On the Structure of Max–Stable Processes

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Graybill VIII: Extreme Value Analysis
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Some references:

• The talk is based on the work:

Wang & S. (2009) On the structure and representations of max–stable processes, Preprint.

• Closely related works:

de Haan & Pickands (1986) Stationary min-stable stochastic processes. *Probab. Theory Relat. Fields*, 72:477–492, 1986.
Kabluchko, Schlather & de Haan (2009) Stationary max–stable fields associated to negative definite functions, Preprint.
Kabluchko (2009) Spectral representations of sum– and max–stable processes, Preprint.
Wang & S. (2009) On the Association of Sum– and Max– Stable Processes, Preprint.

There is a beautiful parallel world out there.

Sum–stable processes:

Hardin (1982) On the spectral representation of symmetric stable processes. *Journal of Multivariate Analysis*, 12:385–401, 1982.

Rosiński (1995) On the structure of stationary stable processes. *Ann. Probab.*, 23(3):1163–1187, 1995.

Rosiński & Samorodnitsky (1996) Classes of mixing stable processes. *Bernoulli*, 2(4):365–377, 1996.

Pipiras & Taqqu (2004) Stable stationary processes related to cyclic flows. *Ann. Probab.*, 32(3A):2222–2260, 2004.

Samorodnitsky (2005) Null flows, positive flows and the structure of stationary symmetric stable processes. *Ann. Probab.*, 33:1782–1803, 2005.

Our goal:

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Our goal: To catch up! Provide tools to achieve similar structural results for max–stable processes.

Preliminaries

Motivation: Consider heavy–tailed i.i.d. processes $\{Y_t^{(i)}\}_{t\in T}$. If for some $a_n \sim n^{1/\alpha}\ell(n)$ $(\alpha > 0)$, we have

$$\Big\{\frac{1}{a_n}\bigvee_{1\le i\le n} Y_t^{(i)}\Big\}_{t\in T} \stackrel{f.d.d.}{\longrightarrow} \{X_t\}_{t\in T}, \quad (n\to\infty),$$

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∘ The margins of X are then $\alpha-$Fréchet $(\alpha > 0)$, namely:

$$\mathbb{P}\{X_t \le x\} = \exp\{-\sigma_t^\alpha x^{-\alpha}\}, \quad x > 0,$$

with $\sigma_t^\alpha > 0$.

Fine print: For simplicity, we focus on max–stable processes with $\alpha-$Fréchet marginals. By transforming the margins, our theory applies to the most general definition of max–stable processes where the margins can be extreme value distributions of different types.

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(b) Extremal integral representation:

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(b) Extremal integral representation:

$$X_t = {}^{e}\!\!\int_U f_t(u) M_\alpha(du), \quad (t \in T)$$

for an $\alpha-$Fréchet random *sup–measure* $M_\alpha(du)$ on (U,μ).
• The deterministic functions $f_t(u) \geq 0$ are called spectral functions of X and satisfy:

$$\int_U f_t(u)^\alpha \mu(du) < \infty, \quad (t \in T).$$

Fine print: The measure space (U,μ) can be chosen $([0,1], dx)$ if the process $\{X_t\}_{t\in T}$ is *separable in probability*, in particular, continuous in probability when T is a separable metric space.

Extremal Integrals

Let M_α be a random $\alpha-$Fréchet sup–measure on (U, μ).

◦ For *simple functions* $f(u) = \sum_{i=1}^n a_i 1_{A_i}(u)$, $f(u) \geq 0$:

$$\int_U^e f(u) M_\alpha(du) := \bigvee_{1\leq i\leq n} a_i M_\alpha(A_i).$$

◦ The def of ${}^e\!\!\int_U f dM_\alpha$ extends to all $f \in L_+^\alpha(\mu)$ and

$$\mathbb{P}\Big\{ \int_U^e f dM_\alpha \leq x \Big\} = \exp\{-\|f\|_{L^\alpha(\mu)}^\alpha x^{-\alpha}\}, \quad x > 0.$$

◦ For $f,\ g \in L_+^\alpha(\mu)$:

$$\int_U^e (af \vee bg) dM_\alpha = a \int_U^e f dM_\alpha \vee b \int_U^e g dM_\alpha \quad \text{(max–linearity)}$$

◦ ${}^e\!\!\int_U f dM_\alpha$ and ${}^e\!\!\int_U g dM_\alpha$ are *independent* if and only if $fg = 0$, (mod μ).

Benefits: For any $f_t \in L_+^\alpha(\mu)$, $t \in T$, we get a max–stable process:

$$X_t := \int_U^e f_t dM_\alpha$$

∘ For the finite–dimensional distributions, we have:

$$\begin{aligned} \mathbb{P}\{X_{t_i} \le x_i,\ 1 \le i \le d\} &= \mathbb{P}\{\int_U^e (\vee_{1\le i\le d} x_i^{-1} f_{t_i}) dM_\alpha \le 1\} \\ &= \exp\{-\int_U (\vee f_{t_i}^\alpha / x_i^\alpha) d\mu\}. \end{aligned}$$

Examples:

- (moving maxima)

$$X_t := \int_{\mathbb{R}}^e f(t-x) M_\alpha(dx), \quad t \in \mathbb{R},$$

with $(U,\mu) \equiv (\mathbb{R}, dx)$ and $f \in L_+^\alpha(dx)$.

∘ Smith's storm processes are moving maxima.

Examples (cont'd)

- (mixed moving maxima) With $(U, \mu) = (\mathbb{R} \times V, dx \times d\nu)$:

$$X_t := {}^{e}\!\!\int_{\mathbb{R}\times V} f(t-x, v) M_\alpha(dx, dv), \quad f(x,v) \in L^\alpha_+(dx, d\nu).$$

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∘ A continuous–time version of the M3 processes. • (doubly stochastic) Let (U, μ) be a probability space and $\xi_t(u) \geq 0$, $t \in \mathbb{R}$ a stochastic process over (U, μ). If $\mathbb{E}_\mu \xi_t^\alpha < \infty$, then

$$X_t := \int^{e}_{U} \xi_t(u) M_\alpha(du), \quad t \in \mathbb{R},$$

is an $\alpha-$Fréchet process on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$.
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◦ Schlater's processes are doubly stochastic with particular ξ_t's.
• (Brown–Resnick) With (U,μ) a probability space and $\{w_t(u)\}_{t\in\mathbb{R}}$ a standard Brownian motion on (U,μ):

$$X_t := \int^e_U e^{w_t(u)-\alpha|t|/2} M_\alpha(du), \quad t\in\mathbb{R}.$$

Max-linear Isometries

Consider a max–stable process:

$$X_t = {\int_U}^{e} f_t dM_\alpha, \quad (t \in T).$$

Some Natural Questions:

• How does the structure of the $f_t(u)$'s determine the structure of $X = \{X_t\}_{t \in T}$ and vice versa?

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• Given another representation $\{g_t\} \subset L_+^\alpha(V,\nu)$

$$\{X_t\}_{t\in T} \stackrel{d}{=} \Big\{ {\int_V}^{e} g_t d\widetilde{M}_\alpha \Big\}_{t\in T},$$

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Some Answers: For all $a_i \geq 0, \ t_i \in T$, we have:

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$$\| \bigvee a_i f_{t_i} \|_{L^\alpha(\mu)}^\alpha = \| \bigvee a_i g_{t_i} \|_{L^\alpha(\nu)}^\alpha.$$

• Thus, there exists a max–linear isometry $\mathbb{I} : L_+^\alpha(U,\mu) \to L_+^\alpha(V,\nu)$, such that $\mathbb{I}(f_t) = g_t$, for all $t \in T$.

Def $\mathbb{I} : L_+^\alpha(U,\mu) \to L_+^\alpha(V,\nu)$ *is a max–linear isometry, if:*

$$\mathbb{I}(af \vee bg) = a\mathbb{I}(f) \vee b\mathbb{I}(g), \quad \forall f, g \in L_+^\alpha(U,\mu),\ a, b \geq 0.$$

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∘ Conversely, any max–linear isometry $\mathbb{I} : L_+^\alpha(U,\mu) \to L_+^\alpha(V,\nu)$ yields an equivalent spectral represenation $g_t := \mathbb{I}(f_t)$ of the process X over (V,ν).

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• In Wang & S., 2009, we extend results of Hardin, 1981/82 on the structure of linear isometries to the max–linear case.

• The importance of these results for max–stable processes is best understood via the notion of minimal representation.

Minimal spectral representations

Def *The spectral representation* $\{f_t\}_{t\in T} \subset L_+^\alpha(U,\mu)$ *of* X *is minimal if:*
(i) (full support) $\mathrm{supp}\{f_t(u),\ t\in T\} = U$ *(mod* μ*)*
(ii) (non-redundancy) For any measurable $A \subset U$*, there exists*

$$B \in \rho\{f_t,\ t\in T\} \equiv \sigma\{f_t/f_s,\ t,s\in T\},$$

such that $\mu(A\Delta B) = 0$.

∘ This def is identical to the one of Rosiński (1995) in the sum–stable case, similar to Hardin (1982), and to the proper pistons of de Haan and Pickands (1986).

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- ◦ Consider the process

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where $Z = {}^{e}\!\!\int_{[0,1]} \sin^2(u) M_\alpha(du)$.

- This representation is clearly redundant! Note that

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and trivial $U = \{0\}$, and $\mu(du) = c\delta_0(du)$, $c := \int_{[0,1]} \sin^{2\alpha}(x) dx$.
• The ratio $\sigma-$algebra $\rho(f_t,\ t \in T)$ captures best the minimal information needed to represent the process.

What are the benefits of minimal representations?

Thm 1. (Wang & S.(2009)) *Let* $\{f_t\}_{t\in T} \subset L^{\alpha}_{+}(\mu)$ *be a minimal measurable rep of* X. *If* $\{g_t\}_{t\in T} \subset L^{\alpha}_{+}(V,\nu)$ *is another measurable rep of* X, *then:*

$$g_t(v) = h(v) f_t(\phi(v)), \quad \nu - a.e.$$

for some measurable $h \geq 0$ *and* $\phi : V \to U$. *The map* ϕ *is unique (mod* ν*).*
If $\{g_t\}$ *is also minimal, then* ϕ *is bi–measurable,* $\nu \sim \mu \circ \phi$ *and*

$$\frac{d\mu \circ \phi}{d\nu}(v) = h^{\alpha}(v) > 0.$$

Note: h and ϕ are independent of 'time' $t \in T$ point mappings.

Minimal representations with standardized support

Consider the sets

$$S_{\mathrm{I},N} = (0,\mathrm{I}) \cup \{1,2,\cdots,N\},$$

where $\mathrm{I} \in \{0,1\}$ and $0 \le N \le \infty$:

∘ For example:

$$S_{1,3} = (0,1) \cup \{1,2,3\}, \quad S_{0,\infty} = \{1,2,\cdots\}, \quad \text{and} \quad S_{1,0} = (0,1).$$

- Equip $S_{\mathrm{I},N}$ with the measure

$$\lambda_{\mathrm{I},N}(x) = dx + \sum_{i=1}^{N} \delta_{\{i\}}(dx).$$

Fine print: Every standard Lebesgue space is isomorphic to some $(S_{I,N}, \lambda_{I,N})$.

Def *A minimal representation $\{f_t\}_{t\in T} \subset L^\alpha_+(U,\mu)$ is said to have* ***standardized support*** *if, for some I, N: $(U,\mu) \equiv (S_{\mathrm{I},N}, \lambda_{\mathrm{I},N})$.*

Thm 2. (Wang & S., 2009) *Every separable in probability $\alpha-$Fréchet process X has a minimal representation with standardized support:*

$$\{X_t\}_{t\in T} \stackrel{d}{=} \Big\{ {}^{e}\!\!\int_{S_{I,N}} f_t dM_\alpha \Big\}_{t\in T}.$$

Continuous–Discrete Decomposition

Consider an $\alpha-$Fréchet process X with the minimal rep of standardized support:

$$\{X_t\}_{t\in T} \stackrel{d}{=} \left\{ {\int^{e}}_{S_{I,N}} f_t dM_\alpha \right\}_{t\in T}.$$

By setting

$$X_t^I := {\int^{e}}_{S_{I,N}\cap(0,1)} f_t dM_\alpha \quad \text{and} \quad X_t^N := {\int^{e}}_{S_{I,N}\cap\mathbb{N}} f_t dM_\alpha,$$

we obtain the continuous–discrete decomposition:

$$\{X_t\}_{t\in T} \stackrel{d}{=} \{X_t^I \vee X_t^N\}_{t\in T}.$$

The components X_t^I and X_t^N are independent.

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Intuition: Suppose $I = 1$ and $N > 0$. Then,

$$X_t^I = {\int^{e}}_{(0,1)} f_t dM_\alpha \quad \text{and} \quad X_t^N = \bigvee_{i=1}^{N} f_t(i) Z_i,$$

where $Z_i = M_\alpha\{i\}$ are i.i.d. standard $\alpha-$Fréchet, independent of X_t^I.

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• The continous–discerete decomposition does not depend on the representation.

Thm (Wang & S., 2009) *Let* $\{g_t\}_{t\in T} \subset L_+^\alpha(S_{I',N'}, \lambda_{I',N'})$ *be another minimal rep of* X *with standardized support, then* $(I,N) \equiv (I',N')$,

$$\{X_t^I\} \stackrel{d}{=} \{X_t^{I'}\}_{t\in T} \quad \mathit{and} \quad \{X_t^N\} \stackrel{d}{=} \{X_t^{N'}\}_{t\in T},$$

where $X_t^{I'} := {}^e\!\!\int_{S_{I,N}\cap(0,1)} g_t dM_\alpha$ *and* $X_t^{N'} = {}^e\!\!\int_{S_{I,N}\cap\mathbb{N}} g_t dM_\alpha$.

• Moreover, for the discrete component, we have that:

$$\{X_t^N\}_{t\in T} \stackrel{d}{=} \Big\{ \bigvee_{i=1}^{N} \phi_t(i) Z_i \Big\}_{t\in T},$$

for some unique set of functions $\{\phi_t(i),\ 1\le i\le N\}$.

Discrete Principal Components

Consider the spectrally discrete component of the process $\{X_t\}_{t\in T}$:

$$X_t^N = \bigvee_{i=1}^{N} \phi_t(i) Z_i, \quad (t \in T),$$

for i.i.d. standard $\alpha-$Fréchet Z_i's.

• The functions $t \mapsto \phi_t(i),\ 1 \le i \le N$ are unique up to permutation of the indices.

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• The functions $t \mapsto \phi_t(i), \ 1 \leq i \leq N$ are unique up to permutation of the indices.

◦ The $\phi_t(i)$'s are the discrete principal components of X.

• Not all sequences of positive functions can be discrete principal components.

Fine print: Prop: (Wang & S., 2009) *A countable set of functions* $\phi := \{\phi_t(i) \geq 0, \ 1 \leq i \leq N\}$ *can be discrete principal components of an* $\alpha-$*Fréchet process if and only if,* ϕ *is a minimal representation. Namely, if (i)* $\sum_{i=1}^N \phi_t^\alpha(i) < \infty$ *and (ii)* $\rho(\phi_t(\cdot), \ t \in T) = 2^{\{1,\cdots,N\}}$.

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Thm (Wang & S., 2009) *Let $\{X_t\}_{t\in\mathbb{R}}$ be a measurable and stationary $\alpha-$Fréchet process. Then, the spectrally discrete component of X is either zero or trivial, i.e.*

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• Certainly, with i.i.d. $\alpha-$Fréchet Z_i's:

$$X_t := \bigvee_{i\in\mathbb{Z}} f(t-i)Z_i, \quad (t \in \mathbb{Z}),$$

is a non–trivial stationary and spectrally discrete process.

Stationary Max–Stable Processes and Flows

Let now $X_t = {}^e\!\!\int_U f_t(u) M_\alpha(du)$, $(t \in \mathbb{R})$ be stationary.
As in the sum–stable case, if $\{f_t\}_{t\in T}$ is minimal:

$$f_t(u) = \Big(\frac{d\mu \circ \phi_t}{d\mu}(u)\Big)^{1/\alpha} f_0(\phi_t(u)),$$

where $\phi_t : U \to U$ is a non–singular flow:

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(i) $\phi_{t+s} = \phi_t \circ \phi_s$, $\forall t, s \in \mathbb{R}$, *(ii)* $\phi_0 = \mathrm{id}$, and *(iii)* $\mu\circ\phi_t \sim \mu$, $\forall t \in \mathbb{R}$.

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Intuition: by stationarity and Thm 1:

$$f_{t+s}(u) = h_{t+s}(u) f_0(\phi_{t+s}(u)) = h_t(u) f_s(\phi_t(u)) = h_t(u) h_s(u) f_0(\phi_t(\phi_s(u))).$$

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• The flow $\{\phi_t\}_{t\in\mathbb{R}}$ associated with X is unique (does not depend on the minimal rep), up to flow–equivalence.
∘ As in the sum–stable case, the structure of $\{\phi_t\}_{t\in\mathbb{R}}$'s motivates classifications of the X's.

Hopf Decomposition

Let $\phi : U \to U$ be non–singular bijection.

- $B \subset U$ is a wandering set for ϕ if $\phi^k(B) \cap \phi^j(B) = \emptyset$, $(\forall k \neq j \in \mathbb{Z})$.

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Hopf decomposition: We have $U = C \cup D$, where:
(i) $C \cap D = \emptyset$ and C and D are $\phi-$invariant.
(ii) C has no wandering sub–set of positive measure (for ϕ).
(iii) $D = \cup_{k \in \mathbb{Z}} \phi^k(B)$ for some wandering set $B \subset D$.

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∘ For a measurable flow $\{\phi_t\}_{t \in \mathbb{R}}$, we have $U = C_t \cup D_t$, where

$$C_t = C \quad \text{and} \quad D_t = D\,, \quad (\bmod \mu).$$

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Characterization: for any strictly positive $f \in L^\alpha_+(U,\mu)$,

$$C = \Big\{u : \int_{\mathbb{R}} f_t(u)^\alpha dt = \infty\Big\} \quad \text{and} \quad D = \Big\{u : \int_{\mathbb{R}} f_t(u)^\alpha dt < \infty\Big\}$$

with $f_t(u) = \big(\frac{d\mu\circ\phi_t}{d\mu}(u)\big)^{1/\alpha} f(\phi_t(u))$.

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Thm (Wang & S., 2009) Let $\{X_t\}_{t\in\mathbb{R}}$ be a stationary α–Fréchet process with spectral representation $\{f_t\}_{t\in\mathbb{R}} \in L_+^\alpha(U, du)$. Then,

$$\{X_t\}_{t\in\mathbb{R}} \stackrel{d}{=} \left\{X_t^C \vee X_t^D\right\}_{t\in\mathbb{R}}$$

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We say $\{X_t^C\}_{t\in\mathbb{R}}$ is generated by a conservative flow and $\{X_t^D\}_{t\in\mathbb{R}}$ is generated by a dissipative flow.

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Fine print: Recently Kabluchko 2009, has independently obtained decompositions of max–stable processes by association with the sum–stable setting.

Moreover, he has shown that a max–stable X is ergodic if and only if it is generated by a null flow. An exact parallel to Samorodnitsky 2005.

Generalized Brown–Resnick Processes

Thm (Kabluchko, Schlather and de Haan, 2009) *Let* $\{w_t(u)\}_{t\in\mathbb{R}}$ *be zero mean, continuous path, Gaussian process with stationary increments on the* prob space (U,μ). *Then the max–stable process*

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  - By our results: X is dissipative, if and only if

$$\int_{\mathbb{R}} e^{\alpha w_t(u)-\alpha^2\sigma_t^2/2} dt < \infty \quad (\mu - a.e.). \tag{1}$$

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- If $\{w_t\}$ is the *fractional Brownian motion*, then X is dissipative and hence a mixed moving maxima.

Fine print: (1) follows from the law of the iterated logarithm of Oodaira, 1972.

A bonus and an open problem

• From Kabluchko, Schlather & de Haan, 2009, we have that a generalized Brown–Resnick process is dissipative if:

$$\lim_{|t|\to\infty} (w_t - \sigma_t^2/2) = -\infty. \tag{2}$$

∘ By combining with our NSC, we get the bonus:
Thm *Consider a Gaussian* $\{w_t\}$ *process with zero mean, continuous paths, and stationary increments. The condition (2) implies*

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Open question: (To me!) Is there a zero–one law:

$$\mu\left\{u\ :\ \int_{\mathbb{R}} e^{w_t(u)-\sigma_t^2/2}dt < \infty\right\} = 1 \quad \text{or} \quad 0.$$

for the $\{w_t\}$ in the above Thm.

Co–spectral Functions

Let now T be a separable metric space, equipped with a Borel measure λ. Consider the $\alpha-$Fréchet process $X = \{X_t\}_{t \in T}$:

$$X_t := \int_U^e f(t,u) M_\alpha(du), \quad (t \in T),$$

where $(t,u) \mapsto f(t,u)$ is measurable.

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• Can show that the co–spectral functions of X do not depend on the representation (up to rescaling)!

Fine print: If (U,μ) is standard Lebesgue, then X is measurable, if and only if, $(t,u) \mapsto f(t,u)$ has a measurable modification.

Co–spectral Functions: Classification

Let $\mathcal{P}$ be a positive (measurable) cone in $L^0_+(T,\lambda)$, i.e. $c\mathcal{P} \subset \mathcal{P}$.
Consider the partition of $U = A \cup B$ into disjoint components:

$$A := \{u \in U \,:\, f(\cdot, u) \in \mathcal{P}\} \quad \text{and} \quad B := U \setminus A = \{u \in U \,:\, f(\cdot, u) \notin \mathcal{P}\}.$$

This yields the decomposition:

$$\{X_t\}_{t\in T} \stackrel{d}{=} \{X_t^A \vee X_t^B\}_{t\in T}, \tag{3}$$

where

$$X_t^A := {\int_A}^{e} f(t,u) M_\alpha(du) \quad \text{and} \quad X_t^B := {\int_B}^{e} f(t,u) M_\alpha(du)$$

are two independent processes.

• The decomposition (3) does not depend on the choice of the measurable rep $\{f(t,u)\}_{(t,u)\in T\times U}$.

Idea of proof: WLOG suppose that $\{f(t,u)\}_{t\in T}$ is minimal and let $\{g(t,v)\}_{t\in T} \subset L^\alpha_+(V,\nu)$ is another measurable rep of $X = \{X_t\}_{t\in T}$.

Then, by Thm 1:

$$g(t, v) = h(v) f(t, \phi(v)), \quad \text{where } h(v) \geq 0.$$

Since $\mathcal{P}$ is a cone,

$$g(\cdot, v) \in \mathcal{P} \quad \Leftrightarrow \quad f(\cdot, \phi(v)) \in \mathcal{P},$$

which shows that the corresponding partition of V is:

$$V = \widetilde{A} \cup \widetilde{B} := \phi^{-1}(A) \cup \phi^{-1}(B)$$

A change of variables, yields:

$$\Big\{ \overset{e}{\int_A} f_t dM_\alpha \Big\}_{t \in T} \overset{d}{=} \Big\{ \overset{e}{\int_{\widetilde{A}}} g_t d\widetilde{M}_\alpha \Big\}_{t \in T},$$

completing the proof.

Applications

Corollary: *Let*

$$\{ {}^{e}\!\!\int_U f(t,u) M_\alpha(du)\} \stackrel{d}{=} \{ {}^{e}\!\!\int_V g(t,v) \widetilde{M}_\alpha(dv)\}.$$

Then, given a cone $\mathcal{P} \subset L^0_+(T)$*,*

$$f(\cdot,u) \in \mathcal{P}, \text{a.e.} \quad \textit{if and only if} \quad g(\cdot,v) \in \mathcal{P}, \text{a.e.}$$

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Thm (Wang & S., 2009) *Let* $(U,\mu) \equiv (\mathbb{R}^d, dx)$ *and* $f,\ g \in L^\alpha_+(\mathbb{R}^d)$. *Consider the moving maxima random fields*

$$X_t := {}^{e}\!\!\int_{\mathbb{R}^d} f(t-u) M_\alpha(du) \quad \text{and} \quad Y_t := {}^{e}\!\!\int_{\mathbb{R}^d} g(t-u) M_\alpha(du).$$

Then,

$$\{X_t\}_{t\in\mathbb{R}^d} \stackrel{d}{=} \{Y_t\}_{t\in\mathbb{R}^d}, \quad \text{iff} \quad g(\cdot) = f(\cdot + \tau),$$

for some $\tau \in \mathbb{R}^d$.

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$$\{X_t\}_{t\in\mathbb{R}^d} \stackrel{d}{=} \{Y_t\}_{t\in\mathbb{R}^d}, \quad \text{iff} \quad g(\cdot) = f(\cdot+\tau),$$

for some $\tau \in \mathbb{R}^d$*.*

Proof: Consider the positive cone: $\mathcal{P} := \{cf(\cdot+\tau),\ c>0,\ \tau\in\mathbb{R}\}$. We have $g(\cdot+x) \in \mathcal{P}$, for almost all x!

Thank you!

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