Modeling Financial Market Returns with a Lognormally Scaled Stable Distribution

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STABLE software by John Nolan
www.robustanalysis.com
Market Model: Stable Mixture Distribution with varying scale factor

Product of Two Variables

Stable random variable

Scaling variable that is not independent
Density close to lognormal
Stable Characteristic Function

\[ \phi(t) = \exp \left( i t \delta - \gamma^\alpha |t|^{\alpha} \left( 1 - i \beta \text{sgn}(t) \tan \left( \frac{\pi \alpha}{2} \right) \right) \right); \quad \alpha \neq 1 \]

\[ \phi(t) = \exp \left( i t \delta - \gamma |t| \left( 1 + \frac{2 i \beta \text{sgn}(t) \log(|t|)}{\pi} \right) \right); \quad \alpha = 1 \]

\( \alpha \) is the shape parameter \( \alpha \in (0, 2] \)

tail exponent \( (\alpha < 2) \)

\( \beta \) is the skewness parameter \( \beta \in [-1, 1] \)

\( \gamma \) is the scale parameter \( \gamma \in (0, \infty) \)

\( \delta \) is the location parameter \( \delta \in \text{Reals} \)

Limiting distribution of sums of i.i.d. RVs

Distribution of sums of stable RVs is stable with same \( \alpha \) and \( \beta \) scaling is by \( n^{\alpha / \alpha} \)
Continuous Double Auction

Under controlled conditions CDA can output stable RVs
SPY ETF Daily Closing Prices
Logarithmic Returns

SPY Close

One Minute Logarithmic Returns
Autocorrelation Absolute Value Returns

One Minute Return Autocorrelation

- raw log returns
- Abs[log returns]
# Daily Stable Parameters

<table>
<thead>
<tr>
<th>α</th>
<th>β</th>
<th>γ</th>
<th>δ</th>
</tr>
</thead>
<tbody>
<tr>
<td>1.8052</td>
<td>0.0370129</td>
<td>0.000483713</td>
<td>5.80342 × 10⁻⁸</td>
</tr>
</tbody>
</table>

SPY Alpha

![Graphic representation of SPY Alpha data]
Stable Gamma
a measure of volatility
Volatility Autocorrelation

Autocorrelation Daily Scale Factor, $\gamma$

Trading Days
Stable Fits

\begin{center}
\begin{tabular}{cccc}
\hline
\textbf{Parameter} & \textbf{Value} \\
\hline
$\alpha$ & 1.41694 \\
$\beta$ & $1.26061 \times 10^{-8}$ \\
$\gamma$ & 0.000387194 \\
$\delta$ & $-3.73386 \times 10^{-6}$ \\
\hline
\end{tabular}
\end{center}

\begin{center}
\begin{tabular}{cccc}
\hline
\textbf{Parameter} & \textbf{Value} \\
\hline
$\alpha$ & 1.79027 \\
$\beta$ & $-3.25178 \times 10^{-9}$ \\
$\gamma$ & 1.014 \\
$\delta$ & $-0.00312431$ \\
\hline
\end{tabular}
\end{center}
Stable Tail Fits

Log Log Stable Distribution
Left Tail Blue, Right Red, (Normal Green)

Rescaled Log Log Stable Distribution
Left Tail Blue, Right Red, (Normal Green)

\[
\begin{array}{cccc}
\alpha & \beta & \gamma & \delta \\
1.41694 & 1.26061 \times 10^{-8} & 0.000387194 & -3.73386 \times 10^{-6} \\
1.79027 & -3.25178 \times 10^{-9} & 1.014 & -0.00312431 \\
\end{array}
\]
Lognormal Density

\[ \lambda(x, \mu, \sigma) = \frac{e^{-\frac{(\log(x) - \mu)^2}{2\sigma^2}}}{\sqrt{2\pi x \sigma}} \]

Stable Characteristic Function

\[ \phi(t, \alpha, \beta) = e^{-|t|^\alpha \left(1 - i \beta \text{sgn}(t) \tan\left(\frac{\pi \alpha}{2}\right)\right)} \]

Lognormal Stable Characteristic Function

\[ \text{lnscf}(t, \alpha, \beta, \gamma, \sigma, \delta) = e^{i\delta t} \int_0^\infty \lambda(s, \log(\gamma), \sigma) \phi(st, \alpha, \beta) \, ds \]
Lognormal Stable Distribution

\[ \lnscdf(x, \alpha, \beta, \gamma, \sigma, \delta) = \int_0^\infty \text{scdf}(x, \alpha, \beta, s, \delta) \lambda(s, \log(\gamma), \sigma) \, ds \]

Lognormal Stable Density

\[ \lnspdf(x, \alpha, \beta, \gamma, \sigma, \delta) = \int_0^\infty \text{spdf}(x, \alpha, \beta, s, \delta) \lambda(s, \log(\gamma), \sigma) \, ds \]
Lognormal Stable Density

LNS and Stable Density Functions

\{1.5, 1, 1, (0.5), 0\} Red

\{1.5, 0, 1, (0.5), 0\} Blue
Lognormally Scaled Stable Distribution
LNS Tail Behavior (CDF)

Stable Blue
LNS Red
{1.8, 0, 1, (0.55), 0}

Stable Blue
LNS Red
{1.8, 0, 1, (1), 0}
Estimating Tail Exponent with GEV

GEV Fit to Left Tail Data

Calculated $\alpha = 1.58033$.

GEV Fit to Right Tail Data

Calculated $\alpha = 1.69657$. 

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Market Model

Product of stable random and a non-random scaling variable

This is a stable mixture distribution with a varying gamma parameter.

For financial data the scaling appears to be constrained; the tail exponent of the distribution can be estimated.

The model solves the fitting problems associated with the assumption of a stationary stable model.

It is possible to rescale daily data, taking advantage of the serial dependence in gamma.

If you can guess the future behavior of volatility, you can make some reasonable estimates of future event probability using stable distributions.
Lognormally Scaled Stable Distribution

Product of a stable random variable and a lognormal random variable.

It is computable.

The maximum domain of attraction of the distribution is determined by stable alpha.

Can be used for simulation.

Serial dependent structure can be added to the lognormal RV to better understand financial data.

For a more detailed technical paper on this presentation visit www.mathestate.com