A TEST OF SIGNIFICANCE FOR THE INDEX OF ORDINAL VARIATION[1]

KENNETH J. BERRY AND PAUL W. MIELKE, JR.
Colorado State University

---

[1] Request reprints from K. J. Berry, Department of Sociology, Colorado State University, Fort Collins, CO 80523

A TEST OF SIGNIFICANCE FOR THE INDEX OF ORDINAL VARIATION

Summary.—An exact test of significance for the Index of Ordinal Variation is presented. A high-speed recursion algorithm is described which generates all possible permutations of objects to categories.

A TEST OF SIGNIFICANCE FOR THE INDEX OF ORDINAL VARIATION

Consider a population of objects classified into a fixed number of predetermined ordered categories. The amount of variation in such a population can be measured with the Index of Ordinal Variation (*IOV*) which was introduced by Berry and Mielke (1992a, 1992b). Often, the research interest is more in measuring lack of variation than in measuring variation per se. A measure closely related to the *IOV* is presented here and is designated as the Index of Ordinal Consensus (*IOC*). The *IOC* yields a value of 1 when all events fall into the same ordered category and a value of 0 when extreme polarization is present. This paper presents exact tests of significance for the sample-based *IOV* and *IOC* measures.

If k predefined, fixed, mutually exclusive categories are ordered from 1 to k with frequencies $n_1, \ldots, n_k$ where

$$N = \sum_{i=1}^{k} n_i , \qquad [1]$$

then the *IOV* is defined as

$$IOV = \frac{T}{T_{\max}} , \qquad [2]$$

and the *IOC* is defined as

$$IOC = 1 - IOV, \qquad [3]$$

where

$$T = \sum_{i<j} n_i n_j (j-i)\,, \qquad [4]$$

$1 \leq i < j \leq k$, and $T_{\max}$ is a standardized norming factor representing the maximum possible variation. When N is odd,

$$T_{\max} = \frac{(N^2-1)(k-1)}{4}\,, \qquad [5]$$

and when N is even,

$$T_{\max} = \frac{N^2(k-1)}{4} \qquad [6]$$

(Berry & Mielke, 1992a). In either case, $0 \leq IOV \leq 1$ and $0 \leq IOC \leq 1$.

The null hypothesis (H_0) implies that each of the N objects may appear in any of the k categories with equal probability. Specifically, the null hypothesis states

$$P(n_1, \ldots, n_k | H_0) = \binom{N}{n_1, \ldots, n_k}\left(\frac{1}{k}\right)^N . \qquad [7]$$

Note that "equal probability" does not imply "equal frequencies." While it is true that, under the stated null hypothesis and in the long run, the expected frequencies of any cell will be N/k, the probability of equal category frequencies occurring in all k cells simultaneously is usually very small. Suppose N is divisible by k, then

$$P\left(\frac{N}{k},\ldots,\frac{N}{k}\middle|H_0\right) = \frac{N!}{k^N\left[\left(\frac{N}{k}\right)!\right]^k} \doteq \left[\frac{k}{\left(\frac{2\pi N}{k}\right)^{k-1}}\right]^{\frac{1}{2}} . \qquad [8]$$

As an example, consider $k = 5$ and $N = 200$. The probability of the observed cell frequencies {40,40,40,40,40} is only 3.5×10^{-5}. In the *IOV*, expectations do not enter into either the calculation of the test statistic or the calculation of the probability value, and the null hypothesis cannot be interpreted as implying simultaneous equal category frequencies of N/k.

When $k \geq 2$, the number of possible arrangements of placing the N objects into the k categories is given by

$$M = \binom{N+k-1}{k-1} . \qquad [9]$$

For each of the M possible arrangements of $n_1,\ldots,n_k$, there are

$$W_j = \left\{\frac{N!}{\prod_{i=1}^{k} n_i!}\right\}_j \qquad [10]$$

possible permutations of the N objects to the k categories ($j = 1,\ldots,M$). Under H_0, the sum of the M permutations

$(W_1, \ldots, W_M)$ is given by

$$Q = \sum_{j=1}^{M} W_j = k^N. \qquad [11]$$

Let IOV_o denote the observed value of the IOV. Under H_0, the probability (P) of an IOV_o this large or larger, is given by

$$P = \frac{S}{Q} \qquad [12]$$

where

$$S = \sum_{j=1}^{M} I_j W_j, \qquad [13]$$

and

$$I_j = \begin{cases} 1, \text{ if } IOV_j \geq IOV_o, \\ 0, \text{ otherwise.} \end{cases} \qquad [14]$$

Under H_0, the probability (P) of an observed IOC this large or larger is given by Equations 12 and 13, but the indicator function is redefined as

$$I_j = \begin{cases} 1, \text{ if } IOV_j \leq IOV_o, \\ 0, \text{ otherwise.} \end{cases} \qquad [15]$$

Exact tests of significance are always computationally intensive and only the most trivial problems can be evaluated with hand calculations. A FORTRAN program to compute the exact probability of an IOV as large or larger than the observed *IOV* and the exact probability of an *IOC* as large or larger than the observed *IOC* is available from the authors. Program SIGIOV utilizes a modification of a routine developed by Dunlap, Myers, and Silver (1984) to generate the *M* possible arrangements of the *N* objects to the *k* categories. The permutations of objects to categories is solved with a recursive routine. Given

$$W_j = \frac{N!}{n_1!\,n_2!\cdots n_{k-1}!\,n_k!}\,, \qquad [16]$$

the next *W* value

$$W_{j+1} = \frac{N!}{n_1!\,n_2!\cdots(n_{k-1}+1)!\,(n_k-1)!} \qquad [17]$$

is defined as

$$W_{j+1} = W_j\frac{n_k}{n_{k-1}+1}\,. \qquad [18]$$

As an example, consider $N = 6$ objects in $k = 3$ categories with $n_1 = 0$, $n_2 = 2$, and $n_3 = 4$, then

$$W_j = \frac{6!}{0!\,2!\,4!} = 15 \qquad [19]$$

and

$$W_{j+1} = \frac{6!}{0!\,3!\,3!} = 20\,. \qquad [20]$$

Alternatively,

$$W_{j+1} = W_j\left(\frac{n_k}{n_{k-1}+1}\right) = 15\left(\frac{4}{2+1}\right) = 20\,. \qquad [21]$$

As the initial arrangement is always of the form $n_1 = n_2 = \cdots = n_{k-1} = 0$ and $n_k = N$, W_1 is always equal to 1. Because the recursive routine accumulates products there is a possibility that the W_j values might exceed the upper range of the computer. For this reason it is advantageous to begin the recursion with a small value near the lower range of the computer (e.g., 10^{-200}), yielding M relative frequency values. The exact probability of an observed *IOV* value can then be obtained by dividing the appropriate sum of the relative frequency values by the total of all relative frequency values.

The *IOV* and the *IOC* have many potential applications, especially in survey research. Consider an example application with $k = 5$ categories and a sample of size $N = 50$. Suppose a Likert question on a survey of a sample of elementary school parents solicits information on introducing a test of motor skills into the elementary school curriculum. The five Likert categories are "Strongly Agree," "Agree," "Neutral," "Disagree," and "Strongly Disagree," and the observed category frequencies are 4, 10, 16, 8, and 12, respectively. While the value of the *IOV* may be interesting in this case (*IOV* = 0.6976 with $P = 0.9537$), the real interest of the research is in the amount of consensus shown by the respondents. For these data, the *IOC* = 0.3025 and the probability of an observed *IOC* this large or larger, under H_0, is $P = 0.0471$. Although the results convey only the probability of an observed *IOC* this large or larger under the null hypothesis of equal category probabilities, if a level of significance is set by the researcher, e.g., $\alpha = 0.05$, then the null hypothesis of equal category probabilities is rejected and an alternative hypothesis of unequal category probabilities is posited, indicating that the observed category frequencies reflect something other than chance assignment, with a five percent chance of a Type I error.

Exact probability solutions are limited to small sample sizes. If *k* is not too large, e.g., five ordered categories representing “Strongly Agree,” “Agree,” “Neutral,” “Disagree,” and “Strongly Disagree,” and *N* is less than or equal to 100, computation time is reasonable. Some representative computing times are given in Table 1.

Insert Table 1 about here

The timing runs were done on an IBM RISC-6000 workstation. The large difference in times is due to the value of *M*. For example, with $N = 25$ and $k = 2$, $M = 26$; with $N = 75$ and $k = 3$, $M = 2{,}926$; but with $N = 100$ and $k = 5$, $M = 4{,}598{,}126$.

Program SIGIOV is written in double precision in FORTRAN-77 and includes a driver program for interactive input. Program SIGIOV is available from Kenneth J. Berry, Department of Sociology, Colorado State University, Fort Collins, CO, or from Internet address: berry@lamar.colostate.edu.

REFERENCES

BERRY, K. J., & MIELKE, P. W. (1992) Assessment of variation in ordinal data. *Perceptual and Motor Skills*, 74, 63-66.

BERRY, K. J., & MIELKE, P. W. (1992) Indices of ordinal variation. *Perceptual and Motor Skills*, 74, 576-578.

DUNLAP, W. P., MYERS, L., & SILVER, N. C. (1984) Exact multinomial probabilities for one-way contingency tables. *Behavior Research Methods, Instruments, & Computers*, 16, 54-56.

TABLE 1

IOV TEST OF SIGNIFICANCE COMPUTING TIME (IN SECONDS) FOR SELECTED COMBINATIONS OF *N* AND NUMBER OF CATEGORIES

N	Number of Categories (*k*)			
	2	3	4	5
25	0.00	0.00	0.01	0.10
50	0.00	0.00	0.07	1.28
75	0.00	0.00	0.21	5.53
100	0.00	0.01	0.49	17.44