Aggregation of Rapidly varying Risks and Asymptotic independence

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(Joint work with S. I. Resnick)
Outline:

1. Introduction and Preliminaries
2. Main Result
3. Examples
Consider a portfolio with two assets. The risks associated with the two assets: $X$ and $Y$. Total risk of the portfolio: $X + Y$. 

Quantities of interest: $\text{VaR}(X + Y)(p)$ for $p$ close to 1, and $P(X + Y > x)$ for large $x$. Direct computation is difficult in most cases. Simulation strategies are not well-known in the case where the marginal distributions are subexponential (defined later).
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The usual approach towards obtaining $P(X + Y > x)$: Check if

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This approach might help in obtaining $P(X + Y > x)$ in two ways:

- The limit can be used to create a direct approximation of $P(X + Y > x)$
- To prove that some simulation algorithms for finding $P(X + Y > x)$ have nice properties like bounded relative error or logarithmic efficiency, often such a limit is used.
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Approach the problem using extreme value theory.
Suppose \((X, Y) \sim F(x, y)\).

**Definition**

\(F\) is in the *maximal domain of attraction* of extreme value distribution \(G\), if there exists normalizing constants \(a_n^{(i)} > 0, b_n^{(i)} \in \mathbb{R}, 1 \leq i \leq 2\), such that as \(n \to \infty\),

\[
F^n(a_n^{(1)}x^{(1)} + b_n^{(1)}, a_n^{(2)}x^{(2)} + b_n^{(2)}) \to G(x^{(1)}, x^{(2)})
\]

for limit distribution \(G\), such that each marginal \(G_i, i = 1, 2\) is non-degenerate.
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**Implications of the definition**: If the marginals of \(F\) are \(F_i\), \(i = 1, 2\),

\[
F_i^n(a_n^{(i)} x^{(i)} + b_n^{(i)}) \to G_i(x^{(i)})
\]
Since \( X \overset{d}{=} Y \), in the present case,
\[
F_1(\cdot) = F_2(\cdot), \quad G_1(\cdot) = G_2(\cdot), \quad a_n^{(1)} = a_n^{(2)}, \quad b_n^{(1)} = b_n^{(2)}.
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Multivariate Extremes

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- From Extreme Value Theory, there are only 3 choices for $G_1(\cdot)$:
  - Frechet: $G_1(x) = \begin{cases} 
  \exp(-x^{-\alpha}) & x \geq 0, \\
  0 & x < 0 
  \end{cases}$
  for some $\alpha > 0$. 

- The last case (Weibull) $\Rightarrow F_1$ has a finite right end point, so the limit makes little sense here. Hence, the case where
  \[ G_1(x) = \begin{cases} 
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- Gumbel: $G_1(x) = \exp(-e^{-x}), \quad x \in \mathbb{R}$. 
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Asymptotic independence

- Suppose, $F$ is in the maximal domain of attraction of extreme value distribution $G$, i.e. there exists normalizing constants $a_n^{(i)} > 0$, $b_n^{(i)} \in \mathbb{R}, 1 \leq i \leq 2$, such that as $n \to \infty$,

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\( X \) and \( Y \) are said to be *asymptotically independent*, if

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- In general, *asymptotic independence* is present in many models used in practice.
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Existing Results

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  the answer to the question is well known both in presence and in absence of asymptotic independence.
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the answer to the question is well known both in *presence* and in *absence* of asymptotic independence.

In the *Gumbel* case, i.e. $G_1(x) = \exp(-e^{-x}), x \in \mathbb{R}$, in *absence* of asymptotic independence, the question is recently answered by Klüppelberg and Resnick (2008).
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Remaining case: When $X \overset{d}{=} Y \in MDA(\Lambda)$, where $\Lambda(\cdot)$ is the Gumbel distribution and $X$ and $Y$ are asymptotically independent. Denote by $\mathcal{C}$ the class of 2 dimensional distributions of $(X, Y)$ for which $X \overset{d}{=} Y, (X, Y) \in MDA(G)$ where $G(x, y) = \Lambda(x)\Lambda(y)$. 
One special case: Independent $X$ and $Y$

- The class of distribution functions $C_1$, for which $X \overset{d}{=} Y \in MDA(\Lambda)$ and $X$ and $Y$ are independent, is a subclass of $C$. 

A distribution $F$ on $\mathbb{R}^+$ is subexponential if 
\[
\lim_{x \to \infty} \frac{1 - F^2(x)}{1 - F(x)} = 2.
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Let us denote the class of subexponential distributions by $S$. 
One special case: Independent $X$ and $Y$

- The class of distribution functions $C_1$, for which $X \overset{d}{=} Y \in MDA(\Lambda)$ and $X$ and $Y$ are independent, is a subclass of $C$.
  - If the distribution of $(X, Y)$ belongs to $C_1$, when is
    \[
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Answering the above question led to new definitions of classes of distribution functions. Most celebrated among these classes is the class of subexponential distributions.

Definition: A distribution $F$ on $\mathbb{R}_+$ is subexponential if
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One special case: Independent $X$ and $Y$

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- Let us denote the class of subexponential distributions by $S$. 
There are 2 distinct behaviors observed within the class $\mathcal{C}$ $(X \overset{d}{=} Y, (X, Y) \in MDA(G)$ where $G(x, y) = \Lambda(x)\Lambda(y))$:

- First, suppose $(X, Y)$ are two iid risks with common distribution $F$ and $F \in MDA(\Lambda) \cap S$. Then $X$ and $Y$ are certainly asymptotically independent and

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\lim_{x \to \infty} \frac{P(X + Y > x)}{P(X > x)} = 2.
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- Very different tail behavior is observed by Albrecher, Asmussen and Kortschak (2006), who exhibit a distribution of $(X, Y)$, with $X$ and $Y$ being identically distributed and asymptotically independent with common distribution $F \in MDA(\Lambda) \cap S$, but

\[
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So, subexponentiality of $X$ and $Y$ is not sufficient to ensure a limit in $(0, \infty)$ when $(X, Y)$ has a distribution belonging to $\mathcal{C}$. 
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Back to the general case: Problems

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- So, subexponentiality of $X$ and $Y$ is \textit{not} sufficient to ensure a limit in $(0, \infty)$ when $(X, Y)$ has a distribution belonging to $\mathcal{C}$. 

What assumptions would ensure the limit $= 2$?

- We provide sufficient conditions under which the limit is 2.
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- We provide sufficient conditions under which the limit is 2.
- To state the conditions, first define the auxiliary function \( f \) of \( F \). For a distribution \( F \in MDA(\Lambda) \), there exists an absolutely continuous function \( f \) with its derivative going to 0, such that

\[
\lim_{t \to \infty} \frac{\bar{F}(t + xf(t))}{\bar{F}(t)} = e^{-x}.
\]  

(1)

Such a function \( f \) is called the auxiliary function of \( F \) (de Haan (1970)). In common cases \( f \) is the reciprocal of the hazard function

\[
f = \frac{1 - F}{F'}.
\]
Assumptions

- We need 3 assumptions:

  1. For all \( z > 0 \), \( \lim_{x \to \infty} P( |Y| > zf(t) | X > t) = 0 \) (2)
  2. For all \( z > 0 \), \( \lim_{x \to \infty} P( |X| > zf(t) | Y > t) = 0 \) (3)
  3. For some \( L > 0 \), \( \lim_{x \to \infty} P( Y > Lf(t), X > Lf(t) ) = 0 \). (4)
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Theorem

Suppose, $X \overset{d}{=} Y \sim F \in MDA(\Lambda)$ and $x_0 = \sup\{x : F(x) < 1\} = \infty$. If (2), (3) and (4) hold, then

\[ \lim_{x \to \infty} \frac{P(X + Y > x)}{P(X > x)} = 2. \]
Find rich classes of distributions which satisfy the assumptions.
Examples

Find rich classes of distributions which satisfy the assumptions.

First model: \((X_1, X_2) \sim N(\mu, \Sigma)\), where \(\rho = \text{Correlation}(X_1, X_2) < 1\). Then, \((X, Y) = (\exp(X_1), \exp(X_2))\) satisfies all our assumptions.

Second Model: Suppose, \(X_i \overset{iid}{\sim} F, i = 1, 2, 3\), where for some \(\alpha > 1\),
\[
\bar{F}(x) = \begin{cases} 
\exp\{-\left(\log x\right)^\alpha\} & \text{if } x > 1 \\
1 & \text{if } x \leq 1 
\end{cases}
\]
Then, \((X, Y) = (X_1 \wedge X_2, X_2 \wedge X_3)\) satisfies all our assumptions.

Third Model: Suppose, \(F \in \text{MDA}(\Lambda)\), concentrated on \([0, \infty)\), such that \(x_0 = \sup\{x : F(x) < 1\} = \infty\), \(x_1 = \inf\{x : F(x) > 0\} = 0\), \(\lim_{x \to \infty} f(x) > 0\). Recall, \(f(x)\) is the auxiliary function as defined in (1). Examples of such \(F\): Exponential, Gamma, Lognormal etc.
Then, \((X, Y) = (F \leftarrow (U), F \leftarrow (1 - U))\), where \(U \sim \text{Uniform}(0, 1)\), satisfies all our assumptions.

The third model shows that subexponentiality of \(X\) and \(Y\) is not a necessary condition to ensure a limit in \((0, \infty)\).
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\(\liminf_{x\to\infty} f(x) > 0.\)
Recall, \(f(x)\) is the auxiliary function as defined in (1).
Examples of such \(F\): Exponential, Gamma, Lognormal etc.
Then, \((X, Y) = (F^\leftarrow(U), F^\leftarrow(1 - U))\), where \(U \sim \text{Uniform}(0, 1)\), satisfies all our assumptions.
Examples

- Find rich classes of distributions which satisfy the assumptions.
  - First model: \((X_1, X_2) \sim \mathcal{N}(\mu, \Sigma)\), where \(\rho = \text{Correlation}(X_1, X_2) < 1\).
    Then, \((X, Y) = (\exp(X_1), \exp(X_2))\) satisfies all our assumptions.
  - Second Model: Suppose, \(X_i \overset{iid}{\sim} F, i = 1, 2, 3\), where for some \(\alpha > 1\),
    \[
    \bar{F}(x) = \begin{cases} 
    \exp\{-(\log x)^\alpha\}, & \text{if } x > 1, \\
    1, & \text{if } x \leq 1.
    \end{cases}
    \]
    Then, \((X, Y) = (X_1 \wedge X_2, X_2 \wedge X_3)\) satisfies all our assumptions.
  - Third Model: Suppose, \(F \in \text{MDA}(\Lambda)\), concentrated on \([0, \infty)\), such that
    \(x_0 = \sup\{x: F(x) < 1\} = \infty, x_1 = \inf\{x: F(x) > 0\} = 0, \lim \inf_{x \to \infty} f(x) > 0\).
    Recall, \(f(x)\) is the auxiliary function as defined in (1).
    Examples of such \(F\): Exponential, Gamma, Lognormal etc.
    Then, \((X, Y) = (F^-((U), F^-((1 - U)))\), where \(U \sim \text{Uniform}(0, 1)\), satisfies all our assumptions.

- The third model shows that subexponentiality of \(X\) and \(Y\) is not a necessary condition to ensure a limit in \((0, \infty)\).
We have provided sufficient conditions under which

$$\lim_{x \to \infty} \frac{P(X + Y > x)}{P(X > x)} = 2.$$
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The assumption (2) implies asymptotic independence, hence the class of distributions satisfying the assumptions of the theorem forms a subclass of the class of distributions $C$ we considered first ($X \overset{d}{=} Y, (X, Y) \in MDA(G)$ where $G(x, y) = \Lambda(x)\Lambda(y)$).
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The assumption \(X \overset{d}{=} Y \in MDA(\Lambda)\) implies that both \(X\) and \(Y\) are rapidly varying, i.e. for all \(t > 1\),
\[
\lim_{x \to \infty} \frac{P(X > tx)}{P(X > x)} = \lim_{x \to \infty} \frac{P(Y > tx)}{P(Y > x)} = 0.
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The conditions are only sufficient. Necessary and sufficient conditions?
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The result is extended to the case where $X$ and $Y$ are not identically distributed, i.e. where $X \sim F, Y \sim G$ and $\lim_{x \to \infty} \frac{\bar{G}(x)}{F(x)} = c \in [0, \infty)$. 
Thank You !