FISHER'S EXACT TEST OF MUTUAL INDEPENDENCE FOR $2 \times 2 \times 2$ CROSS-CLASSIFICATION TABLES

PAUL W. MIELKE, JR. AND KENNETH J. BERRY

Colorado State University

DANIEL ZELTERMAN

University of Minnesota

FISHER'S EXACT TEST OF MUTUAL INDEPENDENCE FOR $2 \times 2 \times 2$ CROSS-CLASSIFICATION TABLES

[Abstract]

A subroutine to calculate Fisher's exact test of mutual independence in 2x2x2 cross-classification tables is presented. The subroutine's speed is due to (*a*) use of an arbitrary constant for the initial table and (*b*) recursively defined values obtained for all remaining tables.

FISHER'S EXACT TEST OF MUTUAL INDEPENDENCE FOR $2\times2\times2$ CROSS-CLASSIFICATION TABLES

An algorithm is developed for computing the exact significance level in testing the hypothesis of mutual independence of the three factors in a $2\times2\times2$ contingency table. The algorithm is useful if the table contains small counts and the dependence on asymptotic approximations for commonly-used test statistics such as Pearson's chi-squared statistic is questionable. Analyses of $2\times2\times2$ and other contingency tables have previously been addressed by many researchers, e.g., Agresti (1992), Bartlett (1935), Fisher (1934), Freeman and Halton (1951), Mielke and Berry (1988), Pomer (1984), and Zelen (1972). An exact significance level can be obtained from an enumeration of all possible permutations of the eight cell frequencies, conditioned on the realized marginals. The purpose of this article is to present a high speed recursive algorithm and the associated FORTRAN-77 subroutine (FEP222) to compute Fisher's exact test for a $2\times2\times2$ cross-classification table.

Subroutine

The logic underlying subroutine FEP222 was initially used by Quetelet (1849, pp. 254-269) to calculate binomial probability values. Beginning with an arbitrary initial value, a recursion procedure is used to generate relative frequency values for all possible 2×2×2 cross-classification tables, given the observed one-way marginal frequency totals. The exact probability value is obtained by summing those relative frequency values equal to or less than the observed relative frequency value, and dividing the sum by the unrestricted frequency total. Subroutine FEP222 is extremely fast because (*a*) no computed initial value is required and (*b*) all remaining values are recursively obtained. In addition, no factorials, logarithms, or log-factorial values are used.

Consider a 2×2×2 cross-classification table where n_{ijk} denotes the cell frequency of the ith row, jth column, and kth slice ($i = 1,2$; $j = 1,2$; $k = 1,2$). Let $A = n_{1..}$, $B = n_{.1.}$, $C = n_{..1}$, and $N = n_{...}$ denote the observed frequency totals of the first row, first column, first slice, and entire table, respectively. Also let $w = n_{111}$, $x = n_{112}$, $y = n_{121}$, and $z = n_{211}$ be the cell frequencies of the contingency table such that $1 \le A \le B \le C \le N/2$. Then the probability for any w, x, y, and z is

$$P(w,x,y,z|A,B,C,N) =$$

$$A!\,(N-A)!\,B!\,(N-B)!\,C!\,(N-C)!\,/$$

$$[(N!)^2\,w!\,x!\,y!\,z!\,(A-w-x-y)!$$

$$(B-w-x-z)!\,(C-w-y-z)!\,(N-A-B-C+2w+x+y+z)!]$$

(Mielke and Berry, 1988). The nested looping structure involves two distinct passes. The first pass yields the exact probability S_o of the observed table and is terminated when S_o is obtained. The second pass yields the exact probability P_o of all possible tables which are as or more extreme than the observed table. The four nested loops within each pass are over the cell frequency indices w, x, y, and z, respectively. The bounds for w, x, y, and z in each pass are

$$0 \le w \le M_w,$$

$$0 \le x \le M_x,$$

$$0 \le y \le M_y, \text{ and}$$

$$L_z \le z \le M_z$$

where $M_w = A$, $M_x = A-w$, $M_y = A-w-x$, $M_z = \min(B-w-x, C-w-y)$, and $L_z = \max(0, A+B+C-N-2w-x-y)$.

The recursion method is illustrated with the fourth (inner) loop over z given w, x, y, A, B, C, and N since this inner loop yields both S_0 during the first pass and P_0 during the second pass. Let $H(w,x,y,z)$ be a recursively defined positive function given A, B, C, and N, satisfying

$$H(w,x,y,z+1) = H(w,x,y,z)\,g(w,x,y,z)$$

where

$$g(w,x,y,z) = \frac{(B-w-x-z)(C-w-z)}{(z+1)(N-A-B-C+2w+x+y+z+1)} .$$

The remaining three loops of each pass initialize $H(w,x,y,z)$ for continued enumerations. Let $I_z = \max(0, A+B+C-N)$ and let the initial value of $H(0,0,0,I_z)$ be equated to an arbitrary small constant such as 10^{-250}. Then the total T over the completely enumerated distribution is found by

$$T = \sum_{w=0}^{M_w} \sum_{x=0}^{M_x} \sum_{y=0}^{M_y} \sum_{z=L_z}^{M_z} H(w,x,y,z) .$$

If w_o, x_o, y_o, and z_o are the values of w, x, y, and z in the observed table, then S_o and P_o are given by

$$S_o = H(w_o, x_o, y_o, z_o) / T$$

and

$$P_o = \sum_{w=0}^{M_w} \sum_{x=0}^{M_x} \sum_{y=0}^{M_y} \sum_{z=L_z}^{M_z} H(w, x, y, z)\, \psi(w, x, y, z) / T$$

where

$$\psi(w, x, y, z) = \begin{cases} 1, \text{ if } H(w, x, y, z) \le H(w_o, x_o, y_o, z_o) \\ 0, \text{ otherwise.} \end{cases}$$

The present algorithm complements the algorithms of Mielke and Berry (1992) to provide efficient recursive Fisher's exact tests for all r-way contingency tables with five or fewer degrees of freedom.

Examples

The small data set in Table 1 (Grizzle, Starmer, and Koch, 1969) involves $N = 46$ subjects with Favorable (1) and

Unfavorable (0) responses to each of the drugs denoted by A, B, and C. The two probabilities associated with Table 1 are $S_o = 0.388\times10^{-4}$ and $P_o = 0.0253$. There are 8,419 tables consistent with the observed one-way marginal totals. Exactly 6,732 of these tables have probabilities equal to or less than S_o.

Insert Table 1 about here

The data in Table 2 is cited in Pomar (1984). A total of N = 1663 respondents were asked if they agreed with the statement that minorities should have equal job opportunity (No, Yes); the respondents were then categorized by region of the country (North, South) and by year of the survey (1946, 1963). The two probabilities associated with Table 2 are $S_o = 0.186\times10^{-72}$ and $P_o = 0.168\times10^{-65}$. There are 3,683,159,504 tables consistent with the observed one-way marginal totals. Exactly 2,761,590,498 of these tables have probabilities equal to or less than S_o.

Insert Table 2 about here

Program Language

Subroutine FEP222 is written in FORTRAN-77 in double precision. Comment lines provide for input/output specification and documentation. Input into subroutine FEP222 consists of the eight observed cell frequencies in the 2×2×2 cross-classification table, in INTEGER mode. Output consists of the observed cell frequencies, the exact probability value S_o, the exact significance level P_o, the number of tables consistent with the observed one-way marginal totals, and the number of tables with probabilities equal to or less than S_o.

Availability

A listing of subroutine FEP222 and an appropriate driver program is available from Kenneth J. Berry, Department of Sociology, Colorado State University, Fort Collins, CO 80523, or by e-mail from Internet address: berry@lamar.colostate.edu.

REFERENCES

Agresti, A. (1992). A survey of exact inference for contingency tables. *Statistical Science*, 7, 131-177.

Bartlett, M. S. (1935). Contingency table interactions. *Journal of the Royal Statistical Society Supplement*, 2, 248-252.

Berry, K. J. and Mielke, P. W. (1989). Analyzing independence in r-way contingency tables. EDUCATIONAL AND PSYCHOLOGICAL MEASUREMENT, 49, 605-607.

Fisher, R. A. (1934). *Statistical methods for research workers*. Edinburgh: Oliver and Boyd.

Freeman, G. H. and Halton, J. H. (1951). Note on an exact treatment of contingency, goodness of fit and other problems of significance. *Biometrika*, 38, 141-149.

Grizzle, J. E., Starmer, C. F. and Koch, G. G. (1969). Analysis of categorical data by linear models. *Biometrics*, 25, 489-504.

Mielke, P. W. and Berry, K. J. (1988). Cumulant methods for analyzing independence of r-way contingency tables. *Biometrika*, 75, 790-793.

Mielke, P. W. and Berry, K. J. (1992). Fisher's exact probability test for cross-classification tables. EDUCATIONAL AND PSYCHOLOGICAL MEASUREMENT, 52, 97-101.

Pomar, M. I. (1984). Demystifying loglinear analysis: four ways to assess interaction in a 2×2×2 table. *Sociological Perspectives*, 27, 111-135.

Quetelet, M. A. (1849). *Letters addressed to H. R. H. the Grand Duke of Saxe Coburg and Gotha on the theory of probabilities as applied to the moral and political sciences*. Translated by O. G. Downes. London: Charles & Edwin Layton.

Zelen, M. (1972). Exact significance tests for contingency tables embedded in a 2^n classification. Pages 737-757 in L. M. LeCam, J. Neyman, and E. L. Scott (Eds.), *Proceedings of the sixth Berkeley symposium on mathematical statistics and probability, Vol. 1*. Berkeley, CA: University of California Press.

TABLE 1

Cross-Classification Table of Favorable (1) and Unfavorable (0) Responses to Drugs A, B, and C

Drug A:		1		0	
Drug B:		1	0	1	0
Drug C:	1:	6	2	2	6
	0:	16	4	4	6

TABLE 2

Cross-Classification Table of Responses (No, Yes)

Categorized by Year (1946, 1963) and Region (North, South)

Region:		North		South	
Response:		No	Yes	No	Yes
Year:	1963:	410	56	126	31
	1946:	439	374	64	163