REPRESENTATION OF THE FRAGILITY INDEX BY NORMS

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EVA 2009

The stability of a network can be characterized by a measure called the

Fragility Index FI

The FI captures the amount of risk in the tail via the limit of a conditional expectation.

Setting the stage

Consider a system of random variables

$$\{Q_1, \dots, Q_d\}$$

and the exceedence over a high threshold s

$$\{Q_j > s\}, \quad j \in \{1, \dots, d\}$$

which is an extreme event.

We are interested in the collapse of the system $\{Q_1, \dots, Q_d\}$.

Denote by

$$N_s := \sum_{j=1}^{d} \mathbb{1}_{(s,\infty)}(Q_j)$$

the number of exceedances among $\{Q_1, \dots, Q_d\}$.

Extension of the Fragility Index

The Fragility Index (FI) is the limit of the expected number of exceedances over a high threshold among d random variables $Q_1, \ldots, Q_d$ as the threshold increases, given that there is at least one exceedance, i.e.

$$FI := \lim_{s\to\infty} E(N_s \,|\, N_s > 0) \,.$$

This is the work of Geluk, de Vries and de Haan (2007).

The ***Extended Fragility Index*** **FI(m)** is the extension of the FI under the condition that there are at least m exceedances, i.e.

$$FI(m) := \lim_{s\to\infty} E(N_s \,|\, N_s \geq m) \,, \quad m \in \{1, \ldots, d\} \,.$$

Hence, we continue the work of Geluk, de Vries and de Haan (2007).

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Note, that $FI(m) := \lim_{s\to\infty} E(N_s \mid N_s \geq m) \ \in [m, d]$.

We call the system $\{Q_1, \dots, Q_d\}$

- *weak fragile* if $FI(m) = m$ (asymptotic stability of the system)
- *strongly fragile* if $FI(m) > m$ (asymptotic stability of the system).

Multivariate Extreme Value Distribution

Proposition

An arbitrary d-dim. Extreme Value Distribution (EVD) G can be represented as

$$G(\boldsymbol{x}) = \exp(-\|(\psi_1(x_1), \ldots, \psi_d(x_d))\|_D)$$

with $\psi_i(x) := \log(G_i(x))$, G_i the i-th margin of G.

The pertaining *Generalized Pareto function* (GPD) is any distribution function W, such that

$$W(\boldsymbol{x}) = 1 + \log(G(\boldsymbol{x}))$$

holds for $G(\boldsymbol{x})$ in a left neighborhood of 1.

Multivariate Domain of Attraction

We assume, that the distribution function F of the random vector $\boldsymbol{Q} := (Q_1, \dots, Q_d)$ is in the *domain of attraction* of an EVD G, abbr.

$$(Q_1, \dots, Q_d) \sim F \in \mathcal{D}(G) \ .$$

Theorem

F is in the domain of attraction of an EVD G, iff the uniform margins F_i of F are in the domain of attraction of the uniform margins G_i of the EVD G, $i \leq d$, and the convergence of the copulas C_F and C_G of F and G holds, i.e.

$$t^{-1}(1 - C_F(\mathbf{1} + t\boldsymbol{x})) \to_{t\downarrow 0} l_G(\boldsymbol{x}) = -\log(C_G(\exp(\boldsymbol{x}))), \ \boldsymbol{x} \leq \mathbf{0}.$$

(Deheuvels (1978,1984), Galambos (1987), de Haan and de Ronde (1998)).

Setting the threshold

The definition of the Extended Fragility Index

$$FI(m) := \lim_{s \to \infty} E(N_s \mid N_s \geq m)$$

offers more than one possible approach. The computation is based on the *type* of the event of exceedance:

- Approach 1: $\{F_j(Q_j) > 1 - c\}$ leads to an individual threshold $\tilde{s}_j = F_j^{-1}(1 - c)$ for Q_j
- Approach 2: $\{Q_j > s\}$ leads to a common threshold s for $Q_j,\ j \in \{1, \dots, d\}$

Approach 1: Individual threshold

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We consider events of exceedance like

$$\{F_j(Q_j) > 1 - c\}, \quad j \in \{1, \dots, d\}.$$

Denote by

$$N_c := \sum_{j \le d} \mathbb{1}_{(-c,0]}(F_j(Q_j) - 1)$$

the number of exceedances over c among the random variables $Q_1, \dots, Q_d$.

Approach 1: Individual threshold

Theorem

Assume that $\boldsymbol{Q} \sim F \in \mathcal{D}(G)$. Suppose each margin F_j is continuous in the neighborhood of $\omega(F_j)$ and the margins do not have to be identical.

Denote by $p_k := \lim_{c \downarrow 0} P(N_c = k \mid N_c > 0)$ the limit of the conditional distribution function of N_c.

Then we get

$$FI(m) = \frac{\sum_{k=m}^{d} k \cdot p_k}{\sum_{k=m}^{d} p_k} =$$

$$\frac{\sum_{k=m}^{d} k \cdot \sum_{0 \le j \le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T \subset \{1,\dots,d\} \\ |T| = d-j}} \|\sum_{i \in T} \boldsymbol{e}_i\|_D}{\sum_{k=m}^{d} \sum_{0 \le j \le k} (-1)^{k-j+1} \binom{d-j}{k-j} \sum_{\substack{T \subset \{1,\dots,d\} \\ |T| = d-j}} \|\sum_{i \in T} \boldsymbol{e}_i\|_D}$$

Approach 1: Individual threshold

$$FI(m) = \frac{\sum_{k=m}^{d} k \cdot \sum_{0\le j\le k}(-1)^{k-j+1}\binom{d-j}{k-j}\sum_{\substack{T\subset\{1,\dots,d\}\\|T|=d-j}}\|\sum_{i\in T}\boldsymbol{e}_i\|_D}{\sum_{k=m}^{d}\sum_{0\le j\le k}(-1)^{k-j+1}\binom{d-j}{k-j}\sum_{\substack{T\subset\{1,\dots,d\}\\|T|=d-j}}\|\sum_{i\in T}\boldsymbol{e}_i\|_D}$$

Remark

- If we suppose an individual threshold $\tilde{s}_j = F_j^{-1}(1-c),\ j \le d$, the value of FI(m) is independent of the type of marginal distribution F_j.
- *Of course* FI(m) still depends on the choice of the D-norm, respectively the Pickands Dependence function D.

Examples: Choices for the D-norm

Example

An obvious choice for $\|\cdot\|_D$ is the arbitrary L_p-norm

$$\|(x_1,\dots,x_d)\|_p := \left(\sum_{i=1}^d |x_i|^p\right)^{1/p}, \quad 1 \le p < \infty \quad \text{and}$$

$$\|(x_1,\dots,x_d)\|_\infty := \max_{1\le i\le d} |x_i| ,$$

which includes the two extreme cases of maximum dependence and full independence in the tail of $(Q_1,\dots,Q_d)$.

Example

Note, that the choice $\|\boldsymbol{x}\|_D := \|\boldsymbol{x}\|_p$ represents $\boldsymbol{Q} \sim F$, which belongs to $\mathcal{D}$ of the negative logistic df in $\mathbb{R}^d$, i.e

$$G_p(\boldsymbol{x}) = \exp\left(-\left(\sum_{i\le d}(-x_i)^p\right)^{1/p}\right) = \exp(-\|\boldsymbol{x}\|_p),\ p \ge 1\ .$$

The negative logistic df has margins $G_i(x) = \exp(x),\ x \le 0$.

Examples: Choices for the D-norm

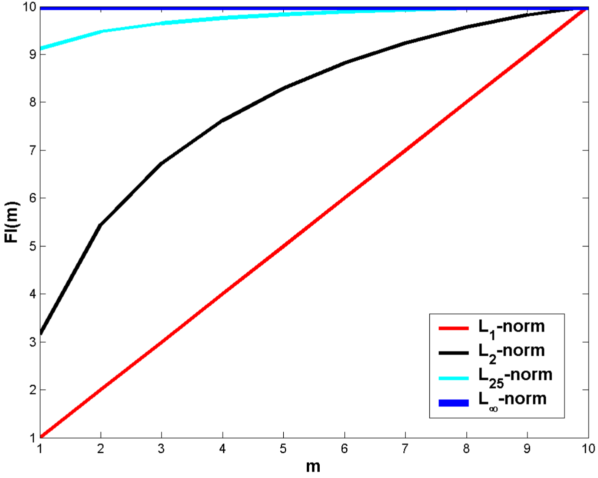
Example

(i) For the L_1-norm we get $FI(m) = m$, which is the case of independence in the tail of $\boldsymbol{Q}$, that means we have asymptotic stability of the system.
$(G_1(\boldsymbol{x}) = \exp(-|x_1| - \ldots - |x_d|))$

(ii) For the L_∞-norm we get $FI(m) = d,\ m \leq d$, which is the case of total dependence in the tail of $\boldsymbol{Q}$, that means we have asymptotic instability of the system.

Examples: Choices for the D-norm

Graphic of the FI(m) in dependence on the number of exceedances and the Lp-norm



Examples: Archimedean Copula

Example

If we model the copula function C_F of the df F with an *Archimedean copula* C_φ, we get under certain regularity conditions

$$C_\phi(\mathbf{1} + t\boldsymbol{x}) = 1 - \|\boldsymbol{x}\|_1 + o(t), \ \boldsymbol{x} \leq \mathbf{0},$$

($\rightarrow$ talk of Michael Falk) and hence we get $FI(m) = m$, which is the case of independence.

Hence, if one assumes $\boldsymbol{Q} \sim F$, where F has arbitrary margins and the model for the dependence structure is an Archimedean Copula, the system $\{Q_1, \dots, Q_d\}$ will be weak fragile, i.e. we have *asymptotic stability* for the system!

Approach 2: Common threshold

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We consider events of exceedance like

$$\{Q_j > s\}, \quad j \in \{1, \ldots, d\}.$$

Denote by

$$N_s := \sum_{j=1}^{d} \mathbb{1}_{(s,\infty)}(Q_j)$$

the number of exceedances over s among the random variables $Q_1, \ldots, Q_d$.

Approach 2: Common threshold

In addition to Approach 1 we need the following condition:

Condition C: There exists an index $\kappa \in \{1, \dots, d\}$ with $\omega^* := \omega(F_\kappa) := \sup\{t \in \mathbb{R} : F_\kappa(t) < 1\}$, such that

$$\lim_{s \uparrow \omega^*} \frac{1 - F_i(s)}{1 - F_\kappa(s)} =: \gamma_i \in [0, \infty), \quad 1 \leq i \leq d$$

holds.

Approach 2: Common threshold

Theorem

Assume $\boldsymbol{Q} \sim F \in \mathcal{D}(G)$. Suppose F is continuous for $\boldsymbol{x}$ close to $\omega(F) := (\omega(F_1), \dots, \omega(F_d))$ and Condition C holds.

Denote by $p_k(\gamma) := \lim_{s \uparrow \omega^*} P(N_s = k \mid N_s > 0)$ the limit of the conditional distribution function of N_s.

Then we get

$$FI(m) = \frac{\sum_{k=m}^{d} k \cdot p_k(\gamma)}{\sum_{k=m}^{d} p_k(\gamma)}$$

$$= \frac{\sum_{k=m}^{d} k \cdot \sum_{j=0}^{k} (-1)^{k-j+1} \begin{pmatrix} d-j \\ k-j \end{pmatrix} \sum_{\substack{T \subset \{1,\dots,d\} \\ |T| = d-j}} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D}{\sum_{k=m}^{d} \sum_{j=0}^{k} (-1)^{k-j+1} \begin{pmatrix} d-j \\ k-j \end{pmatrix} \sum_{\substack{T \subset \{1,\dots,d\} \\ |T| = d-j}} \left\| \sum_{i \in T} \gamma_i \boldsymbol{e}_i \right\|_D}$$

Approach 2: Common threshold

Remark

- In the situation of a common treshold s, the value of FI(m) depends on the margins of the df F, because γ_i is the limit of $\frac{1-F_i(s)}{1-F_\kappa(s)}$ for $s \uparrow \omega^*$.
- If we suppose identical margins of F in the situation of a common treshold s, we get $\gamma_i = 1$ for $i = 1, \dots, d$. Hence, the resulting FI(m) is independent of the margins of the df F.

The case $\boldsymbol{Q} \sim GPD$

In contrast to the assumption $\boldsymbol{Q} \sim F \in \mathcal{D}(G)$, which is underlying Approach 1 and 2, we can model a situation where the random vector $\boldsymbol{Q}$ follows a Generalized Pareto distribution function.

Note:

If we suppose $\boldsymbol{Q} \sim GPD\ W \quad \ldots$

- the FI(m) attains its limit for a finite threshold!
- the resulting FI(m) is independent of the margins of the GPD W.

Summary of our results

- The cases of ...
  (i) an individual threshold (Approach 1),
  (ii) identical margins of F (Approach 1 or 2) and
  (iii) $\boldsymbol{Q} \sim GPD$

  lead to a FI(m) which is independent of the type of the margins of F.
- These cases have in common that the FI(m) still depends on the dependence structure of the multivariate distribution F, that means on the D-norm or the Copula C_F!!

What do I want to do next??

- Given a specific df $F \quad \rightarrow \quad$ *Easy* criteria for the corresponding $\mathcal{D}(G)$??
- Given a specific Copula function C_F of $F \quad \rightarrow \quad$ Value of the FI(m) ??
- If I only have information about the margins of F, what do I know about $\mathcal{D}(G)$?

Thank you for your attention!