Fiducial Intervals for Variance Components in an Unbalanced Two-component Normal Mixed Linear Model

Lidong E, Jan Hannig, and Hari Iyer
Colorado State University, Fort Collins,CO 80523

October 13, 2006

Abstract

In this paper, a new method for constructing confidence intervals for σ_α^2, σ_ε^2 and the intra-class correlation $\rho = \sigma_\alpha^2/(\sigma_\alpha^2 + \sigma_\varepsilon^2)$ in a two component mixed effects linear model is proposed. This method is based on an extension of R. A. Fisher's fiducial argument. A simulation study is conducted to compare the resulting interval estimates with other competing confidence interval procedures from the literature. Our results demonstrate that the proposed fiducial intervals have satisfactory performance in terms of coverage probability. In addition these intervals have shorter average confidence interval lengths overall. We also prove that these fiducial intervals have asymptotically exact frequentist coverage probability. The computations for the proposed procedures are illustrated using examples from animal breeding applications.

Keywords: Fiducial Density, Fiducial Generalized Confidence Interval (FGCI), Unbalanced One-Way Random Effects Model, Variance Component.

1 Introduction

Random effects and mixed effects linear models are useful in applications that require accounting for components of variability arising from multiple sources. For example, in animal breeding studies, mixed linear models with two variance components are often used. One variance component accounts for genetic variability and the other accounts for variability due to environmental factors. In industrial applications where one is interested in understanding process variability mixed models with multiple variance components are used to account for variability due to operators, due to batches of raw material, due to machine differences, due to measurement errors, and so on. In such situations it is of interest to estimate the components of variance and provide lower and upper confidence bounds for them.

Confidence intervals for variance components have been an important topic of research for over 70 years. Interestingly, the first published work on interval estimation for the between groups variance component in the standard one-way normal random model is by R. A. Fisher (1935) who gave a solution to this problem using his then new method of fiducial argument. Bross (1950) provided further computational details for the fiducial approach and informally compared it with approximate frequentists methods available at the time. Numerous subsequent articles have been written on this topic by many authors. See for instance, Green (1954), Huitson (1955), Graybill et al. (1956), Welch (1956), Healy (1961, 1963), Williams (1962), Broemeling (1969), Burdick and Sielken (1978), Venables and James (1978), Jeyaratnam and Graybill (1980), Graybill and Wang (1980), Seely (1980), Burdick and Graybill (1984), Harville and Fenech (1985), Wild (1981), among others. Most of this work is devoted to developing exact or approximate confidence intervals for specified linear functions of variance components or their ratios. Some of the work was carried out in the context of inference on a heritability coefficient in animal breeding contexts. Healy (1963), Venables and James (1978), and Wild (1981) consider fiducial approaches to the problem in the case of balanced data.

Our focus in this paper is on two variance component unbalanced normal mixed linear models. A fiducial solution to the interval estimation problem in this context is not currently available. Here we develop such a fiducial solution and demonstrate via a simulation study that the resulting procedure has better overall frequentist performance than competing methods. We also establish the asymptotic exactness of the coverage probability of fiducial intervals for variance components of interest.

More specifically, let $\boldsymbol{Y}$ denote a $N \times 1$ vector of observable random variables. Suppose $\boldsymbol{Y}$ has a distribution described by the following mixed linear model with two variance components

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{u} + \boldsymbol{\varepsilon} \tag{1}$$

where $\boldsymbol{X}$ and $\boldsymbol{Z}$ are known incidence matrices of sizes $N \times p$ and $N \times a$, respectively, $\boldsymbol{\beta}$ is a $p \times 1$ vector of unknown parameters, $\boldsymbol{u} \sim N(\boldsymbol{0}, \sigma_\alpha^2 \boldsymbol{A})$ is a $a \times 1$ vector of random effects, $\boldsymbol{\varepsilon} \sim N(\boldsymbol{0}, \sigma_\varepsilon^2 \boldsymbol{I}_N)$ is the error vector of size $N \times 1$, and $\boldsymbol{u}$ and $\boldsymbol{\varepsilon}$ are independent. Without loss of generality we assume $\text{rank}(\boldsymbol{X}) = p$. Also $\boldsymbol{A}$ is a known matrix often referred to as a *relationship matrix* in animal breeding context since it describes the degree to which the elements $u_1, \ldots, u_a$ of the vector $\boldsymbol{u}$ covary. For example, if the elements u_1 and u_2 of $\boldsymbol{u}$ are the (additive) genetic effects corresponding to a parent and an offspring, respectively, then $Cov(u_1, u_2) = \sigma_\alpha^2/2$ (Falconer, 1989). Note that, when the standard unbalanced one-way random effects model given by

$$Y_{ij} = \mu + u_i + \varepsilon_{ij}, \qquad i = 1, \ldots, a; \quad j = 1, \ldots, n_i, \tag{2}$$

is written in matrix form the matrix $\boldsymbol{X}$ is simply $\boldsymbol{1}_N$, a column vector whose elements are all equal to one, $\boldsymbol{Z} = diag(\boldsymbol{I}_{n_1}, \ldots, \boldsymbol{I}_{n_a})$ is a block diagonal matrix whose i^{th} block is $\boldsymbol{I}_{n_i}$, and $\boldsymbol{A}$ is $\boldsymbol{I}_N$. Here $\boldsymbol{I}_r$ denotes the $r \times r$ identity matrix. Under the model in (1), $E[\boldsymbol{Y}] = \boldsymbol{X}\boldsymbol{\beta}$ and $Var[\boldsymbol{Y}] = \boldsymbol{V} = \sigma_\alpha^2 \boldsymbol{Z}\boldsymbol{A}\boldsymbol{Z}^T + \sigma_\varepsilon^2 \boldsymbol{I}_N$. In this paper, we focus on constructing confidence intervals for the variance component σ_α^2, σ_ε^2 and the ratio $\rho = \sigma_\alpha^2/(\sigma_\alpha^2 + \sigma_\varepsilon^2)$. In the special case of a one-way random effects model, σ_α^2 is the between-groups variance component and ρ is the intraclass correlation coefficient. Our proposed methods follow the fiducial generalized confidence interval (FGPQ) procedures discussed in Hannig et al. (2006) and the generalizations of the fiducial method given in Hannig (2006).

The paper is organized as follows. Section 2 provides a brief review of published confidence interval procedures for σ_α^2, σ_ε^2 and ρ. In Section 3 we outline the fiducial method for obtaining confidence intervals for general situations. We then apply this method to derive fiducial confidence intervals for σ_α^2, σ_ε^2 and ρ. Our procedure is applicable to the two component mixed model given in (1). We compare our proposed procedures with competing methods described in Section 2 using a simulation study. Details of the simulation study are described in Section 4 along with a discussion of the simulation results. Tables containing numerical results from the simulation study are given in the Appendix. In Section 5 we consider some data examples using previously published data and illustrate how our proposed procedures are applied. Asymptotic exactness of the proposed fiducial intervals is proved in Section 6. Finally, we conclude with summary discussions in Section 7.

2 Discussion of Published Confidence Intervals for Two Component Mixed Models

This section provides an overview of various published confidence intervals for σ_α^2, σ_ε^2, and $\rho = \sigma_\alpha^2/(\sigma_\alpha^2 + \sigma_\varepsilon^2)$ in a two-component mixed model. First we discuss the unbalanced one-way random effects model.

One-way unbalanced random effects model

Here we consider the unbalanced one-way random effects model, see Equation (2), which is a special case of (1). The standard ANOVA table for this model is shown in Table 1 where the following notation is used.

$$\overline{Y}_{i\star} = \frac{\sum_{j=1}^{n_i} Y_{ij}}{N}, \quad \overline{Y}_{\star\star} = \frac{\sum_{i=1}^{a} n_i \overline{Y}_{i\star}}{N}, \quad SS_1 = \sum_{i=1}^{a} n_i(\overline{Y}_{i\star} - \overline{Y}_{\star\star})^2,$$

$$SS_2 = \sum_{i=1}^{a}\sum_{j=1}^{n_i}(Y_{ij} - \overline{Y}_{i\star})^2, N = \sum_{i=1}^{a} n_i, \text{ and } n_0 = \frac{N - \frac{\sum_{i=1}^{a} n_i^2}{N}}{a-1}.$$

Table 1: ANOVA for the One-way Unbalanced Random Effects Model.

Source	DF	MS	EMS
Among groups	$a-1$	$S_1^2 = SS_1/(a-1)$	$\theta_1 = \sigma_\varepsilon^2 + n_0\sigma_\alpha^2$
Within groups	$N-a$	$S_2^2 = SS_2/(N-a)$	$\theta_2 = \sigma_\varepsilon^2$
Total	$N-1$		

When n_i are all equal, the one-way random effects model is said to be balanced. In this case we write $n_i = n$. It follows that, in the balanced case $n_0 = n$, and SS_1/θ_1 and SS_2/θ_2 are independent chi-squared random variables with $a-1$ and $N-a$ degrees of freedom, respectively. It is well known that, in the balanced one-way random effects model, exact confidence intervals are available for σ_ε^2, $\sigma_\alpha^2/\sigma_\varepsilon^2$ and ρ. An exact confidence interval for the between group variance σ_α^2 is not available even in the balanced case. Approximate confidence intervals for σ_α^2 include the well-known Tukey-Williams

(Tukey (1951), Williams (1962)) methods and a host of other methods including the method based on Satterthwaite's (1946) approximation. See Burdick and Graybill (1992) for details.

In the case of an unbalanced design, SS_1 and SS_2 are still independent and $SS_2/\sigma_\varepsilon^2$ still has a chi-squared distribution with $N-a$ degrees of freedom, but SS_1/θ_1 does not have a chi-squared distribution unless $\sigma_\alpha^2 = 0$ (Burdick and Graybill, 1992).

2.1 Confidence intervals for σ_α^2 in an unbalanced one-way random effects model

Several methods are available in the literature for constructing approximate confidence intervals for σ_α^2 in the unbalanced one-way random effects model. Five different confidence interval procedures for σ_α^2 that have previously appeared in the literature are discussed below.

Burdick-Graybill (BG) confidence interval

In an unbalanced design, SS_1/θ_1 has a chis-squared distribution if and only if $\sigma_\alpha^2 = 0$. If it is known that σ_α^2 is close to zero, then treating SS_1/θ_1 as a chi-squared random variable may be appropriate. Using this idea, Burdick and Graybill (1992) developed an approximate confidence interval for σ_α^2 based on the reasoning that SS_1/θ_1 has, approximately, a chi-squared distribution with $a-1$ degrees of freedom when σ_α^2 is close to zero. They obtained this approximate confidence interval by appropriately modifying the corresponding confidence interval in balanced case. The resulting approximate two-sided $(1-\alpha)100\%$ confidence interval is given by

$$\left[\max\left(\frac{S_1^2 - S_2^2 - \sqrt{V_L}}{n_0},\ 0\right),\ \ \max\left(\frac{S_1^2 - S_2^2 + \sqrt{V_U}}{n_0},\ 0\right)\right]$$

where

$$\begin{aligned}
V_L &= G_1^2 S_1^4 + H_2^2 S_2^4 + G_{12} S_1^2 S_2^2, \quad V_U = H_1^2 S_1^4 + G_2^2 S_2^4 + H_{12} S_1^2 S_2^2,\\
G_1 &= 1 - \frac{1}{F_{1-\alpha/2;a-1,\infty}}, \quad G_2 = 1 - \frac{1}{F_{1-\alpha/2;N-a,\infty}},\\
H_1 &= \frac{1}{F_{\alpha/2;a-1,\infty}} - 1, \quad H_2 = \frac{1}{F_{\alpha/2;N-a,\infty}} - 1,\\
G_{12} &= \frac{\left(F_{1-\alpha/2;a-1,N-a} - 1\right)^2 - G_1^2 F_{1-\alpha/2;a-1,N-a}^2 - H_2^2}{F_{1-\alpha/2;a-1,N-a}},\\
H_{12} &= \frac{\left(1 - F_{\alpha/2;a-1,N-a}\right)^2 - H_1^2 F_{\alpha/2;a-1,N-a}^2 - G_2^2}{F_{\alpha/2;a-1,N-a}}
\end{aligned}$$

and $F_{\alpha;v_1,v_2}$ represents the α-quantile of the F-distribution with v_1 and v_2 degrees of freedom. Since this procedure is based on the assumption that σ_α^2 is close to zero, it might result in very liberal intervals when σ_α^2 is far from zero (Burdick and Graybill, 1992).

Thomas-Hultquist (TH) confidence interval

Thomas and Hultquist (1978) derived an approximate pivotal quantity for θ_1 that can be used for constructing confidence intervals for σ_α^2 in the unbalanced one-way random effects model. This quantity is SS_3/θ_3 where

$$SS_3 = \sum_{i=1}^{a} \left(\overline{Y}_{i\star} - \frac{1}{a} \sum_{i=1}^{a} \overline{Y}_{i\star} \right)^2, \quad \theta_3 = \sigma_\alpha^2 + \frac{\sigma_\varepsilon^2}{\tilde{n}}, \quad \text{and } \tilde{n} = \frac{a}{\sum_{i=1}^{a}(1/n_i)}.$$

We define $S_3^2 = SS_3/(a-1)$. Note that SS_3 is the unweighted sum of squares of the treatment means and $\tilde{n}$ denotes the harmonic mean of n_i values. Thomas and Hultquist (1978) showed that the moment generating function of SS_3/θ_3 approaches that of a chi-squared random variable with $a-1$ degrees of freedom as all n_i approach a constant value or infinity, or if the ratio $\eta = \sigma_\alpha^2/\sigma_\varepsilon^2$ approaches infinity. Furthermore, SS_3 is independent of SS_2. Therefore, $(SS_3/\theta_3)/(SS_2/\theta_2)$ has an approximate $F_{a-1,N-a}$ distribution. Using these facts and modifying the Tukey-Williams confidence interval formula for σ_α^2 developed for the balanced case, Thomas and Hultquist (1978) proposed the following approximate two-sided $(1-\alpha)100\%$ confidence interval for σ_α^2

$$\left[\frac{\tilde{n}SS_3 - (a-1)S_2^2 F_{1-\alpha/2;a-1,N-a}}{\tilde{n}\chi^2_{1-\alpha/2;a-1}}, \frac{\tilde{n}SS_3 - (a-1)S_2^2 F_{\alpha/2;a-1,N-a}}{\tilde{n}\chi^2_{\alpha/2;a-1}} \right], \tag{3}$$

where $\chi^2_{\alpha;v}$ represents the α-quantile of the chi-squared distribution with v degrees of freedom. Results of their simulation study indicated that SS_3/θ_3 is not well approximated by a chi-squared random variable when $\eta < 0.25$ and the design is extremely unbalanced. In these cases, the confidence interval in (3) can be quite liberal.

Burdick-Eickman (BE) confidence interval

Williams (1962) constructed an interval for σ_α^2 in the balanced one-way random effects model by solving for the intersection of exact $(1-\alpha)100\%$ confidence intervals on $\sigma_\varepsilon^2 + n\sigma_\alpha^2$ and the ratio η. Burdick and Eickman (1986) followed this strategy and combined approximate intervals for θ_3 and η. The approximate $(1-\alpha)100\%$ confidence interval for θ_3 they used is based on the Thomas-Hultquist (1978) approximation, and is given by

$$\left[\frac{SS_3}{\chi^2_{1-\alpha/2;a-1}}, \frac{SS_3}{\chi^2_{\alpha/2;a-1}} \right]. \tag{4}$$

The approximate $(1-\alpha)100\%$ confidence interval on η they used is the one developed by Burdick et al. (1986). This interval is $[L_{BM}, U_{BM}]$ where

$$\begin{aligned} L_{BM} &= \max\left(0,\ \frac{S_3^2}{S_2^2 F_{1-\alpha/2;a-1,N-a}} - \frac{1}{\min(n_1,\ldots,n_a)}\right), \\ U_{BM} &= \max\left(0,\ \frac{S_3^2}{S_2^2 F_{\alpha/2;a-1,N-a}} - \frac{1}{\max(n_1,\ldots,n_a)}\right). \end{aligned} \tag{5}$$

The interval in (5) has a confidence coefficient at least as great as $1-\alpha$. By finding the intersection region of (4) and (5), Burdick and Eickman arrived at an approximate two-sided $(1-\alpha)100\%$ confidence interval for σ_α^2. This interval is

$$\left[\left(\frac{\tilde{n} L_{BM}}{1 + \tilde{n} L_{BM}} \right) \frac{SS_3}{\chi^2_{1-\alpha/2;a-1}}, \; \left(\frac{\tilde{n} U_{BM}}{1 + \tilde{n} U_{BM}} \right) \frac{SS_3}{\chi^2_{\alpha/2;a-1}} \right]. \tag{6}$$

The confidence coefficient of the interval in (6) is at least $1-\alpha$.

Hartung-Knapp (HK) confidence interval

In the unbalanced one-way random effects model Wald (1940) showed that the quantity SS_4 defined by

$$SS_4 = \sum_{i=1}^{a} w_i \left(\overline{Y}_{i\star} - \frac{\sum_{i=1}^{a} w_i \overline{Y}_{i\star}}{\sum_{i=1}^{a} w_i} \right)^2$$

where $w_i = n_i/(1+\eta n_i)$ is a pivotal quantity for $\eta = \sigma_\alpha^2/\sigma_\varepsilon^2$. Specifically, $SS_4/\sigma_\varepsilon^2$ follows chi-squared distribution with $a-1$ degrees of freedom. Furthermore, SS_4 and SS_2 are independent. Therefore, letting $S_4^2 = SS_4/(a-1)$, it follows that

$$R(\eta) = \frac{S_4^2}{S_2^2} \sim F_{a-1,N-a}$$

and an exact confidence interval for η may be obtained from an interval for $R(\eta)$. Wald (1940) showed that SS_4 is a strictly monotonic decreasing function in η, so the bounds of a $100(1-\alpha)\%$ confidence interval for η are given as the unique solutions to the equations

$$\begin{aligned} R(\eta) &= F_{1-\alpha/2;a-1,N-a}, \\ R(\eta) &= F_{\alpha/2;a-1,N-a}. \end{aligned} \tag{7}$$

Hartung and Knapp (2000) considered the solutions, η_L, η_U, to equations (7) and used these to construct an approximate two-sided $(1-\alpha)100\%$ confidence interval for σ_α^2. Their interval is given by

$$[S_2^2 \eta_L', S_2^2 \eta_U'],$$

where

$$\eta_L' = \begin{cases} \eta_L & \text{if } 0 \le \eta_L \le R(0) \\ 0 & \text{otherwise} \end{cases} \quad \text{and} \quad \eta_U' = \begin{cases} \eta_U & \text{if } 0 \le \eta_U \le R(0) \\ 0 & \text{otherwise} \end{cases}.$$

It is important to note that the four interval procedures discussed above apply only for the one-way random model. They do not apply to the general two-component mixed model in (1).

2.2 Confidence intervals for σ_α^2 in a two variance components mixed model

Arendacká (2005) proposed a method that is applicable in some instances of the model in (1). We describe her method next. Before doing so, we review some well known results concerning minimal sufficient statistics for the mixed model in (1).

Let $\boldsymbol{H}$ be a $N \times (N-p)$ matrix such that $\boldsymbol{H}\boldsymbol{H}^T = \boldsymbol{I}_N - \boldsymbol{X}(\boldsymbol{X}^T\boldsymbol{X})^{-}\boldsymbol{X}^T$ and $\boldsymbol{H}^T\boldsymbol{H} = \boldsymbol{I}_{N-p}$. Using the fact that $\boldsymbol{Y} \sim N(\boldsymbol{X\beta}, \sigma_\varepsilon^2\boldsymbol{I}_N + \sigma_\alpha^2\boldsymbol{ZAZ}^T)$, it follows that

$$\boldsymbol{H}^T\boldsymbol{Y} \sim N(\boldsymbol{0}, \sigma_\varepsilon^2\boldsymbol{I}_{N-p} + \sigma_\alpha^2\boldsymbol{G}) \tag{8}$$

where $\boldsymbol{G} = \boldsymbol{H}^T\boldsymbol{ZAZ}^T\boldsymbol{H}$. Let $\lambda_1 >, \dots, > \lambda_d \geq 0$ be the distinct eigenvalues of $\boldsymbol{G}$ having multiplicities $r_1, \dots, r_d$, respectively. Let $\boldsymbol{P} = [\boldsymbol{P}_1, \dots, \boldsymbol{P}_d]$ be a $(N-p) \times (N-p)$ orthogonal matrix such that $\boldsymbol{P}^T\boldsymbol{G}\boldsymbol{P} = diag(\lambda_1\boldsymbol{1}_{r_1}^T, \dots, \lambda_d\boldsymbol{1}_{r_d}^T)$. where $\boldsymbol{P}_i$ corresponding to λ_i is of size $(N-p) \times r_i$. Define

$$V_i = \boldsymbol{Y}^T\boldsymbol{H}\boldsymbol{P}_i\boldsymbol{P}_i^T\boldsymbol{H}^T\boldsymbol{Y}, \qquad i = 1, \dots, d. \tag{9}$$

Olsen et al. (1976) showed that $(V_1, \dots, V_d)$ is minimal sufficient for $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ under (8). Furthermore,

$$U_i = \frac{V_i}{\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2} \sim \chi_{r_i}^2, \qquad i = 1, \dots, d, \tag{10}$$

and U_is' are mutually independent.

Note that, when λ_d is zero, a *pure error* estimate of σ_ε^2 is given by V_d/r_d. An exact $100(1-\alpha)\%$ confidence interval for σ_ε^2 exists and is given by

$$\left[\frac{V_d}{\chi^2_{1-\alpha/2;r_d}}, \frac{V_d}{\chi^2_{\alpha/2;r_d}}\right] \tag{11}$$

We refer to the interval in (11) as EXACT (**EX**) confidence interval for σ_ε^2. When $\lambda_d > 0$ a pure error estimate of σ_ε^2 is not available. In particular, an exact confidence interval for σ_ε^2 is unavailable.

Arendacká (Ar) confidence interval

Arendacká (2005) considered the special case of $\lambda_d = 0$ and constructed a confidence interval for σ_α^2 using generalized test variables and generalized p-values. For a discussion of generalized p-values, see Weerahandi (1991). Arendacká showed that the quantity T defined by

$$T = \sum_{i=1}^{d-1}\left(U_i - \frac{v_iU_d}{v_d + \lambda_i\sigma_\alpha^2U_d}\right), \tag{12}$$

is a generalized test variable, where $(v_1, \dots, v_d)$ is a realization of $(V_1, \dots, V_d)$. She further defined the function

$$\pi_T(v_1, \dots, v_d, \sigma_\alpha^2) = \int_0^\infty \left(1 - F_W\left(\sum_{i=1}^{d-1}\frac{v_iu}{v_d + \lambda_i\sigma_\alpha^2u}\right)\right)f_{U_d}(u)du,$$

where $W = \sum_{i=1}^{d-1} U_i$ and $f_{U_d}(u)$ is the p.d.f. of U_d. She showed that

$$L_{BA} \leq \sigma_\alpha^2 \leq U_{BA} \tag{13}$$

is a generalized confidence interval for σ_α^2, where L_{BA} and U_{BA} are obtained by solving the equations

$$\pi_T(v_1, \ldots, v_d, L_{BA}) = \alpha/2, \quad \text{and}$$
$$\pi_T(v_1, \ldots, v_d, U_{BA}) = 1 - \alpha/2.$$

In particular, $[L_{BA}, U_{BA}]$ has coverage probability approximately $(1 - \alpha)$. It is worth noting that Arendacká's method is closely related to the generalized pivotal quantity for σ_α^2 derived in Iyer et al. (2004) in an unbalanced one-way random model with heterogeneous variances.

Arendacká (2005) also considered three other test variables based on the results in Zhou and Mathew (1994). Her simulation study showed that all the test variables perform equally well in terms of empirical coverages. But when comparing the average lengths of the intervals, the test variable T in (12) performed better overall than the other three test variables. Thus we use the interval in (13) for comparing with our proposed fiducial method.

2.3 Confidence intervals for σ_ε^2 in a two variance components mixed model

As mentioned earlier, an exact confidence interval for σ_ε^2 is available when $\lambda_d = 0$, i.e., a pure error estimate of σ_ε^2 is available. However, for the case $\lambda_d > 0$, to our knowledge, no confidence interval procedure has been proposed in the literature for σ_ε^2. Here we propose a fiducial interval estimate for σ_ε^2 that appears to have satisfactory coverage properties. The fiducial approach is discussed in Section 3.

2.4 Confidence intervals for ρ in a two variance components mixed model

In many applications the quantity $\rho = \sigma_\alpha^2/(\sigma_\alpha^2 + \sigma_\varepsilon^2)$ is of interest. For example, in plant and animal breeding, ρ represents the proportion of the total variance that is explainable by additive genetic effects. It is often referred to as the heritability of the trait under study. Burch and Iyer (1997) proposed a class of confidence intervals for ρ under the general two-component mixed effects linear model in (1). We describe their method below.

Burch and Iyer (BI) confidence intervals for ρ

Burch and Iyer (1997) proposed a method to construct an exact confidence interval on ρ based on a class of pivotal quantities R_k, $k = 1, \ldots, d-1$, given by

$$R_k = \frac{\sum_{i=k+1}^{d} \frac{V_i}{1+\rho(\lambda_i - 1)} \Big/ \sum_{i=k+1}^{d} r_i}{\sum_{j=1}^{k} \frac{V_j}{1+\rho(\lambda_j - 1)} \Big/ \sum_{j=1}^{k} r_j}. \tag{14}$$

They observed that $R_k \sim F\left(\sum_{i=k+1}^{d} r_i, \sum_{j=1}^{k} r_j\right)$ for each $k = 1, \ldots, d-1$. The pivotal quantity in (14) is a monotone decreasing function of ρ. Therefore, a confidence interval for ρ can be obtained by numerically inverting suitable probability limits for the pivotal quantity R_k. An exact $(1-\alpha)100\%$ confidence region for ρ is given by the set

$$\left\{ \rho \in [0,1) : F_{\alpha/2} \le \frac{\sum_{i=k+1}^{d} \frac{V_i}{1+\rho(\lambda_i-1)} / \sum_{i=k+1}^{d} r_i}{\sum_{j=1}^{k} \frac{V_j}{1+\rho(\lambda_j-1)} / \sum_{j=1}^{k} r_j} \le F_{1-\alpha/2} \right\},$$

where $F_{\alpha/2}$ and $F_{1-\alpha/2}$ are the $100(\alpha/2)-$percentile and $100(1-\alpha/2)-$percentile of the F distribution having numerator and denominator degrees of freedom equal to $\sum_{i=k+1}^{d} r_i$ and $\sum_{j=1}^{k} r_j$, respectively. Let L denote the infimum of this set and U the supremum. Then $[L, U]$ is a confidence interval for ρ with confidence coefficient equal to $(1-\alpha)$. Note that there are $d-1$ possible versions of the pivotal quantity R_k and consequently $d-1$ possible confidence intervals. Burch and Iyer (1997) showed that a confidence interval for ρ obtained from the pivotal quantity R_k is locally unbiased if the numerator degrees of freedom for R_k is equal to the denominator degrees of freedom, i.e., $\sum_{i=k+1}^{d} r_i = \sum_{j=1}^{k} r_j$. When such a k does not exist, they suggest using a k for which the numerator degrees of freedom, $\sum_{i=k+1}^{d} r_i$, and the denominator degrees of freedom $\sum_{j=1}^{k} r_j$, are as close to each other as possible. We refer to the resulting interval as **BI** confidence interval. We will compare our proposed fiducial interval for ρ with the **BI** intervals.

3 Fiducial Intervals for σ_α^2, σ_ε^2 and ρ

It is worth noting that generalized confidence intervals such as those proposed by Arendacká (2005) are closely related to fiducial intervals. This connection between generalized inference and fiducial inference is discussed in detail by Hannig et al. (2006). They also provide a recipe for constructing fiducial intervals when $\boldsymbol{X}$ has a continuous distribution. Hannig (2006) generalizes this to arbitrary distributions. We use the term *weak fiducial inference* to emphasize the fact that the version of fiducial inference discussed in Hannig et al. (2006) and Hannig (2006) is a generalization of R. A. Fisher's fiducial argument.

In this section we describe fiducial interval (**FI**) procedures for σ_α^2, σ_ε^2 and ρ that are applicable under the general two-component mixed model in (1). The intervals we propose are obtained using the fiducial method described in Hannig et al. (2006) and Hannig (2006). We first briefly describe weak fiducial inference.

3.1 The Fiducial approach

Let $\boldsymbol{X}$ be a (possibly discrete) random vector with a distribution indexed by a (possibly vector) parameter $\xi \in \Xi$. Hannig (2006) defines a weak fiducial distribution for ξ as follows. Assume that $\boldsymbol{X}$ has a *structural representation* given by

$$\boldsymbol{X} = G(U, \xi)$$

where U is a uniform $(0,1)$ random variable and G is a jointly measurable function of U and ξ. Let $R(\boldsymbol{x}, u)$ be a set-valued function defined by

$$R(\boldsymbol{x}, u) = \{\xi : \boldsymbol{x} = G(u, \xi)\}$$

The set $\{\xi : \boldsymbol{x} = G(u, \xi)\}$ may be empty, may consist of a single element, or, when the distribution of $\boldsymbol{X}$ is not continuous, may consist of more than one element (possibly uncountably many elements). The function $R(\boldsymbol{X}, U)$ may be viewed as an inverse of the function G. Here G defines u as an implicit function of ξ and $\boldsymbol{x}$ is regarded as fixed. Finally, for any measurable set S, let $V(S)$ denote a random variable with support $\overline{S}$, where $\overline{S}$ is the closure of S. Hannig (2006) defined a weak fiducial distribution of ξ as a conditional distribution of $V(R(\boldsymbol{x}, U^*))$ given $R(\boldsymbol{x}, U^*) \neq \emptyset$. Here $\boldsymbol{x}$ is the observed value of $\boldsymbol{X}$ and U^* is an independent copy of U. Note, without loss of generality, that U could be taken as any random variable or random vector whose distribution is free of unknown parameters, since any such distribution can be generated starting from a uniform $[0,1]$ variate.

In the next subsection we apply the fiducial method to construct a confidence interval for σ_α^2.

3.2 A Fiducial confidence interval for σ_α^2

We first derive a weak fiducial distribution for σ_α^2 and use it to construct confidence intervals for it. We begin with the statistics $Q_i = V_i/r_i$, $i = 1, \ldots, d$ where V_i, r_i are defined in (9). Observe that they are minimal sufficient for $\{\sigma_\alpha^2, \sigma_\varepsilon^2\}$ under the model in (8). When $d = 2$, the relationship between $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ and (Q_1, Q_2) is invertible. This makes fiducial inference for the case $d = 2$ quite straightforward and is not considered here. Hereafter we assume $d > 2$ which is the more general and challenging case. We rewrite the expressions in (10) as follows.

$$Q_1 = \frac{(\lambda_1 \sigma_\alpha^2 + \sigma_\varepsilon^2)\, U_1}{r_1},\ Q_2 = \frac{(\lambda_2 \sigma_\alpha^2 + \sigma_\varepsilon^2)\, U_2}{r_2}, \ldots,\ Q_d = \frac{(\lambda_d \sigma_\alpha^2 + \sigma_\varepsilon^2)\, U_d}{r_d}. \tag{15}$$

Observe that (15) provide a structural representation for the observable random vector $\boldsymbol{Q} = (Q_1, \ldots, Q_d)$ in terms of the random vector $\boldsymbol{U} = (U_1, \ldots, U_d)$ whose distribution is completely known. We denote realized values of Q_i and U_i by q_i and u_i, respectively, for $i = 1, \dots, d$. Solving the first two equations in (15) for σ_α^2 and σ_ε^2 gives

$$\sigma_\alpha^2 = \frac{1}{(\lambda_1 - \lambda_2)} \left(\frac{r_1 Q_1}{U_1} - \frac{r_2 Q_2}{U_2} \right), \quad \sigma_\varepsilon^2 = \frac{1}{(\lambda_1 - \lambda_2)} \left(-\frac{\lambda_2 r_1 Q_1}{U_1} + \frac{\lambda_1 r_2 Q_2}{U_2} \right). \tag{16}$$

Therefore the system of equations in (15) is consistent if and only if the values of σ_α^2 and σ_ε^2 in (16) also satisfy the remaining equations in (15). This requirement leads to the following set of constraints that must be satisfied by $\boldsymbol{q}$ and $\boldsymbol{u}$, the realized values of $\boldsymbol{Q}$ and $\boldsymbol{U}$, respectively.

$$q_j = \frac{u_j}{r_j(\lambda_1 - \lambda_2)} \left(\frac{r_1 q_1 (\lambda_j - \lambda_2)}{u_1} - \frac{r_2 q_2 (\lambda_j - \lambda_1)}{u_2} \right) \text{ for } j = 3, \dots, d.$$

The set-valued function $R(\boldsymbol{q}, \boldsymbol{U}^\star)$ in the fiducial recipe is the set of all σ_α^2, σ_ε^2, with $\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2 > 0$, $i = 1, \dots, d$ for which the equations

$$q_i = \frac{(\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2)\, U_i^\star}{r_i}, \quad i = 1, \dots, d,$$

are satisfied. The set $R(\boldsymbol{q}, \boldsymbol{U}^\star)$ is nonempty if and only if

$$q_j = \frac{U_j^\star}{r_j(\lambda_1 - \lambda_2)} \left(\frac{r_1 q_1 (\lambda_j - \lambda_2)}{U_1^\star} - \frac{r_2 q_2 (\lambda_j - \lambda_1)}{U_2^\star} \right) \text{ for } j = 3, \dots, d, \tag{17}$$

and, when (17) holds, the set $R(\boldsymbol{q}, \boldsymbol{U}^\star)$ consists of the single element

$$\left(\frac{1}{(\lambda_1 - \lambda_2)} \left(\frac{r_1 q_1}{U_1^\star} - \frac{r_2 q_2}{U_2^\star} \right), \frac{1}{(\lambda_1 - \lambda_2)} \left(-\frac{\lambda_2 r_1 q_1}{U_1^\star} + \frac{\lambda_1 r_2 q_2}{U_2^\star} \right) \right).$$

This leads us to define the random variables $W_1, \dots, W_d$ as follows.

$$\begin{aligned}
W_1 &= \frac{1}{(\lambda_1 - \lambda_2)} \left(\frac{r_1 q_1}{U_1^\star} - \frac{r_2 q_2}{U_2^\star} \right), \\
W_2 &= \frac{1}{(\lambda_1 - \lambda_2)} \left(-\frac{\lambda_2 r_1 q_1}{U_1^\star} + \frac{\lambda_1 r_2 q_2}{U_2^\star} \right), \text{ and} \\
W_j &= \frac{U_j^\star}{r_j(\lambda_1 - \lambda_2)} \left(\frac{r_1 q_1 (\lambda_j - \lambda_2)}{U_1^\star} - \frac{r_2 q_2 (\lambda_j - \lambda_1)}{U_2^\star} \right), \qquad j = 3, \dots, d.
\end{aligned}$$

The fiducial distribution of $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ is then the same as the conditional distribution of (W_1, W_2) given $W_3 = q_3, \dots, W_d = q_d$. Routine calculation shows that the joint density of $W_1, \dots, W_d$ is given by

$$\begin{aligned}
f_{\boldsymbol{W}}(w_1, \dots, w_d) \;\; &= \;\; \frac{(\lambda_1 - \lambda_2)(r_1 q_1)^{\frac{r_1}{2}} (r_2 q_2)^{\frac{r_2}{2}}}{2^{\sum_{i=1}^d \frac{r_i}{2}} \left(\prod_{i=1}^d \Gamma\left(\frac{r_i}{2}\right) \right) (\lambda_1 w_1 + w_2)^{\frac{r_1}{2}+1} (\lambda_2 w_1 + w_2)^{\frac{r_2}{2}+1}} \\
&\quad \times \exp\left[-\frac{1}{2} \left(\frac{r_1 q_1}{\lambda_1 w_1 + w_2} + \frac{r_2 q_2}{\lambda_2 w_1 + w_2} + \sum_{i=3}^d \frac{r_i w_i}{\lambda_i w_1 + w_2} \right) \right] \\
&\quad \times \prod_{i=3}^d \frac{r_i^{\frac{r_i}{2}} w_i^{\frac{r_i}{2}-1}}{(\lambda_i w_1 + w_2)^{\frac{r_i}{2}}} \prod_{i=1}^d I_{\{\lambda_i w_1 + w_2 > 0\}}.
\end{aligned}$$

The fiducial distribution of $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ has therefore the density

$$f_{W_1,W_2}(w_1,w_2) = \frac{f_{\boldsymbol{W}}(w_1,w_2,q_3,\ldots,q_d)}{\int\int f_{\boldsymbol{W}}(w_1',w_2',q_3,\ldots,q_d)\,dw_1'\,dw_2'}.$$

It follows that the fiducial distribution of σ_α^2 is given by

$$f_{W_1}(w_1) = \frac{\int f_{\boldsymbol{W}}(w_1,w_2,q_3,\ldots,q_d)\,dw_2}{\int\int f_{\boldsymbol{W}}(w_1',w_2',q_3,\ldots,q_d)\,dw_1'\,dw_2'}.$$

Observe that the fiducial distribution of σ_α^2 is not unique. A different choice of the two equations used for solving for σ_α^2 and σ_ε^2 in (15) will result in different joint density for $W_1, W_2, \ldots, W_d$ and a different fiducial distribution for σ_α^2. The nonuniqueness comes from the fact we are conditioning on a set of measure zero since $P(R(\boldsymbol{q}, \boldsymbol{U}^\star)) = 0$. This is related to the well known Borel's paradox described, for example, in Casella and Berger (2002), Section 4.9.3. We partially resolve this by averaging over all $\binom{d}{2}$ possibilities for choosing the two equations used for solving σ_α^2 and σ_ε^2 to derive the joint density of $W_1, W_2, \ldots, W_d$. Following this approach we get

$$\begin{aligned} f_{\boldsymbol{W}}(w_1,w_2,\ldots,w_d) &= \sum_{i<j} \frac{(\lambda_i-\lambda_j)(r_iq_i)^{\frac{r_i}{2}}(r_jq_j)^{\frac{r_j}{2}}}{\binom{d}{2}2^{\sum_{i=1}^d \frac{r_i}{2}}\left(\prod_{i=1}^d \Gamma(\frac{r_i}{2})\right)(\lambda_iw_1+w_2)^{\frac{r_i}{2}+1}(\lambda_jw_1+w_2)^{\frac{r_j}{2}+1}} \\ &\quad \times \exp\left[-\frac{1}{2}\left(\frac{r_iq_i}{\lambda_iw_1+w_2}+\frac{r_jq_j}{\lambda_jw_1+w_2}+\sum_{k\neq i,\,k\neq j}\frac{r_kw_k}{\lambda_kw_1+w_2}\right)\right] \\ &\quad \times \prod_{k\neq i,\,k\neq j} \frac{r_k^{\frac{r_k}{2}} w_k^{\frac{r_k}{2}-1}}{(\lambda_kw_1+w_2)^{\frac{r_k}{2}}} \prod_{i=1}^d I_{\{\lambda_iw_1+w_2>0\}}. \end{aligned}$$

The resulting fiducial distribution of $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ has the density given by

$$f_{W_1,W_2}(w_1,w_2) = C \cdot g(w_1,w_2) \tag{18}$$

where

$$g(w_1,w_2) = \left(\sum_{i<j}\frac{(\lambda_i-\lambda_j)q_iq_j}{(\lambda_iw_1+w_2)(\lambda_jw_1+w_2)}\right)\left(\frac{\exp(-\frac{1}{2}\sum_{i=1}^d \frac{r_iq_i}{\lambda_iw_1+w_2})}{\prod_{i=1}^d(\lambda_iw_1+w_2)^{\frac{r_i}{2}}}\right)\prod_{i=1}^d I_{\{\lambda_iw_1+w_2>0\}}.$$

and

$$C^{-1} = \int_{-\infty}^0\int_{-\lambda_1w_1}^\infty g(w_1,w_2)dw_2dw_1 + \int_0^\infty\int_{-\lambda_dw_1}^\infty g(w_1,w_2)dw_2dw_1.$$

It follows that the fiducial distribution of σ_α^2 is given by

$$f_{W_1}(w_1) = \begin{cases} C\int_{-\lambda_1w_1}^\infty g(w_1,w_2)\,dw_2 & \text{if } w_1<0 \\ C\int_{-\lambda_dw_1}^\infty g(w_1,w_2)\,dw_2 & \text{otherwise.}\end{cases}$$

Let $\mathcal{R}_{\sigma^2_\alpha,\gamma}$ be the $100\gamma-$percentile of the fiducial distribution of σ^2_α. Then a two-sided $(1-\alpha)100\%$ fiducial confidence interval for σ^2_α is given by

$$\left[\max(0,\mathcal{R}_{\sigma^2_\alpha,\alpha/2}),\max(0,\mathcal{R}_{\sigma^2_\alpha,1-\alpha/2})\right].$$

3.3 A Fiducial confidence interval for σ^2_ε

A fiducial distribution for σ^2_ε can be easily derived from the joint fiducial distribution of $(\sigma^2_\alpha,\ \sigma^2_\varepsilon)$ in (18) and is given by

$$f_{W_2}(w_2)=\begin{cases} C\int_{-w_2/\lambda_d}^{\infty} g(w_1,w_2)\,dw_1 & \text{if } w_2<0 \text{ and } \lambda_d>0\\ C\int_{-w_2/\lambda_1}^{\infty} g(w_1,w_2)\,dw_1 & \text{if } w_2>0\\ 0 & \text{otherwise.}\end{cases}$$

Where C and $g(w_1,w_2)$ are the same as C and $g(w_1,w_2)$ in (18), respectively.

Let $\mathcal{R}_{\sigma^2_\varepsilon,\gamma}$ be the $100\gamma-$percentile of the fiducial distribution of σ^2_ε. Then a two-sided $(1-\alpha)100\%$ fiducial confidence interval for σ^2_ε is given by

$$\left[\max(0,\mathcal{R}_{\sigma^2_\varepsilon,\alpha/2}),\max(0,\mathcal{R}_{\sigma^2_\varepsilon,1-\alpha/2})\right].$$

3.4 A Fiducial confidence interval for ρ

A fiducial distribution for ρ can be easily derived from the joint fiducial distribution of $(\sigma^2_\alpha,\sigma^2_\varepsilon)$ in (18). In fact, we obtain the fiducial density for ρ as the density of $X=W_1/(W_1+W_2)$ given by

$$f_X(x)=\begin{cases} C\int_{-\infty}^{0} g(x,y)\,dy & \text{if } \dfrac{x}{1-x}<-\dfrac{1}{\lambda_d} \text{ and } \lambda_d>0\\ C\int_{0}^{\infty} g(x,y)\,dy & \text{if } \dfrac{x}{1-x}>-\dfrac{1}{\lambda_1}\\ 0 & \text{otherwise}\end{cases}$$

where

$$\begin{aligned} g(x,y) &= \left(\sum_{i<j}\frac{(\lambda_i-\lambda_j)q_iq_j}{((\lambda_i-1)xy+y)\,((\lambda_j-1)xy+y)}\right)\left(\frac{(1-x)^{(\sum_{i=1}^d r_i)/2}|y|}{\prod_{i=1}^d((\lambda_i-1)xy+y)^{\frac{r_i}{2}}}\right)\\ &\quad\times\exp\left(-\frac{1}{2}\sum_{i=1}^{d}\frac{(1-x)r_iq_i}{(\lambda_i-1)xy+y}\right)\prod_{i=1}^{d} I_{\{\frac{(\lambda_i-1)xy+y}{1-x}>0\}},\end{aligned}$$

and

$$C^{-1} = \begin{cases} \int_{-\infty}^{1/(1-\lambda_d)} \int_{-\infty}^{0} g(x,y)\,dy\,dx + \int_{1}^{\infty} \int_{-\infty}^{0} g(x,y)\,dy\,dx + \int_{1/(1-\lambda_1)}^{1} \int_{0}^{\infty} g(x,y)\,dy\,dx, & \text{if } \lambda_d > 1 \\ \int_{1}^{\infty} \int_{-\infty}^{0} g(x,y)\,dy\,dx + \int_{1/(1-\lambda_1)}^{1} \int_{0}^{\infty} g(x,y)\,dy\,dx, & \text{if } \lambda_d = 1 \\ \int_{1}^{1/(1-\lambda_d)} \int_{-\infty}^{0} g(x,y)\,dy\,dx + \int_{-\infty}^{1} \int_{0}^{\infty} g(x,y)\,dy\,dx + \int_{1/(1-\lambda_1)}^{\infty} \int_{0}^{\infty} g(x,y)\,dy\,dx, & \text{if } 0 < \lambda_1 < 1 \\ \int_{1}^{1/(1-\lambda_d)} \int_{-\infty}^{0} g(x,y)\,dy\,dx + \int_{-\infty}^{1} \int_{0}^{\infty} g(x,y)\,dy\,dx, & \text{if } \lambda_1 = 1 \\ \int_{1}^{1/(1-\lambda_d)} \int_{-\infty}^{0} g(x,y)\,dy\,dx + \int_{1/(1-\lambda_1)}^{1} \int_{0}^{\infty} g(x,y)\,dy\,dx, & \text{if } \lambda_1 > 1 \text{ and } 0 \le \lambda_d < 1. \end{cases}$$

Let $\mathcal{R}_{\rho,\gamma}$ be the 100γ-percentile of the fiducial distribution of ρ. Then a two-sided $(1-\alpha)100\%$ fiducial confidence interval for ρ is given by

$$\left[\max(0, \min(\mathcal{R}_{\rho,\alpha/2}, 1)), \max(0, \min(\mathcal{R}_{\rho,1-\alpha/2}, 1))\right]. \tag{19}$$

In the next section we describe the details of a simulation study we conducted to compare the proposed fiducial intervals for σ_α^2, σ_ε^2 and ρ with previously proposed methods.

4 Simulation Study and Discussion

The coverage probability of a confidence interval on σ_α^2 depends on the design (e.g. number of within group measurements, $n_1, \ldots, n_a$) as well as the values of σ_α^2 and σ_ε^2. The degree of imbalance of the design, in the case of a one-way random effects model, has been quantified by Ahrens and Pincus (1981) using the measure Φ defined as

$$\Phi = \frac{a}{\frac{N}{a}\sum_{i=1}^{a}(1/n_i)} = \frac{a\tilde{n}}{N}$$

Note that $0 < \Phi \le 1$ and that Φ equals one if and only if n_i are all equal. The smaller the value of Φ is, the larger is the degree of imbalance. For our simulation study we selected seven different unbalanced patterns shown in Table 2. Patterns 1, 2 and 5 were also considered in Hartung and Knapp (2000). Pattern 4 was also considered in Arendacká (2005). We added the additional patterns 3, 6, and 7 to study the performance of confidence intervals in small sample situations. Without loss of generality, we assumed that $\mu = 0$. The values selected for $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ are $(0.1, 10)$, $(0.5, 10)$, $(1, 10)$, $(0.5, 2)$, $(1, 1)$, $(2, 0.5)$, $(5, 0.2)$, and $(10, 0.1)$, where the settings $(0.1, 10)$, $(0.5, 2)$, $(1, 1)$, $(2, 0.5)$, $(5, 0.2)$ were used by Arendacká (2005). Three more settings were added to our study to better investigate the performance of confidence intervals under extremely large and small values of the ratio $\sigma_\alpha^2/\sigma_\varepsilon^2$.

For each setting of sample sizes n_i and values of $(\sigma_\alpha^2, \sigma_\varepsilon^2)$, 3000 independent data sets were generated and two-sided 95% confidence intervals for σ_α^2 were computed for each method. The methods compared were (a) **BG** interval, (b) **TH** interval, (c) **BE** interval, (d) **HK** interval, (e) **Ar** interval, and (f) **FI** interval. The criteria for judging the performance of the methods are (i) the empirical coverage probabilities and (ii) the average lengths of the confidence intervals. The

Table 2: Unbalanced Patterns Used in the Simulation Study.

Pattern	Φ	a	n_i
1	0.068	6	1 1 1 1 1 100
2	0.130	6	2 2 2 2 2 100
3	0.187	3	2 5 60
4	0.410	5	4 4 4 8 48
5	0.700	6	5 10 15 20 25 30
6	0.807	4	2 2 4 6
7	0.957	6	6 6 8 8 10 10

normal approximation to the binomial distribution suggests that, when the true coverage probability is 95%(90%), then there is less than a 2.5% chance that the empirical coverage based on 3000 simulations will be less than 94.2%(88.9%) and a 2.5% chance that it will exceed 95.8%(91.1%). Therefore, values for the empirical coverages within the binomial confidence limits may be deemed to maintain the declared coverage levels.

The results of our simulation study are shown in Appendix. The results show that **BG** procedure is very liberal when the ratio $\eta = \sigma_\alpha^2/\sigma_\varepsilon^2$ is large. The **TH** procedure is liberal for small values of η and very unbalanced designs. This finding agrees with the findings of Burdick and Eickman (1986). The **BE** procedure is conservative and its behavior for large η is similar to that of the **TH** procedure. The **HK** procedure becomes more conservative as the value of η becomes large. The **Ar** procedure appears to always maintain the stated confidence coefficient. The **FI** interval is conservative when the ratio η is less than 1, but maintains the stated confidence coefficient when η is greater than or equal to 1.

Comparing average interval lengths, we observe that all the intervals behave very similarly except the **BG** interval and the **FI** interval. Although the **BG** interval has small average lengths, it does not adequately maintain the stated coverage probabilities when η is large. Therefore the **BG** interval is not recommended. When compared with procedures other than the **BG** procedure, the **FI** interval always has the smallest average lengths and standard deviations, even when it is conservative. The average lengths of **FI** intervals are 10% to 25% smaller than the average lengths of other intervals, except **BG** interval.

Based on the above results, we recommend the **FI** intervals for σ_α^2 as the most suitable choice for practical applications.

5 Examples

As noted earlier, a fiducial interval for σ_α^2, σ_ε^2 and ρ is available in the general mixed model (1) with two variance components. In this section we give two examples both of which arise from animal breeding studies. The first example uses a model that might be referred to as a *sire model*. The degrees of freedom for error is positive and the eigenvalue λ_d is zero in this example. The second example uses a model that may be referred to as a *full animal model*. All eigenvalues $\lambda_j, j = 1, \ldots, d$, are positive and hence there are no degrees of freedom available for error.

5.1 Sire model

This data set was used in Harville and Fenech (1985) and Burch (1996). The data consist of the birth weight of male lambs which were obtained from five distinct population lines (two control lines and three selection lines). Sixty-two observations were made on progeny of twenty-three rams and each lamb came from a different dam. The age of each dam was recorded as belonging to one of three categories: 1-2 years, 2-3 years, and over 3 years. The fixed effects in this case are population line and age of dam. The random effects are the ram's (additive) genetic effects (within lines) and error (which includes environmental effects).

The mixed linear model we consider is

$$Y_{ijkl} = \mu + \alpha_i + \beta_j + \gamma_{k(j)} + \varepsilon_{ijkl}, \quad i = 1, ..., 3, \quad j = 1, ..., 5, \quad k = 1, ..., 23,$$

where Y_{ijkl} is the birthweight of the l^{th} lamb of the k^{th} ram in the j^{th} population line from a dam belonging to the i^{th} age category. Assume that the ram's genetic effects $\gamma_{k(j)}$ are distributed independently as $N(0, \sigma_\alpha^2)$ and the errors ε_{ijkl} are distributed as $N(0, \sigma_\varepsilon^2)$ independently of each other and of the ram's genetic effects. The quantity μ is the general mean, α_i are fixed effects due to the age group of the dam, and β_j are fixed effects due to the different population lines. The relationship matrix $\boldsymbol{A}$ is $\boldsymbol{I}_{56}$.

The number of distinct eigenvalues of $\boldsymbol{G} = \boldsymbol{H}^T \boldsymbol{Z} \boldsymbol{A} \boldsymbol{Z}^T \boldsymbol{H}$ is $d = 18$. The eigenvalues range in magnitude from $\lambda_1 = 5.087479$ to $\lambda_{18} = 0$. The eigenvalue $\lambda_{18} = 0$ with multiplicity $r_{18} = 37$, $\lambda_8 = 2.0$ with multiplicity $r_8 = 2$, and all remaining eigenvalues have a multiplicity of one. The method of moments (MOM) estimates of σ_α^2 and σ_ε^2 are 0.7676 and 2.7631, respectively. The corresponding estimate of ρ is 0.2174. We refer to this estimate as MOM estimate of ρ. The REML estimates of σ_α^2 and σ_ε^2 are 0.5171 and 2.9616, respectively. The corresponding estimate of ρ is 0.1486. We refer to this estimate as REML estimate of ρ.

The ANOVA table for this data set is shown in Table 3. Figure 1 shows a plot of the fiducial density of σ_α^2.

Table 3: ANOVA Table for the Lamb Birth-weight Data.

Source	df	SS	MS	EMS
Population	4	17.8687	4.4672	σ_ε^2+1.729σ_α^2+Q(population)
Age	2	2.0345	1.0172	σ_ε^2+Q(age)
Ram(Population)	18	80.2978	4.4610	$\sigma_\varepsilon^2 + 2.2119\sigma_\alpha^2$
Residual	37	102.2341	2.7631	σ_ε^2

Note: The terms Q(age) and Q(population) are known functions of the fixed effects parameters.

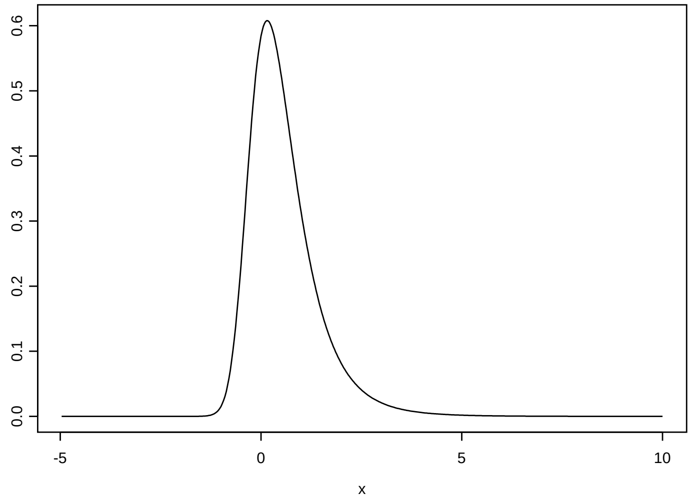

Figure 1: Fiducial density plot for σ_α^2 for the lamb birth-weight data.

Table 4: Nominally 90% and 95% Confidence Intervals on σ_α^2 for the Lamb Birth-weight Data.

Method	90%	95%
Ar	(0, 3.557)	(0, 4.346)
FI	(0, 2.150)	(0, 2.688)

Table 5: Empirical Coverages and Average Lengths (± Standard Deviation) of Nominally 90% and 95% Two-sided Confidence Intervals on σ_α^2 for the lamb birth-weight Data Using MOM Estimates and REML Estimates of σ_α^2 and σ_ε^2 as their True Values, respectively (based on 5000 simulations).

Method	90%		95%	
	MOM	REML	MOM	REML
Ar	0.899	0.898	0.948	0.946
	2.779±1.355	2.461±1.341	3.469±1.655	3.089±1.614
FI	0.903	0.907	0.953	0.959
	2.228±1.075	1.921±1.024	2.782±1.298	2.418±1.220

Note that the support of the fiducial density for σ_α^2, σ_ε^2 and ρ might be a proper superset of their natural boundaries. When calculating fiducial confidence intervals, we replace negative confidence bounds with 0 and when a confidence bound for ρ happens to be bigger than 1 we replace it with 1. Table 4 shows the **Ar** and the **FI** confidence intervals for σ_α^2 with 90% and 95% nominal confidence coefficients. Simulated empirical coverages associated with the nominally 90% and 95% confidence intervals for σ_α^2, along with their average lengths, using MOM and REML estimates of σ_α^2 and σ_ε^2 as their true values, respectively, are shown in Table 5. The results show that the **FI** method gives shorter confidence intervals for this data set. Comparing the average lengths of the intervals, the **FI** confidence interval has smaller average lengths, despite being more conservative than the **Ar** confidence interval. In summary, the **FI** procedure performs better than the **Ar** method for this lamb birth-weight data set.

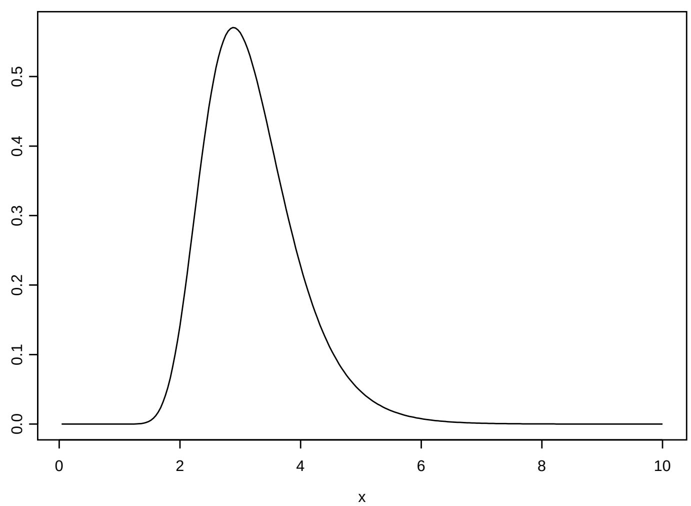

Figure 2: Fiducial density plot for σ_ε^2 for the lamb birth-weight data.

Table 6: Nominally 90% and 95% Confidence Intervals on σ_ε^2 for the Lamb Birth-weight Data.

Method	90%	95%
EX	(1.959, 4.246)	(1.836, 4.625)
FI	(2.135, 4.633)	(1.996, 5.023)

Figure 2 shows a plot of the fiducial density of σ_ε^2. The support of this density is $(0, \infty)$. Table 6 shows the **EX** and the **FI** confidence intervals for σ_ε^2 with 90% and 95% nominal confidence coefficients. Table 7 shows simulated empirical coverages associated with the nominally 90% and

Table 7: Empirical Coverages and Average Lengths (± Standard Deviation) of Nominally 90% and 95% Two-sided Confidence Intervals on σ_ε^2 for the Lamb Birth-weight Data Using MOM Estimates and REML Estimates of σ_α^2 and σ_ε^2 as their True Values, respectively (based on 5000 simulations).

Method	90%		95%	
	MOM	REML	MOM	REML
EX	0.899	0.892	0.949	0.949
	2.300±0.541	2.450±0.573	2.802±0.654	2.978±0.688
FI	0.900	0.902	0.948	0.949
	2.237±0.472	2.349±0.488	2.713±0.571	2.847±0.590

95% confidence intervals for σ_ε^2, along with their average lengths, using MOM and REML estimates of σ_α^2 and σ_ε^2 as their true values, respectively. The results show that the fiducial interval has smaller average length, although **FI** method gives a slightly wider confidence interval for this data set.

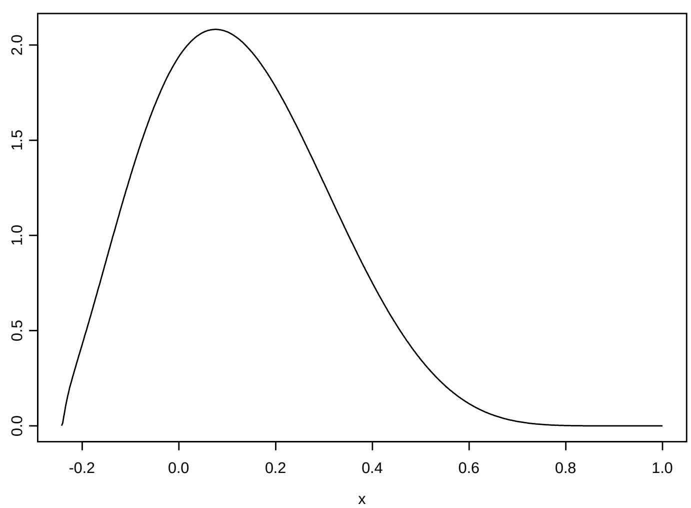

Figure 3: Fiducial density plot for ρ for the lamb birth-weight data.

Figure 3 shows a plot of the fiducial density of ρ for this data set. Note that the range of ρ is $(1/(1-\lambda_1), 1), i.e., (-0.2446, 1)$. Observe that there does not exist an unbiased **BI** confidence interval for ρ. In this case, we take $k = 17$ in the **BI** procedure which gives us the pivotal quantity having the closest "balance" between the numerator and the denominator degrees of freedom where $r_{18} = 37$ and $\sum_{i=1}^{17} r_i = 18$.

Table 8: Nominally 90% and 95% Confidence Intervals on ρ for the Lamb Birth-weight Data.

Method	90%	95%
BI	(0, 0.592)	(0, 0.643)
FI	(0, 0.451)	(0, 0.512)

Table 9: Empirical Coverages and Average Lengths ($\pm$ Standard Deviation) of the Nominally 90% and 95% Two-sided Confidence Intervals on ρ for the Lamb Birth-weight Data Using MOM Estimates and REML Estimates of σ_α^2,σ_ε^2 and ρ as their True Values, respectively (based on 5000 simulations).

Method	90%		95%	
	MOM	REML	MOM	REML
BI	0.900	0.900	0.951	0.951
	0.471±0.125	0.436±0.145	0.538±0.128	0.501±0.146
FI	0.909	0.919	0.962	0.965
	0.428±0.121	0.389±0.133	0.495±0.123	0.451±0.135

Table 8 shows the **FI** confidence interval and the **BI** confidence interval for ρ with 90% and 95% nominal confidence coefficients. Table 9 shows empirical coverages corresponding to these intervals along with their average lengths. These simulations are conducted with the MOM and REML estimates of σ_α^2, σ_ε^2 and ρ, respectively, as their true values. The results show that the **FI** method gives a shorter confidence interval for ρ in this data set. Comparing the average lengths of the intervals, the **FI** confidence interval has a smaller average length although it is more conservative than the **BI** confidence interval. In summary, the **FI** procedure performs better than the **BI** method for this lamb birth-weight data set.

5.2 Full animal model

This data was used in Burch (1996) and Burch and Iyer (1997). Data were obtained on one hundred and seventy-one yearling bulls from a Red Angus seed stock in Montana. A trait of interest was the loin eye (i.e., ribeye) muscle area measured in square inches. Ultrasound techniques were used to obtain these measurements. The fixed effect was age of dam, which belongs to one of five categories: 2 years, 3 years, 4 years, 5-9 years, and 10 or more years. The random effects are animal's (additive) genetic effect and error. The mixed linear model being considered can be represented by

$$\boldsymbol{Y} = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{Z}\boldsymbol{u} + \boldsymbol{\varepsilon},$$

where $\boldsymbol{Y}$ is a 171×1 vector of observable random variables, $\boldsymbol{X}$ is a 171×5 design matrix, $\boldsymbol{\beta}$ is a 5×1 vector of unknown parameters, $\boldsymbol{Z} = \boldsymbol{I}_{171}$, and $\boldsymbol{u}$ and $\boldsymbol{\varepsilon}$ are vectors of unobservable random variables of size 171×1. The relationship matrix $\mathbf{A}$ was determined using a recursive method given in Henderson (1976). This means $Var(\boldsymbol{u}) = \sigma_\alpha^2 \boldsymbol{A}$.

The number of distinct eigenvalues of $\boldsymbol{G} = \boldsymbol{H}^T \boldsymbol{Z} \boldsymbol{A} \boldsymbol{Z}^T \boldsymbol{H}$ is $d = 165$. Eigenvalues range in magnitude from $\lambda_1 = 8.5692472$ to $\lambda_{165} = 0.5656916$. Except for $\lambda_{105} = 0.6718750$ having $r_{105} = 2$, all eigenvalues have a multiplicity of one. The REML estimates of σ_α^2 and σ_ε^2 are 0.2994 and 2.6539, respectively. The corresponding estimate of ρ is 0.1014. We refer to this estimate as REML estimate of ρ.

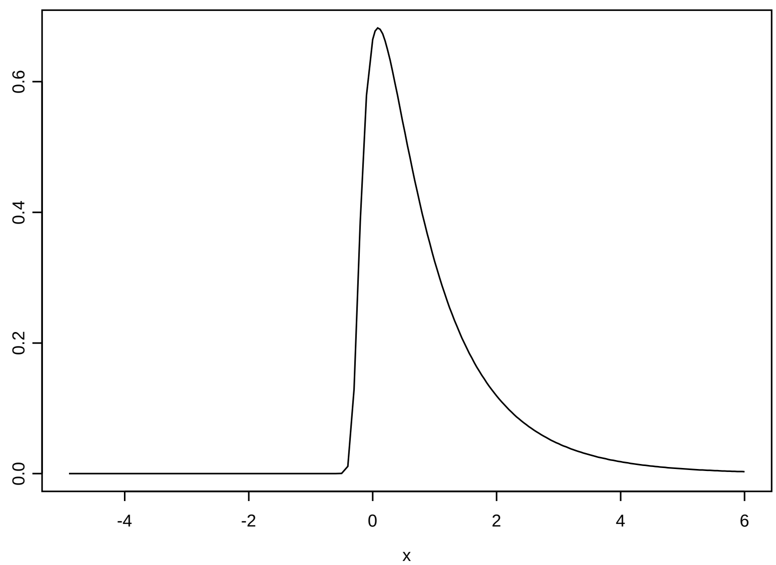

Figure 4: Fiducial density plot for σ_α^2 for the loin-eye data.

Table 10: Nominally 90% and 95% Confidence Intervals on σ_α^2 for the Loin-eye Data.

Method	90%	95%
FI	(0, 3.000)	(0, 3.750)

Figure 4 is a plot of the fiducial density of σ_α^2. The support of this density is $(-\infty, \infty)$. Table 10 shows the **FI** confidence intervals for σ_α^2 with both 90% and 95% nominal confidence coefficients. Table 11 shows simulated empirical coverages corresponding to the nominally 90% and

Table 11: Simulated Empirical Coverages for the Nominally 90% and 95% Two-sided Confidence Intervals on σ_α^2 for the Loin-eye Data Using REML Estimates of σ_α^2 and σ_ε^2 as their True Values (based on 2000 simulations).

Method	90%	95%
FI	0.935	0.975

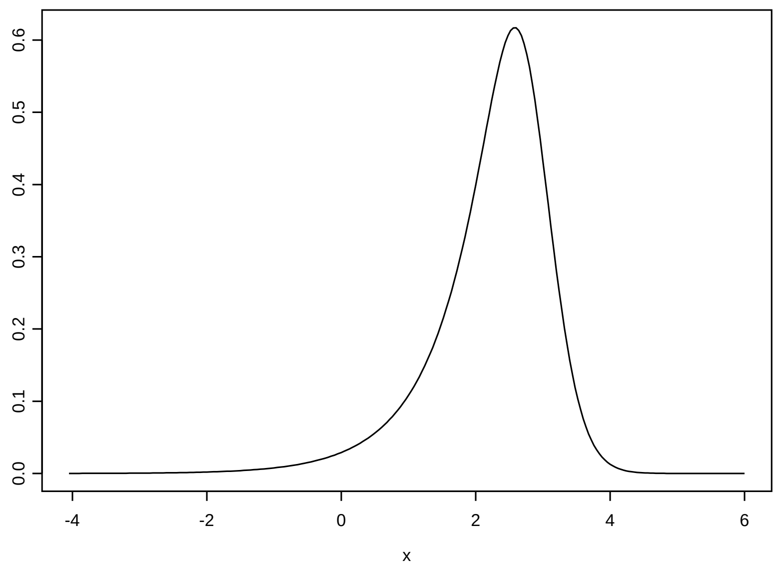

Figure 5: Fiducial density plot for σ_ε^2 for the loin-eye data.

95% confidence intervals on σ_α^2 using REML estimates of σ_α^2 and σ_ε^2 as their true values.

The fiducial density plot of σ_ε^2 is shown in Figure 5. The range of σ_ε^2 is $(-\infty, \infty)$. Table 12 shows the **FI** confidence intervals for σ_ε^2 with 90% and 95% nominal confidence coefficients. Table 13 shows simulated empirical coverages corresponding to the nominally 90% and 95% confidence intervals on

Table 12: Nominally 90% and 95% Confidence Intervals on σ_ε^2 for the Loin-eye Data.

Method	90%	95%
FI	(0.625, 3.341)	(0.100, 3.513)

Table 13: Simulated Empirical Coverages for the Nominally 90% and 95% Two-sided Confidence Intervals on σ_ε^2 for the Loin-eye Data Using REML Estimates of σ_α^2 and σ_ε^2 as their True Values (based on 2000 simulations).

Method	90%	95%
FI	0.923	0.959

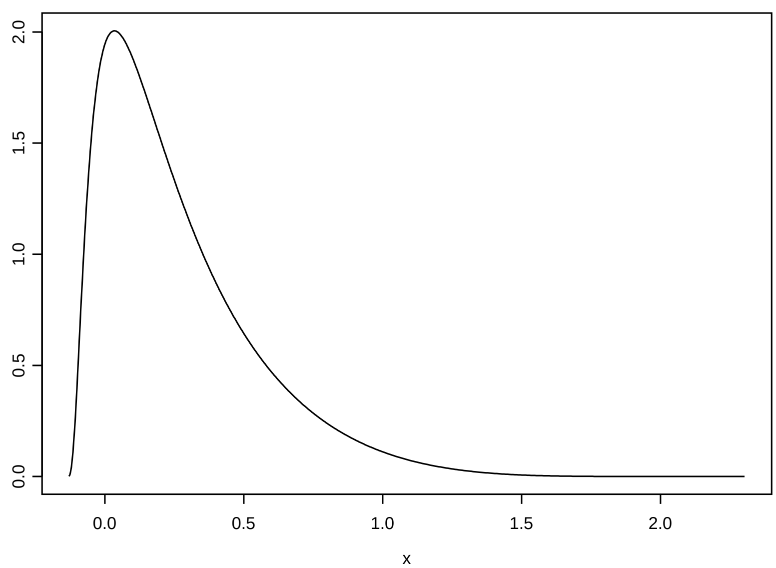

Figure 6: Fiducial density plot for ρ for the loin-eye data.

σ_ε^2 using REML estimates of σ_α^2 and σ_ε^2 as their true values.

Figure 6 shows the plot of the fiducial density for ρ for this data set. The support of this density is

$$\left\{\rho : \rho \in \left(\frac{1}{1-\lambda_1}, 1\right) \cup \left(1, \frac{1}{1-\lambda_d}\right)\right\},$$

i.e., $\left\{\rho : \rho \in (-0.1321, 1) \cup (1, 2.3025)\right\}$. The **BI** pivotal quantity that results in an unbiased confidence interval corresponds to $k = 83$. In this case, $\sum_{i=1}^{83} r_i = \sum_{j=84}^{165} r_j = 83$. We will use this unbiased confidence interval as the **BI** confidence interval in the following discussion. It is interesting to note that the **BI** confidence interval covers the entire parameter space. Inverting the pivotal quantity in (14) results in a confidence interval whose endpoints fall outside of the parameter space.

Harville and Fenech (1985) attribute this to lack of sufficient information in the data about the parameter of interest in such cases.

Table 14 shows the **FI** confidence interval and the **BI** confidence interval for ρ with 90% and 95% nominal confidence coefficients. Table 15 shows the empirical coverages of these interval procedures for ρ using REML estimates of σ_α^2, σ_ε^2 and ρ as their true values, respectively. The results show that the **FI** method leads to a shorter confidence interval for ρ in this data set. Comparing the empirical coverages, the **FI** confidence interval is more conservative than the **BI** confidence interval. In summary, the **FI** method performs better than the **BI** method for this loin-eye data set.

Table 14: Nominally 90% and 95% Confidence Intervals on ρ for the Loin-eye Data.

Method	90%	95%
BI	(0, 1.000)	(0, 1.000)
FI	(0, 0.824)	(0, 0.972)

Table 15: Empirical Coverages of the Nominally 90% and 95% Two-sided Confidence Intervals on ρ for the Loin-eye Data Using REML Estimates of σ_α^2, σ_ε^2 and ρ as their True Values (based on 2000 simulations).

Method	90%	95%
BI	0.900	0.951
FI	0.939	0.977

6 Asymptotic exactness of fiducial intervals

When deriving the **FI** procedure we considered the following structural equations for σ_α^2 and σ_ε^2.

$$Q_1 = \frac{(\lambda_1\sigma_\alpha^2 + \sigma_\varepsilon^2)U_1}{r_1}, \quad Q_2 = \frac{(\lambda_2\sigma_\alpha^2 + \sigma_\varepsilon^2)U_2}{r_2}, \ldots, \quad Q_d = \frac{(\lambda_d\sigma_\alpha^2 + \sigma_\varepsilon^2)U_d}{r_d}$$

Here λ_i is an eigenvalue of $\boldsymbol{G}$ in (8), r_i its multiplicity and U_i are independent random variables with χ^2-distribution on r_i degrees of freedom, respectively, for $i = 1, \ldots d$. The fiducial distribution based on this structural equation was then derived to be a constant multiple of

$$g(w_1, w_2) = \left(\sum_{i<j} \frac{(\lambda_i - \lambda_j)q_i q_j}{(\lambda_i w_1 + w_2)(\lambda_j w_1 + w_2)}\right) \left(\frac{\exp(-\frac{1}{2}\sum_{i=1}^d \frac{r_i q_i}{\lambda_i w_1 + w_2})}{\prod_{i=1}^d (\lambda_i w_1 + w_2)^{\frac{r_i}{2}}}\right) \prod_{i=1}^d I_{\{\lambda_i w_1 + w_2 > 0\}}. \tag{20}$$

Hannig (2006) has proved that this fiducial distribution leads to asymptotically correct frequentist inference if $d = 3$ and $r_i \to \infty$. His method can be easily generalized to any d fixed. However, this is not sufficient for many applications, where we have a large number of different eigenvalues with multiplicities that are relatively small, such as the loin-eye data set discussed above. We now prove that under some natural conditions related to the Fisher's information the fiducial distribution leads to asymptotically correct frequentist inference even when $d \to \infty$.

Theorem 1. *Denote $n = \sum_{j=1}^{d} r_j$ and assume that the limits*

$$\lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{d} \frac{\lambda_j^k r_j}{(\lambda_j \sigma_\alpha^2 + \sigma_\varepsilon^2)^2} = m_k \quad \textit{for} \quad k = 0, 1, 2$$

are such that the matrix

$$\Sigma = \begin{pmatrix} m_0 & m_1 \\ m_1 & m_2 \end{pmatrix}$$

is positive definite. Then the frequentist coverage probability of the $(1-\alpha)$ equal tailed fiducial interval based on the joint fiducial distribution of $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ approaches the stated value as $n \to \infty$.

Proof. We will use the ideas contained in the proof of Theorem 1 of Hannig (2006). Define the random vectors

$$\mathbb{S} = \left(\sum_{i=1}^{d} \frac{r_i Q_i}{(\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2)^2}, \sum_{i=1}^{d} \frac{\lambda_i r_i Q_i}{(\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2)^2} \right),$$

$$\mathbf{t} = \left(\sum_{i=1}^{d} \frac{r_i}{\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2}, \sum_{i=1}^{d} \frac{\lambda_i r_i}{\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2} \right).$$

We will show that $(\mathbb{S} - \mathbf{t})/\sqrt{n}$ converges in distribution to a normal random vector.

To this end assume without loss of generality that $r_i = 1$ for all i, possibly repeating some eigenvalues several times. We can then write

$$\mathbb{S} - \mathbf{t} = \left(\sum_{i=1}^{n} \frac{(U_i - 1)}{\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2}, \sum_{i=1}^{n} \frac{\lambda_i (U_i - 1)}{\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2} \right),$$

where U_i are i.i.d. chi-squared random variables with one degree of freedom. To prove the convergence we will use the Cramèr-Wold device. Fix a, b and denote

$$c = \max_{j=1,\dots,n} \frac{(a + b\lambda_j)^2}{(\lambda_j \sigma_\alpha^2 + \sigma_\varepsilon^2)^2}.$$

By our assumptions $c/n \to 0$. Next, we verify the Lindeberg-Feller condition

$$\begin{aligned}
\lim_{n\to\infty} \sum_{i=1}^{n} E & \left[\frac{(a + b\lambda_i)^2 \, (U_i - 1)^2}{n(\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2)^2}; \ \sum_{j=1}^{n} \frac{(a + b\lambda_j)^2}{n(\lambda_j \sigma_\alpha^2 + \sigma_\varepsilon^2)^2} \varepsilon < \frac{(a + b\lambda_i)^2 (U_i - 1)^2}{n(\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2)^2} \right] \\
& \le \lim_{n\to\infty} E \left[c(U_i - 1)^2; \ (a^2 m_0 + 2ab m_1 + b^2 m_2) \frac{\varepsilon}{2} < c(U_i - 1)^2 \right] \\
& = 0.
\end{aligned}$$

Thus we conclude $(\mathbb{S}-\mathbf{t})/\sqrt{n} \xrightarrow{\mathcal{D}} \mathbf{H} = (H_1, H_2) \sim N(0, 2\Sigma)$. By Skorokhod's representation theorem (Billingsley, 1995) this convergence can be taken a.s. The a.s. convergence is assumed in the rest of the proof.

We will now investigate the distribution of $\sqrt{n}(\mathcal{R}_{(\sigma_\alpha^2,\sigma_\varepsilon^2)} - (\sigma_\alpha^2, \sigma_\varepsilon^2))$. The density of this random variable is a constant multiple of $r(z_1, z_2) = g(\sigma_\alpha^2 + z_1/\sqrt{n}, \sigma_\varepsilon^2 + z_2/\sqrt{n})$ where g is defined in (20). For future reference denote this constant C_n, i.e., the density is $C_n^{-1} r(z_1, z_2)$. Set $w_1 = \sigma_\alpha^2 + z_1/\sqrt{n}$ and $w_2 = \sigma_\varepsilon^2 + z_2/\sqrt{n}$ and consider

$$\begin{aligned}\log r(z_1, z_2) \quad &= \quad -\frac{1}{2}\sum_{i=1}^{d} r_i \left(\frac{q_i}{\lambda_i w_1 + w_2} + \log(\lambda_i w_1 + w_2)\right) \\ &\quad + \log\left(\binom{d}{2}^{-1} \sum_{i<j} \frac{(\lambda_i - \lambda_j) q_i q_j}{(\lambda_i w_1 + w_2)(\lambda_j w_1 + w_2)}\right). \end{aligned} \tag{21}$$

Applying Taylor series to each term of the the first sum in (21) we get

$$\begin{aligned}\sum_{i=1}^{d} r_i \left(\frac{q_i}{\lambda_i w_1 + w_2} + \log(\lambda_i w_1 + w_2)\right) &= -n^{-1/2} z_1 \sum_{i=1}^{d}\left(\frac{r_i q_i}{(\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2)^2} - \frac{r_i}{\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2}\right) \\ & - n^{-1/2} z_2 \sum_{i=1}^{d}\left(\frac{\lambda_i r_i q_i}{(\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2)^2} - \frac{\lambda_i r_i}{\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2}\right) + n^{-1} z_1^2 \sum_{i=1}^{d}\left(\frac{r_i q_i}{(\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2)^3} - \frac{r_i}{2(\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2)^2}\right) \\ & + n^{-1} 2 z_1 z_2 \sum_{i=1}^{d}\left(\frac{\lambda_i r_i q_i}{(\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2)^3} - \frac{\lambda_i r_i}{2(\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2)^2}\right) + n^{-1} z_2^2 \sum_{i=1}^{d}\left(\frac{\lambda_i^2 r_i q_i}{(\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2)^3} - \frac{\lambda_i^2 r_i}{2(\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2)^2}\right) \\ & + \sum_{i=1}^{d} r_i \left(\frac{q_i}{\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2} + \log(\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2)\right) + o_{as}(1). \end{aligned} \tag{22}$$

As noted above, the first two terms on the right hand side of (22) converge a.s. as $n \to \infty$ to $-z_1 H_1 - z_2 H_2$. By Slutsky's theorem the next three terms converge a.s. to $z_1^2 m_0 + 2z_1 z_2 m_1 + z_2^2 m_2$. Similarly set

$$L_n = \binom{d}{2}^{-1} \sum_{i<j} \frac{(\lambda_i - \lambda_j) q_i q_j}{(\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2)(\lambda_j\sigma_\alpha^2 + \sigma_\varepsilon^2)}$$

and notice

$$\log\left(\binom{d}{2}^{-1} \sum_{i<j} \frac{(\lambda_i - \lambda_j) q_i q_j}{(\lambda_i w_1 + w_2)(\lambda_j w_1 + w_2)}\right) - \log(L_n) \to 0 \quad a.s.$$

Define

$$K_n = \frac{\exp\left(\frac{1}{2}\sum_{i=1}^{d} r_i \left(\frac{q_i}{\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2} + \log(\lambda_i\sigma_\alpha^2 + \sigma_\varepsilon^2)\right) - \frac{1}{4}\mathbf{H}^{\mathrm{T}}\Sigma^{-1}\mathbf{H}\right)}{2\pi L_n \sqrt{\det(2\Sigma^{-1})}}$$

and notice that

$$\begin{aligned} h(z_1, z_2) &= \lim_{n\to\infty} K_n r(z_1, z_2) \\ &= K \exp\left\{-\frac{1}{4}\left(z_1^2 m_0 + 2z_1 z_2 m_1 + z_2^2 m_2 - 2z_1 H_1 - 2z_2 H_2\right)\right\} \quad a.s. \end{aligned}$$

Here the constant K is chosen so that $h(z_1, z_2)$ integrates to one. Notice that conditionally on $\mathbf{H}$, $h(z_1, z_2)$ is a density of a multivariate normal distribution $N(\Sigma^{-1}\mathbf{H}, 2\Sigma^{-1})$. We also notice that unconditionally $\Sigma^{-1}\mathbf{H} \sim N(0, 2\Sigma^{-1})$.

Recall that the density of $\sqrt{n}(\mathcal{R}_{(\sigma_\alpha^2,\sigma_\varepsilon^2)} - (\sigma_\alpha^2, \sigma_\varepsilon^2))$ is $C_n^{-1} r(z_1, z_2)$. Furthermore, the functions $\sqrt{\det(2\Sigma^{-1})} K_n r(\sqrt{2}\Sigma^{-1/2}\mathbf{z} + \Sigma^{-1}\mathbf{H})$ are dominated by $C(1 + z_1^2 + z_2^2)^{-1}$ for C large enough. Thus the Lebesgue dominated convergence theorem, the fact that densities integrate to one, and Fatou's lemma imply that $K_n C_n \to 1$. We conclude that the density $C_n^{-1} r(z_1, z_2)$ converges to the density of $N(\Sigma^{-1}\mathbf{H}, 2\Sigma^{-1})$.

This verifies the crucial assumption 1.2 of Hannig (2006). Also, the equal tailed region satisfies the assumption 1.3 of Hannig (2006). The rest of the proof is identical to the proof of Theorem 1 in Hannig (2006). □

Remark 1. It is worth noting that the Fisher information matrix $\mathcal{F}$ for $(\sigma_\alpha^2, \sigma_\varepsilon^2)$ based on Q_i, $i = 1, \ldots, d$, is the 2 by 2 matrix whose (j, k) element is given by

$$\sum_{i=1}^{d} \frac{r_i \lambda_i^{j+k-2}}{2(\lambda_i \sigma_\alpha^2 + \sigma_\varepsilon^2)^2}$$

for $j, k = 1, 2$. Hence the conditions of the theorem is a statement of the requirement that $\frac{1}{n}\mathcal{F}$ converge to a positive definite matrix as $n \to \infty$.

7 Closing Remarks

In this paper, we proposed confidence interval procedures for σ_α^2, σ_ε^2 and ρ in a two component mixed effects linear model using the fiducial approach. A simulation study was carried out to compare the proposed confidence interval for σ_α^2 with five other confidence intervals from the literature, the proposed confidence interval for σ_ε^2 with exact confidence interval, and the proposed confidence interval for ρ with the method due to Burch and Iyer (1997). The results of a simulation study show that the proposed fiducial intervals for σ_α^2 are satisfactory in terms of coverage probability. Although they are conservative for small values of the variance ratio $\eta = \sigma_\alpha^2/\sigma_\varepsilon^2$, they have the smallest average interval lengths among all confidence intervals. Two examples in the context of animal breeding applications are given to illustrate the use of the proposed procedures. The results confirm that the fiducial intervals can be recommended for practical use over the methods previously discussed in the literature.

Fiducial methods have a rich history beginning from the 1930s. While these methods failed to become part of mainstream statistics, recently there has been a resurgence of these methods either under the label of fiducial inference or as generalized inference. Details of connections between fiducial inference and generalized inference may be found in Hannig et. al (2006). An important point to note is that, because the fiducial method is very generally applicable, the methods discussed in this paper, in principle, can be straightforwardly extended to more complex mixed linear models with multiple variance components. However, computational details need to be sorted out. This is ongoing research.

A Numerical Results from the Simulation Study

Table 16: Empirical Coverages of Nominally 95% Two-sided Confidence Intervals on σ_α^2

Design	Method	$(\sigma_\alpha^2, \sigma_\varepsilon^2)$							
		(0.1,10)	(0.5,10)	(1,10)	(0.5,2)	(1,1)	(2,0.5)	(5,0.2)	(10,0.1)
1	BG	0.949	0.947	0.949	0.945	0.926	0.907	0.896	0.895
	TH	0.937	0.936	0.941	0.945	0.947	0.950	0.951	0.950
	BE	0.985	0.984	0.981	0.979	0.959	0.952	0.951	0.950
	HK	0.950	0.949	0.951	0.954	0.956	0.958	0.960	0.958
	Ar	0.947	0.944	0.954	0.947	0.949	0.951	0.950	0.956
	FI	0.986	0.982	0.987	0.988	0.982	0.944	0.948	0.949
2	BG	0.949	0.947	0.946	0.938	0.917	0.905	0.898	0.898
	TH	0.937	0.938	0.943	0.948	0.948	0.950	0.952	0.950
	BE	0.986	0.982	0.978	0.971	0.954	0.951	0.952	0.950
	HK	0.951	0.951	0.954	0.957	0.957	0.958	0.960	0.957
	Ar	0.947	0.950	0.950	0.947	0.953	0.947	0.954	0.954
	FI	0.985	0.985	0.987	0.986	0.957	0.946	0.951	0.947
3	BG	0.948	0.952	0.949	0.940	0.937	0.934	0.931	0.928
	TH	0.931	0.940	0.941	0.943	0.951	0.953	0.950	0.948
	BE	0.991	0.983	0.976	0.966	0.955	0.953	0.950	0.948
	HK	0.949	0.949	0.954	0.959	0.950	0.958	0.954	0.950
	Ar	0.948	0.953	0.950	0.951	0.953	0.946	0.949	0.944
	FI	0.988	0.981	0.986	0.985	0.980	0.961	0.958	0.954

Table 17: Empirical Coverages of Nominally 95% Two-sided Confidence Intervals on σ_α^2 (continued)

Design	Method	$(\sigma_\alpha^2, \sigma_\varepsilon^2)$							
		(0.1,10)	(0.5,10)	(1,10)	(0.5,2)	(1,1)	(2,0.5)	(5,0.2)	(10,0.1)
4	BG	0.952	0.949	0.940	0.938	0.926	0.921	0.922	0.922
	TH	0.938	0.943	0.945	0.947	0.952	0.951	0.956	0.954
	BE	0.991	0.981	0.971	0.962	0.954	0.951	0.956	0.954
	HK	0.950	0.960	0.958	0.958	0.958	0.958	0.958	0.957
	Ar	0.950	0.952	0.947	0.947	0.952	0.950	0.950	0.952
	FI	0.987	0.987	0.986	0.985	0.950	0.949	0.947	0.948
5	BG	0.951	0.949	0.943	0.941	0.936	0.932	0.929	0.935
	TH	0.931	0.945	0.948	0.953	0.949	0.950	0.952	0.952
	BE	0.990	0.982	0.975	0.965	0.951	0.950	0.952	0.952
	HK	0.955	0.953	0.958	0.958	0.957	0.959	0.960	0.958
	Ar	0.952	0.949	0.947	0.946	0.947	0.947	0.954	0.953
	FI	0.990	0.976	0.965	0.950	0.946	0.946	0.946	0.950
6	BG	0.947	0.949	0.948	0.949	0.948	0.943	0.944	0.938
	TH	0.944	0.949	0.951	0.955	0.960	0.953	0.953	0.947
	BE	0.977	0.976	0.971	0.969	0.964	0.953	0.953	0.947
	HK	0.947	0.958	0.961	0.974	0.971	0.972	0.974	0.973
	Ar	0.955	0.947	0.944	0.953	0.950	0.947	0.951	0.945
	FI	0.990	0.989	0.991	0.989	0.976	0.955	0.947	0.950
7	BG	0.950	0.951	0.950	0.954	0.948	0.952	0.948	0.948
	TH	0.947	0.953	0.954	0.955	0.952	0.954	0.951	0.950
	BE	0.975	0.969	0.965	0.960	0.952	0.954	0.951	0.950
	HK	0.954	0.955	0.961	0.962	0.963	0.963	0.966	0.962
	Ar	0.951	0.948	0.953	0.951	0.950	0.953	0.951	0.949
	FI	0.973	0.971	0.966	0.953	0.949	0.957	0.957	0.947

Table 18: Average Lengths (± Standard Deviation) of Nominally 95% Two-sided Confidence Intervals on σ_α^2

Design	Method	$(\sigma_\alpha^2, \sigma_\varepsilon^2)$							
		(0.1,10)	(0.5,10)	(1,10)	(0.5,2)	(1,1)	(2,0.5)	(5,0.2)	(10,0.1)
1	BG	26.0±19.6	28.7±21.3	31.6±23.4	8.1±5.9	8.3±6.5	12.6±10.4	28.4±24.5	57.1±50.8
	TH	42.1±34.0	44.8±35.6	47.8±37.5	11.3±8.3	10.0±6.8	13.6± 8.7	28.7±18.1	57.2±36.0
	BE	43.7±33.3	46.4±35.0	49.3±36.9	11.6±8.2	10.1±6.8	13.7± 8.6	28.8±18.1	57.2±36.0
	HK	41.6±33.8	44.3±35.5	47.6±36.9	11.3±8.6	10.1±7.0	13.7± 8.7	29.4±18.1	57.1±35.9
	Ar	41.5±33.4	44.0±36.1	47.4±36.6	11.2±8.5	9.9±6.8	13.6± 8.6	29.1±18.8	56.3±35.4
	FI	31.0±22.6	32.9±25.0	34.5±25.5	8.6±6.2	7.9±5.6	11.0± 7.3	25.2±16.0	48.5±31.1
2	BG	13.7±10.2	16.1±11.9	19.1±14.1	5.6±4.1	7.0±5.6	12.0±10.0	28.1±24.1	56.8±49.8
	TH	21.4±17.2	23.9±18.6	26.9±20.5	7.1±5.0	7.9±5.1	12.5±7.9	28.3±17.8	56.9±35.8
	BE	22.2±16.8	24.7±18.3	27.6±20.2	7.2±5.0	8.0±5.1	12.5±7.9	28.3±17.8	56.9±35.8
	HK	21.1±17.0	23.7±18.6	26.9±20.1	7.2±5.2	7.9±5.3	12.5±7.9	28.9±17.8	56.8±35.8
	Ar	21.2±16.9	23.5±18.3	26.6±20.3	7.1±5.1	7.8±5.1	12.4±8.0	28.8±18.0	56.0±35.2
	FI	16.0±11.6	20.1±15.3	20.5±14.9	5.5±4.0	5.6±3.5	10.6±6.8	24.8±15.6	49.0±31.9
3	BG	63.6± 66.0	80.0± 83.7	98.8±101.3	32.0±34.5	46.2±52.2	81.1±93.1	194.3±226.2	388.7±457.9
	TH	95.3±113.6	111.9±130.4	131.4±145.4	38.6±40.1	49.0±49.4	83.1±83.1	196.2±199.0	388.8±392.3
	BE	95.5±113.6	112.1±130.5	131.6±145.5	38.7±40.1	49.1±49.4	83.2±83.0	196.2±199.0	388.8±392.3
	HK	93.8±110.3	111.8±130.7	131.4±153.1	38.0±39.5	48.3±48.1	83.2±83.5	199.5±201.7	403.2±392.2
	Ar	93.5±105.6	111.9±126.4	130.1±142.7	37.9±40.7	47.8±48.3	82.2±82.0	196.5±195.8	398.8±402.0
	FI	73.5± 70.2	87.1± 84.7	99.5± 90.1	29.1±28.9	35.8±37.1	62.4±67.3	149.6±150.1	296.4±298.3
4	BG	9.6± 7.7	12.9±10.1	17.3±13.6	5.9±4.6	8.8±7.4	16.2±13.9	39.6±34.2	78.4±69.3
	TH	13.6±11.8	16.9±13.7	21.5±16.8	6.7±5.0	9.3±6.7	16.5±11.6	39.7±27.7	78.1±55.3
	BE	13.9±11.7	17.2±13.6	21.7±16.8	6.8±5.0	9.3±6.7	16.5±11.6	39.7±27.7	78.1±55.3
	HK	13.7±12.0	16.8±13.8	21.0±16.3	6.7±4.9	9.5±6.6	17.1±12.2	40.6±28.7	79.7±56.8
	Ar	13.8±11.9	17.1±14.0	21.2±16.7	6.6±5.0	9.1±6.4	16.7±11.7	39.9±27.6	77.9±54.3
	FI	10.2± 7.4	12.8± 9.8	15.5±11.7	5.0±3.8	7.2±5.4	12.7± 9.0	32.0±22.7	62.9±44.8

Table 19: Average Lengths (± Standard Deviation) of Nominally 95% Two-sided Confidence Intervals on σ_α^2 (continued)

Design	Method	$(\sigma_\alpha^2, \sigma_\varepsilon^2)$							
		(0.1,10)	(0.5,10)	(1,10)	(0.5,2)	(1,1)	(2,0.5)	(5,0.2)	(10,0.1)
5	BG	3.5±2.6	5.9±4.1	8.8±6.0	3.5±2.3	5.9±4.1	11.5±8.0	28.4±19.9	55.7±38.3
	TH	4.6±3.9	7.0±5.2	9.9±6.8	3.7±2.4	6.0±3.8	11.5±7.3	27.9±17.6	55.6±34.8
	BE	4.7±3.9	7.1±5.3	10.1±7.0	3.8±2.4	6.1±3.8	11.5±7.3	27.9±17.6	55.6±34.8
	HK	4.5±3.8	6.9±5.1	10.0±6.8	3.7±2.4	6.1±3.9	11.6±7.3	28.5±17.5	56.4±35.4
	Ar	4.6±3.8	6.9±5.2	10.1±7.1	3.7±2.4	6.0±3.8	11.5±7.2	28.3±17.9	57.2±36.1
	FI	3.2±2.2	5.1±3.5	7.4±5.0	2.7±1.8	4.7±3.0	8.8±5.6	21.7±13.9	43.7±28.1
6	BG	40.7±36.7	46.2±40.4	53.4±46.3	14.7±12.7	17.7±15.1	28.7±25.2	67.5±58.7	138.3±120.9
	TH	47.0±43.7	52.2±47.4	59.9±53.5	15.9±13.7	18.4±15.1	29.1±23.9	68.0±55.3	138.6±113.1
	BE	47.0±43.6	52.4±47.4	60.1±53.5	15.9±13.7	18.5±15.1	29.2±23.9	67.9±55.2	138.6±113.1
	HK	48.0±44.4	54.2±50.6	61.9±56.0	16.3±14.0	19.1±15.9	31.0±25.5	71.8±58.1	140.2±115.0
	Ar	45.3±41.6	50.6±47.3	58.6±53.1	16.4±14.3	18.1±15.4	29.2±24.1	68.9±56.3	135.5±113.2
	FI	39.1±31.2	43.4±34.4	47.9±38.3	12.9±10.6	14.5±12.1	23.0±19.2	54.8±45.6	106.3± 88.7
7	BG	6.8±5.1	9.2±6.6	12.1±8.4	4.2±2.7	6.3±4.0	11.6±7.3	27.9±17.8	56.0±35.8
	TH	7.1±5.3	9.5±6.8	12.4±8.6	4.2±2.7	6.3±4.0	11.6±7.2	28.0±17.6	55.9±35.2
	BE	7.2±5.3	9.6±6.8	12.5±8.6	4.2±2.7	6.3±4.0	11.6±7.2	28.0±17.6	55.9±35.2
	HK	7.3±5.5	9.7±6.9	12.8±8.8	4.4±2.9	6.5±4.2	11.9±7.5	29.2±18.1	57.8±36.4
	Ar	7.1±5.4	9.4±6.8	12.3±8.5	4.2±2.8	6.4±4.1	11.5±7.3	28.4±18.2	57.0±36.1
	FI	6.1±4.6	8.2±5.7	10.8±7.6	3.7±2.5	5.5±3.5	10.2±6.3	24.9±15.4	49.8±31.7

References

Ahrens, H. and Pincus, R. (1981). On two measures of unbalancedness in a one-way model and their relation to efficiency. *Biometrical Journal*, 23:227–235.

Arendacká, B. (2005). Generalized confidence intervals on the variance component in mixed linear models with two variance components. *Statistics*, 39:275–286.

Billingsley, P. (1995). *Probability and Measure*. New York: Wiley, 2nd edition.

Broemeling, L. D. (1969). 263 note: confidence intervals for measures of heritability. *Biometrics*, 25:424–427.

Bross, I. (1950). Fiducial intervals for variance components. *Biometrics*, 6:136–144.

Burch, B. D. (1996). *Confidence Intervals and Prediction Intervals in a Mixed Linear Model*. PhD thesis, Colorado State University, Dept. of Statistics.

Burch, B. D. and Iyer, H. K. (1997). Exact confidence intervals for a variance ratio (or heritability) in a mixed linear model. *Biometrics*, 53:1318–1333.

Burdick, R. K. and Eickman, J. (1986). Confidence intervals on the among group variance component in the unbalanced one-fold nested design. *Journal of Statistical Computation and Simulation*, 26:205–219.

Burdick, R. K. and Graybill, F. A. (1984). Confidence intervals on linear combinations of variance components in the unbalanced one-way classification. *Technometrics*, 26:131–136.

Burdick, R. K. and Graybill, F. A. (1992). *Confidence Intervals on Variance Components*. New York: Dekker.

Burdick, R. K., Maqsood, F., and Graybill, F. A. (1986). Confidence intervals on the intraclass correlation in the unbalanced one-way classification. *Communications in Statistics - theory and methods*, 15:3353–3378.

Burdick, R. K. and Sielken, J. R. L. (1978). Exact confidence intervals for linear combinations of variance components in nested classifications. *Journal of the American Statistical Association*, 73:632–635.

Casella, G. and Berger, R. L. (2002). *Statistical Inference*. CA: Duxbury, 2nd edition.

Falconer, D. S. (1989). *Introduction to Quantitative Genetics*. Essex, England: Longman Scientific and Technical.

Fisher, R. A. (1935). The fiducial argument in statistical inference. *Annals of Eugenics*, 6:391–398.

Graybill, F. A., Martin, F., and Godfrey, G. (1956). Confidence intervals for variance ratios specifying genetic heritability. *Biometrics*, 12:99–109.

Graybill, F. A. and Wang, C. M. (1980). Confidence intervals on nonnegative linear combinations of variances. *Journal of the American Statistical Association*, 75:869–873.

Green, J. R. (1954). A confidence interval for variance components. *The Annals of Mathematical Statistics*, 25:671–686.

Hannig, J. (2006). On fiducial inference, the good, the bad, and the ugly. Technical Report 2006/3, Colorado State University, Dept. of Statistics.

Hannig, J., Iyer, H. K., and Patterson, P. (2006). Fiducial generalized confidence intervals. *Journal of the American Statistical Association*, 101:254–269.

Hartung, J. and Knapp, G. (2000). Confidence intervals for the between group variance in the unbalanced one-way random effects model of analysis of variance. *Journal of Statistical Computation and Simulation*, 65:311–323.

Harville, D. A. and Fenech, A. P. (1985). Confidence interval for a variance ratio, or for heritability, in an unbalanced mixed linear model. *Biometrics*, 41:137–152.

Healy, J. W. R. (1961). Limits for a variance component with an exact confidence coefficient. *The Annals of Mathematical Statistics*, 32:466–476.

Healy, J. W. R. (1963). Fiducial limits for a variance component. *Journal of the Royal Statistical Society, Series B (Methodological)*, 25:128–130.

Henderson, C. R. (1976). A simple method for computing the inverse of a numerator relationship matrix used in prediction of breeding values. *Biometrics*, 32:69–83.

Huitson, A. (1955). A method of assigning confidence limits to linear combinations of variances. *Biometrika*, 42:471–479.

Iyer, H. K., Wang, C. M., and Mathew, T. (2004). Models and confidence intervals for true values in interlaboratory trials. *Journal of the American Statiscal Association*, 99:1060–1071.

Jeyaratnam, S. and Graybill, F. A. (1980). Confidence intervals on variance components in three-factor cross-classification models. *Technometrics*, 22:375–380.

Olsen, A., Seely, J., and Birkes, D. (1976). Invariant quadratic unbiased estimation for two variance components. *The Annals of Statistics*, 4:878–890.

Satterthwaite, F. E. (1946). An approximate distribution of estimates of variance components. *Biometrics Bulletin*, 2:110–114.

Seely, J. F. (1980). Some remarks on exact confidence intervals for positive linear combinations of variance components. *Journal of the American Statistical Association*, 75:372–374.

Thomas, J. D. and Hultquist, R. A. (1978). Interval estimation for the unbalanced case of the one-way random effects model. *The Annals of Statistics*, 6:582–587.

Tukey, J. W. (1951). Components in regression. *Biometrics*, 7:33–69.

Venables, W. and James, A. T. (1978). Interval estimates for variance components. *The Canadian Journal of Statistics*, 6:103–111.

Wald, A. (1940). A note on the analysis of variance with unequal class frequencies. *Annals of Mathematical Statistics*, 11:96–100.

Weerahandi, S. (1991). Testing variance components in mixed models with generalized p values. *Journal of the American Statistical Association*, 86:151–153.

Welch, B. L. (1956). On linear combinations of several variances. *Journal of the American Statistical Association*, 51:132–148.

Wild, C. J. (1981). Interval estimates for variance components. *The Canadian Journal of Statistics*, 9:195–201.

Williams, J. S. (1962). A confidence interval for variance components. *Biometrika*, 49:278–281.

Zhou, L. and Mathew, T. (1994). Some tests for variance components using generalized p values. *Technometrics*, 36:394–402.