Geometric characteristics of the excursion sets over highl levels of non-Gaussian infinitely divisible random fields

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Let $\big(X(\mathbf{t}),\, \mathbf{t} \in T\big)$ be a random field indexed by some parameter space T.

In many applications, including statistical hypothesis testing, one is interested in, whether or not, the random field crosses a level u, often a high level $u > 0$.

Specifically: suppose we use a test statistic $\sup_{\mathbf{t}\in T} X(\mathbf{t})$.

Let $u > 0$ be so large that, under the null hypothesis on the probability law of the random field $\big(X(\mathbf{t}),\, \mathbf{t} \in T\big)$, the level crossing probability $P\big(\sup_{\mathbf{t}\in T} X(\mathbf{t}) > u\big)$ is small.

Decision rule: reject the null hypothesis if $\sup_{\mathbf{t}\in T} X(\mathbf{t}) > u$.

The excursion set

The *excursion set* of the random field $\big(X(\mathbf{t}), \mathbf{t} \in T\big)$ over the level u is the random set

$$A_u = \big\{\mathbf{t} \in T : X(\mathbf{t}) > u\big\}.$$

The statisical decision rule $\sup_{\mathbf{t}\in T} X(\mathbf{t}) > u$ can be reformulated:

reject the null hypothesis if A_u is non-empty.

More powerful statistical tests can be potentially constructed if one uses more delicate geometric properties of the excursion set A_u other than it being empty or not.

One of the very useful geometric characteristics of a set is its Euler characteristic.

For a “nice” set A the Euler characteristic $\chi(A)$ depends only the *homotopy type* of the set A:

- for a “nice” one-dimensional set A, $\chi(A) =$ number of connected components (intervals) in A;
- for a “nice” two-dimensonal set A, $\chi(A) =$ number of connected components in A minus the number of “holes”;
- for a three-dimensonal set A, $\chi(A) =$ number of connected components in A minus the number of “handles” plus the number of “holes’, etc.

Smooth Gaussian random fields

Let T be a "nice" manifold, and let $\big(X(\mathbf{t}), \, \mathbf{t} \in T\big)$ be a smooth zero mean constant variance σ^2 Gaussian random field on T.

Smoothness assumption includes existence of two continuous partial derivatives plus other verifiable technical assumptions.

Consider the excursion set

$$A_u = \big\{\mathbf{t} \in T : \, X(\mathbf{t}) \geq u\big\} \, .$$

An explicit non-asymptotic expression for the expected Euler characteristic of the excursion set exists (Adler and Taylor (2007)).

The expected Euler characteristic of the excursion set of a smooth Gaussian randon field provides an excellent approximation to the level crossing probability $P\big(\sup_{\mathbf{t}\in T} X(\mathbf{t}) > u\big)$.

Let T be a "nice" manifold, and let $\big(X(\mathbf{t}), \mathbf{t} \in T\big)$ be a smooth zero mean constant variance σ^2 Gaussian random field on T. Then there is a number $a > 0$ such that for $u > 0$ large enough,

$$\Big| P\big(\sup_{\mathbf{t}\in T} X(\mathbf{t}) > u\big) - E\big(\chi(A_u)\big) \Big| \le e^{-(1+a)u^2/2\sigma^2}$$

(Taylor, Takemura, Adler (2005)).

That is, the approximation is superexponentially good.

The reason the approximation of the level crossing probability by the expected Euler characteristic of the excursion set is so good is that, for smooth Gaussian random fields, an excursion set over a high level is very likely to be a "nice ball-like" neighbourhood of the point where the maximum is achieved.

Therefore, the excursion set A_u over a high level u, if non-empty, is very likely to have Euler characteristic equal to 1. In particular,

$$E\big(\chi(A_u)\big) = E\Big(\chi(A_u)\mathbf{1}\big(A_u \neq \emptyset\big)\Big)$$

$$\approx P\big(A_u \neq \emptyset\big) = P\Big(\sup_{\mathbf{t}\in T} X(\mathbf{t}) > u\Big),$$

and the approximation $\approx$ is very good.

Non-Gaussian infinitely divisible random fields

In order to obtain a tractable class of models with a richer structure of high level excursion sets, we will look at non-Gaussian infinitely divisible random fields. We will look at random fields on a unit cube, $\left(X(\mathbf{t}),\, \mathbf{t} \in [-1,1]^d\right)$ and study their excursion sets over a high level.

For technical reasons, we would like to have a random field defined on a neigborhood of the unit cube, and have "nice" properties there.

Let G be an open set in $\mathbb{R}^d$ such that $[-1,1]^d \subset G$. We consider an infinitely divisible random field of the form

$$X(\mathbf{t}) = \int_S f(s;\mathbf{t})\, M(ds),\ \mathbf{t} \in G,$$

where

- M is an infinitely divisible random measure on a measurable space $(S, \mathcal{S})$;
- $f(\cdot;\mathbf{t})$, $\mathbf{t} \in G$ is a family of nonrandom measurable functions on S satisfying certain integrability assumptions.

An infinitely divisible random measure is characterized by its Lévy measure. This is a σ-finite measure on $S \times \big(\mathbb{R} \setminus \{0\}\big)$ of the form

$$F(A) = \int_S \rho\big(s; A_s\big)\, m(ds)$$

for a measurable $A \subset S \times \big(\mathbb{R} \setminus \{0\}\big)$, where

- $A_s = \big\{x \in \mathbb{R} \setminus \{0\} : (s,x) \in A\big\}$ is the s-section of the set A;
- m is a σ-finite measure on $(S, \mathcal{S})$, the so-called control measure of the random measure M;
- $\big(\rho(s;\cdot)\big)$ is a measurable family of Lévy measures on $\mathbb{R}$, the so-called local Lévy measures.

We will assume that the local Lévy measures of the infinitely divisible random measure M possess the following regular variation property:

asume that there exists a a function $H : (0, \infty) \to (0, \infty)$ *that is regularly varying at infinity with exponent* $-\alpha$, $\alpha > 0$, *and nonnegative measurable functions* w_+ *and* w_- *on* S *such that*

$$\lim_{u \to \infty} \frac{\rho\big(s; (u, \infty)\big)}{H(u)} = w_+(s), \quad \lim_{u \to \infty} \frac{\rho\big(s; (-\infty, -u)\big)}{H(u)} = w_-(s).$$

Critical points

For a smooth non-Gaussian infinitely divisible random field we will study the Euler characterstic of its excursion A_u by first studying the critical points of the random field over the level u.

- Informally, a critical point of a smooth function f on $\mathbb{R}^d$ is a point where the gradient of f vanishes.
- Critical points can be of different types, depending on the Hessian matrix at the critical point.
- Numbers of *relevant critical points* of a function above a level are related to the Euler characteristic of the excursion set above the level through the *Morse theory*.

Let g be a "nice" (Morse) function. For level $u \in \mathbb{R}$, dimension $k = 0, 1, \ldots, d$, face J of dimension k and type $i = 0, 1, \ldots, k$ of the critical point, let

$$\mu_g(J; i : u) = \mathrm{Card}\big(\mathcal{C}_g(J; i) \cap \{\mathbf{t} : \, g(\mathbf{t}) > u\}\big)$$

be the numbers of the critical points of different types and on different faces of the cube, of function g over the level u.

Let

$$A_u(g) = \big\{\mathbf{t} \in [-1, 1]^d : \, g(\mathbf{t}) \geq u\big\}$$

be the excursion set of the function g over the level u. Then

$$\chi\big(A_u(g)\big) = \sum_{k=0}^{d} \sum_{J \in \mathcal{J}_k} \sum_{i=0}^{k} (-1)^i \mu_g(J; i : u) \, .$$

The goal: obtain the limiting conditional joint distribution of the numbers of the critical points of different types and on different faces of the cube of a non-Gaussian infinitely divisible random field over a high level u given that the random field does exceed that level.

This will allow us to compute, for example, the limiting conditional *distribution* of the Euler characteristic of the excursion set of the level u, given that the level is exceeded.

Knowing **the full limiting conditional distribution** we can compute the conditional mean of the Euler characteristic, conditional variance, etc.

Theorem Assume that an infinitely divisible random field $\big(X(\mathbf{t}),\, \mathbf{t} \in [-1,1]^d\big)$ is smooth, satisfies the assumption of regular variation and additional technical assumtions. Then for any numbers
$n(J;i) = 0,1,2,\ldots,\ J \in \mathcal{J}_k,\ k = 0,1,\ldots,d,\ i = 0,1,\ldots,k$

$$\lim_{u\to\infty} P\Big(\mu_X(J;i:u) \geq n(J;i),\ J \in \mathcal{J}_k,$$

$$k = 0,1,\ldots,d,\ i = 0,1,\ldots,k \Big| \sup_{\mathbf{t}\in[-1,1]^d} X(\mathbf{t}) > u\Big)$$

$$= \frac{\int_S \Big[w_+(s)\Big(\min_{J,i} f^{(J;i:+)}_{[n(J;i)]}(s)\Big)^\alpha + w_-(s)\Big(\min_{J,i} f^{(J;i:-)}_{[n(J;i)]}(s)\Big)^\alpha \Big]\, m(ds)}{\int_S \Big[w_+(s) \sup_{\mathbf{t}\in[-1,1]^d} f(s,\mathbf{t})_+^\alpha + w_-(s) \sup_{\mathbf{t}\in[-1,1]^d} f(s,\mathbf{t})_-^\alpha \Big]\, m(ds)}$$

Moving average random fields

A moving average random field is the random field

$$X(\mathbf{t}) = \int_{\mathbb{R}^d} g(\mathbf{s}+\mathbf{t})\, M(d\mathbf{s}),\ \ t \in G,$$

where the control measure of the infinitely divisible random measure M is the d-dimensional Lebesgue measure, and the local Lévy measure $\rho(\mathbf{s},\cdot) = \rho(\cdot)$ is independent of $\mathbf{s} \in \mathbb{R}^d$.

The measure $\rho(\cdot)$ is assumed to be regularly varying.

Choosing the kernel g of different shapes, one can obtain very different geometrical properties of the high level excursion sets.

Example 1. Let $g(\mathbf{t}) = \exp\{-\|\mathbf{t}\|^2/2\}$, $\mathbf{t} \in \mathbb{R}^d$. Because of the rotaional invariance of this kernel and its radial monotonicity, the high level excursion set A_u, if not empty, looks, geometrically, "like a ball" and, hence, Euler characteristic equal to 1, as in the case of smooth Gassian random fields.

Example 2. Let us take $d = 1$ and

$$g(t) = \big(1 + \cos\gamma t\big)e^{-t^2/2},\ t \in \mathbb{R}\,.$$

In this case the high level excursion set will have a random number of "holes", and the Euler characteristic a non-degenerate conditional distribution.