and $h \geq 1$,

$$P\left(a_n^{-1} \max_{1\leq \ell \leq h} \ell^{-\gamma} \max_{0\leq k\leq n-\ell} \|S_{k+\ell} - S_k\| \leq x\right) \to \Phi_\alpha(x)\,, \quad x > 0\,.$$

(2) Assume in addition that $EX = 0$ for $\alpha > 1$ and that (X_n) satisfies the condition (1). Then, for any $\delta > 0$ and $\gamma > \max(0,\, 0.5 - \alpha^{-1})$ we have

$$\lim_{h\to\infty} \limsup_{n\to\infty} P\left(\max_{h\leq \ell\leq n} \ell^{-\gamma} \max_{0\leq k\leq n-\ell} \|S_{k+\ell} - S_k\| > \delta a_n\right) = 0\,.$$

Lemma. Let (X_n) be an iid sequence with values in $\mathcal{B}$. Then for any $\delta, \gamma > 0$, $h \geq 1$, and $H \leq n$,

$$P\Big(\max_{h\leq \ell \leq H} \ell^{-\gamma} \sup_{k\leq n} \|S_{k+\ell} - S_k\| > \delta a_n\Big)$$

$$\leq 2 \sum_{j=J_1}^{J_0} 2^j P\Big(\max_{1\leq k\leq 2n2^{-j}} \|S_k\| > \delta (n2^{-j})^{\gamma}\, a_n\Big),$$

where $J_0 = \log_2(n/h)$, $J_1 = \log_2(n/H) + 1$, and $\log_2 x$ denotes the dyadic logarithm.

Lemma. Let (X_i) be an iid sequence of regularly varying random elements with values in $\mathcal{B}$, index $\alpha > 0$ and limit measure μ. Then for any $h \geq 1$,

$$\widehat{N}_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1}(X_t,\ldots,X_{t+h-1})}$$

$$\xrightarrow{d} \widehat{N} = \sum_{i=1}^{\infty} \varepsilon_{(J_i,0,\ldots,0)} + \sum_{i=1}^{\infty} \varepsilon_{(0,J_i,0,\ldots,0)} + \cdots + \sum_{i=1}^{\infty} \varepsilon_{(0,\ldots,0,J_i)}\,,$$

where ε_x is Dirac measure at x, $J_1, J_2, \ldots$ are the points of a Poisson random measure with mean measure μ on $\mathcal{B}_0$ equipped with the Borel σ-field. Here convergence in distribution is in the space of point measures M_p on $\mathcal{B}_0^h$ equipped with the vague topology.

A combination of this lemma and the continuous mapping argument analogous to Davis and Resnick (1985) yields

$$N_n = \sum_{t=1}^{n} \varepsilon_{a_n^{-1}(X_t, X_t+X_{t+1}, \ldots, X_t+\cdots+X_{t+h-1})}$$
$$\stackrel{d}{\to} \sum_{i=1}^{\infty} \varepsilon_{(J_i, \ldots, J_i)} + \sum_{i=1}^{\infty} \varepsilon_{(0, J_i, \ldots, J_i)} + \cdots + \sum_{i=1}^{\infty} \varepsilon_{(0, \ldots, 0, J_i)} = N .$$

Write

$$B(y) = \{(x_1, \ldots, x_h) \in \mathcal{B}^h : \|x_i\| \le y \, , i = 1, \ldots, h\}$$

and for $\gamma \ge 0$,

$$\widetilde{M}_{n\ell}^{(\gamma)} = \ell^{-\gamma} \max_{0 \le k \le n} \|S_{k+\ell} - S_k\| \, , \quad \ell = 1, 2, \ldots \, .$$

Then for $y > 0$,

$$P(N_n(B(y)^c) = 0)$$

$$= P(a_n^{-1}\widetilde{M}_{n1}^{(0)} \le y\,, \ldots, a_n^{-1}\widetilde{M}_{nh}^{(0)} \le y)$$

$$\to P(N(B(x)^c) = 0)$$

$$= P\Big(\sup_{i\ge 1} \|J_i\| \le y\,, \sup_{i\ge 1} \|J_i\| \le y\,, \ldots, \sup_{i\ge 1} \|J_i\| \le y\Big)\,.$$

Since (J_i) constitute a Poisson random measure on $\mathcal{B}_0$ with mean measure μ, the transformed points $(\|J_i\|)$ constitute a Poisson random measure on $(0,\infty)$ with mean measure $\nu(y,\infty) = y^{-\alpha}$, $y > 0$. Writing the points $\|J_i\|$ in descending order, they have the representation $\Gamma_1^{-1/\alpha} > \Gamma_2^{-1/\alpha} > \cdots$, where (Γ_i) are the points of a unit rate homogeneous Poisson process on $(0,\infty)$.

Therefore for $h \geq 1$,

$$P(a_n^{-1} \max_{\ell \leq h} \widetilde{M}_{n\ell}^{(\gamma)} \leq x) = P(a_n^{-1} \max_{\ell \leq h} \ell^{-\gamma} \widetilde{M}_{n\ell}^{(0)} \leq x)$$

$$\to P(\sup_{i \geq 1} \Gamma_i^{-1/\alpha} \leq x \,, 2^{-\gamma} \sup_{i \geq 1} \Gamma_i^{-1/\alpha} \leq x \,, \ldots, h^{-\gamma} \sup_{i \geq 1} \Gamma_i^{-1/\alpha} \leq x)$$

$$= P(\Gamma_1^{-1/\alpha} \leq x)$$

$$= \mathrm{e}^{-x^{-\alpha}} = \Phi_\alpha(x) \,, \quad x > 0 \,.$$

Remarks

- For $\gamma \geq 1$ the results are trivially satisfied

$$\max_{1\leq k\leq n} \|X_k\| \leq \widetilde{M}_n^{(\gamma)} \leq \max_{1\leq \ell\leq n} \ell^{-\gamma} \max_{0\leq k\leq n-\ell} \sum_{i=k+1}^{k+\ell} \|X_k\| \leq \max_{1\leq k\leq n} \|X_k\| \,.$$

- The condition $\gamma > \max(0, 0.5 - \alpha^{-1})$ divides the α-values into two sets. For $\alpha \leq 2$ this condition is satisfied for all $\gamma > 0$, whereas it restricts γ to $(0.5 - \alpha^{-1}, \infty)$ for $\alpha > 2$. Under the assumption (1) this condition is a natural one. Indeed, by definition of (a_n), $a_n = n^{1/\alpha}/\ell(n)$ for some slowly varying function ℓ, and hence

$$a_n^{-1}\widetilde{M}_n^{(\gamma)} \geq n^{-\alpha^{-1}-\gamma+0.5}\ell(n)\|n^{-0.5}S_n\| \,.$$

- **With a stronger normalization a non-degenerate limit distribution of the sequence $(\widetilde{M}_n^{(\gamma)})$ can be achieved by an application of the invariance principle in Hölder space; see Račkauskas and Suquet (2006) assume $\gamma < 0.5 - \alpha^{-1}$ for some $\alpha > 2$ and that the central limit theorem holds for (X_n). Then**

$$n^{-0.5+\gamma}\widetilde{M}_n^{(\gamma)} \xrightarrow{d} R_{W,Q} = \sup_{s,t\in[0,1],s\neq t} \frac{\|W_Q(t) - W_Q(s)\|}{|t-s|^\gamma},$$

where $(W_Q(t))_{0\le t\le 1}$ is the $\mathcal{B}$-valued Wiener process corresponding to the covariance operator $Q = \mathrm{cov}(X)$.

- **The limit distribution of the normalized sequence $(\widetilde{M}_n^{(0.5-\alpha^{-1})})$ is in general unknown; it very much depends on the asymptotic behavior of the slowly varying function L in the tail $P(\|X\| > x) = x^{-\alpha}L(x)$.**

One-sided results for real-valued random variables

- (X_i) iid regularly varying with index $\alpha > 0$ and tail balance condition

$$P(X > x) = p\,\frac{L(x)}{x^{\alpha}} \quad \text{and} \quad P(X \le -x) = q\,\frac{L(x)}{x^{\alpha}}\,,$$

for a slowly varying function L and $p \in (0,1)$.

- The two-sided relations for $\widetilde{M}_n^{(\gamma)}$ and $\widetilde{T}_n^{(\gamma)}$ follow immediately.
- In the real-valued case one can also study the asymptotic behavior of the one-sided quantities

$$M_n^{(\gamma)} = \max_{1\le \ell\le n} \ell^{-\gamma} \max_{0\le k\le n-\ell} (S_{k+\ell} - S_k)\,,$$

$$m_n^{(\gamma)} = \min_{1\le \ell\le n} \ell^{-\gamma} \min_{0\le k\le n-\ell} (S_{k+\ell} - S_k)\,.$$

Theorem. Assume that (X_i) is an iid sequence of real-valued random variables regularly varying with index $\alpha > 0$. In addition, assume that $EX = 0$ for $\alpha > 1$. Then for $\gamma > \max(0, 0.5 - \alpha^{-1})$,

$$P(b_n^{-1} m_n^{(\gamma)} \le -x \,, b_n^{-1} M_n^{(\gamma)} \le y) \to \Phi_\alpha(y)(1 - \Phi_\alpha^{q/p}(x))\,, \quad x, y > 0\,,$$

where $b_n = \inf\{x \in \mathbb{R} : P(X \le x) \ge 1 - 1/n\}$.

We notice that

$$b_n^{-1}(m_n^{(\gamma)}, M_n^{(\gamma)}) \xrightarrow{d} (y^{(\gamma)}, Y^{(\gamma)})\,,$$

where the limit distribution is given above . In particular, $y^{(\gamma)}$ is independent of $Y^{(\gamma)}$ and the range statistic $M_n^{(\gamma)} - m_n^{(\gamma)}$ has the limit

$$b_n^{-1}(M_n^{(\gamma)} - m_n^{(\gamma)}) \xrightarrow{d} Y^{(\gamma)} - y^{(\gamma)}\,.$$

Consider the one-sided version of the statistics $\widetilde{T}_n^{(\gamma)}$:

$$T_n^{(\gamma)} = \max_{1\le \ell<n} (\ell(1-\ell/n))^{-\gamma} \max_{0\le k\le n-\ell} (S_{k+\ell} - S_k - \ell \overline{X}_n)\,, \quad n \ge 1.$$

Theorem. **Assume that (X_i) is an iid sequence of real-valued random variables with distribution F and X is regularly varying with index $\alpha > 0$. In addition, assume that $EX = 0$ for $\alpha > 1$. Then for any $\gamma > \max(0, 0.5 - \alpha^{-1})$,**

$$P(b_n^{-1} T_n^{(\gamma)} \le x) \to \Phi_\alpha(x)\,, \quad x > 0\,,$$

The following quantity has a structure similar to $M_n^{(\gamma)}$:

$$\widehat{M}_n^{(\gamma)} = \max_{\ell=1,\dots,n} \ell^{-\gamma} \max_{k=\ell+1,\dots,n-\ell} (S_{k+\ell} + S_{k-\ell} - 2\,S_k)$$

Theorem. **Assume that (X_i) is an iid sequence of real-valued random variables regularly varying with index $\alpha > 0$. Then for $\gamma > \max(0,\ 0.5 - \alpha^{-1})$,**

$$P(a_n^{-1}\widehat{M}_n^{(\gamma)} \le x) \to \Phi_\alpha^2(x)\,, \quad x > 0\,.$$

FIGURE 1. Thank you.