Short Communication
The Geometric Convergence Rate of a Lindley Random Walk

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Abstract

Let $\{X_n\}$ be the Lindley random walk on $[0,\infty)$ defined by $X_n = \max[X_{n-1} + A_n, 0]$ for $n \geq 1$ with $X_0 = x \geq 0$. Here, $\{A_n\}$ is a sequence of independent and identically distributed random variables. When $E[A_1] < 0$ and $E[r^{A_1}] < \infty$ for some $r > 1$, $\{X_n\}$ converges at a geometric rate in total variation to an invariant distribution π; that is, there exists $r > 1$ such that

$$\lim_{n \to \infty} r^n \sup_B \left| P_x[X_n \in B] - \pi(B) \right| = 0$$

for every initial state $x \geq 0$. In this communication, we supply a short proof and some extensions of a result initially due to Veraverbeke and Teugels (1975 and 1976): the largest r satisfying the above relationship is $\phi(r_0)^{-1}$ where $\phi(r) = E[r^{A_1}]$ and $r_0 > 1$ satisfies $\phi'(r_0) = 1$.

MARKOV CHAIN; GEOMETRIC CONVERGENCE; TOTAL VARIATION; QUEUES.

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1. Introduction. Consider the Lindley random walk $\{X_n\}$ driven by an independent and identically distributed (iid) random sequence $\{A_n\}$:

$$X_n = \max[X_{n-1} + A_n, 0] \quad \text{for } n \geq 1, \tag{1.1}$$

with a possibly random initial state $X_0 \geq 0$. The recursion in (1.1) governs customer waiting times in queues [2,5] and arises in discrete storage modeling [5].

Equation (1.1) is also useful for simulating a probability distribution π on $[0, \infty)$ that satisfies the Wiener-Hopf equation

$$\pi([0,z]) = \int_{[0,\infty)} F(z-y)\pi(dy) \quad \text{for } z \geq 0,$$

where F is a known cumulative distribution function supported on $(-\infty, \infty)$ with a strictly negative mean. For this, one generates an iid $\{A_n\}$ with distribution function F, selects an initial state $X_0 = x$, and then recurses with (1.1) to generate $\{X_n\}$. The convergence rate results presented below can be used to find a natural number n where the distribution of X_n is sufficiently close to π.

It is well known that $\{X_n\}$ is a Markov chain that has a unique invariant probability distribution π if and only if $E[A_1] < 0$ [2,7] which we henceforth assume. To avoid degeneracy in $\{X_n\}$, we assume that A_1 is nondegenerate with $P(A_1 > 0) > 0$. Let $\phi(r) = E[r^{A_1}]$ and suppose that

(*i*) There exists $r_0 > 1$ with $\phi(r_0) < \infty$ satisfying $\phi'(r_0) = 0$; and

(*ii*) If A_1 is lattice, then $P(A_1 = 0) > 0$.

A classic result, proven in [8] and [9], states that when $X_0 = 0$ and (*i*) and (*ii*) hold, $P[X_n \leq z]$ converges geometrically fast to $\pi([0,z])$ uniformly in $z \geq 0$. The result also identifies the exact geometric convergence rate as $\phi(r_0)^{-1}$. We note that r_0 is unique and $\phi(r_0) < 1$; these properties follow from $E[A_1] < 0$, the convexity of ϕ, and (*i*). Our objective here is to give a short proof of this result based on sample path orderings; we also upgrade the converge mode to total variation and consider the case where $X_0 > 0$. Specifically, we show that when $E[A_1] < 0$ and (*i*) and (*ii*) hold,

$$\lim_{n\to\infty} r^n \sup_A \left| P_x[X_n \in A] - \pi(A) \right| = 0 \tag{1.2}$$

for every $x \geq 0$ if and only if $r \leq \phi(r_0)^{-1}$. In (1.2), the notation P_x indicates that $X_0 \equiv x$. In addition, we will examine geometric convergence of process moments; that is, for a function f, we investigate the values of $r > 1$ where

$$\lim_{n\to\infty} r^n \left| E_x[f(X_n)] - \pi(f) \right| = 0 \tag{1.3}$$

for all $x \geq 0$. In (1.3), $\pi(f)$ denotes the fth moment of π:

$$\pi(f) = \int_{[0,\infty)} f(x)\pi(dx).$$

By stationarity, $\pi(f) = E_\pi[f(X_n)]$ for all $n \geq 1$; in the above, E_x and E_π denote expectation when $X_0 = x$ and X_0 has distribution π respectively.

2. Results. From (1.1), if $\{X_n\}$ and $\{X'_n\}$ are trajectories of the chain driven by the same $\{A_n\}$ with $X_0 \geq X'_0$, then $X_n \geq X'_n$ for all $n \geq 1$; hence $\{X_n\}$ is pathwise ordered and the results of [6] apply. Let $\tau_0 = \inf\{n > 0 : X_n = 0\}$ and $G_x(r) = E_x[r^{\tau_0}]$. From $P(A_1 > 0) > 0$ and the Markov property of $\{X_n\}$, it is easy to see that $G_x(r)$ has the same radius of convergence in r, say r^*, for each $x \geq 0$.

LEMMA 1. Let $\{X_n\}$ be the Lindley random walk in (1.1) with $E[A_1] < 0$ that satisfies (i) and (ii).

$a)$ If r is such that $G_0(r) < \infty$, then (1.2) holds for all $x \geq 0$.

$b)$ Equation (1.2) fails when $x = 0$ and $r > r^*$.

PROOF: Statement $a)$ is Corollary 5.2 of [6]. To prove $b)$, the argument in Theorem 6.2 of [6] must be modified as Equation (6.1) of [6] does not hold. Towards this, consider two copies of the chain, say $\{X_n\}$ and $\{X'_n\}$, driven by the same $\{A_n\}$, but with the different (possibly) initial conditions $X_0 = 0$ and $X'_0 = D$, where D is random with distribution π and is independent of $\{A_n\}$. Since $\{X'_n\}$ is stationary, $\pi(B) = P[X'_n \in B]$ for all $n \geq 1$ and all measurable B. Hence,

$$\sup_B \left| P_0[X_n \in B] - \pi(B) \right| \geq \left| P[X_n = 0] - P[X'_n = 0] \right| \tag{2.1}$$

Now define the coupling time $T = \inf\{n \geq 0 : X_n = X'_n\}$. Since $X_n \leq X'_n$ for all n, $X_n = X'_n$ whenever $T \leq n$; hence, $P[X_n = 0 \cap T \leq n] = P[X'_n = 0 \cap T \leq n]$ and (2.1) is

$$\sup_B \left| P_0[X_n \in B] - \pi(B) \right| \geq \left| P[X_n = 0 \cap T > n] - P[X'_n = 0 \cap T > n] \right|. \tag{2.2}$$

Further, if $X'_n = 0$, then $X_n = 0$ and $T \leq n$; hence, the right hand side of (2.2) is $P[X_n = 0 \cap T > n]$. Suppose that $\tau_0 = \inf\{k > 0 : X_k = 0\} = n$ and $S_n = A_1 + \ldots + A_n \in [-\Delta, 0]$ for some fixed $\Delta \geq 0$.

Then $X_n = 0$; furthermore, if $D > \Delta$, then $S'_n = D + A_1 + \ldots + A_n > 0$, $\tau'_0 = \inf\{k > 0 : X'_k = 0\} > n$, and $T > n$. Thus, (2.2) gives

$$\sup_B \left| P_0[X_n \in B] - \pi(B) \right| \geq P[\tau_0 = n \cap S_n \geq -\Delta \cap D > \Delta] \tag{2.3}$$
$$= \pi((\Delta, \infty)) P[S_n \geq -\Delta \mid \tau_0 = n] P[\tau_0 = n],$$

where the last line in (2.3) follows from the independence of D and $\{A_n\}$.

When (i) and (ii) hold, [1] shows that $\lim_{n \to \infty} P[S_n \geq -\Delta \mid \tau_0 = n] = G(\Delta)$ for all $\Delta \geq 0$ where G is a proper cumulative distribution function; hence, there is a $\Delta < \infty$ such that $G(\Delta) > 0$. Since $P(A_1 > 0)$, the support set of π is $[0, \infty)$ and $\pi((\Delta, \infty)) > 0$ for all $\Delta > 0$. Selecting such a $\Delta > 0$, multiplying (2.3) by r^n, and taking a limit supremum gives

$$\limsup_{n \to \infty} r^n \sup_B \left| P_0[X_n \in B] - \pi(B) \right| \geq \pi((\Delta, \infty)) G(\Delta) \limsup_{n \to \infty} r^n P[\tau_0 = n] = \infty$$

when $r > r^*$; this proves b). We remark that the subscripts on P have been suppressed; this causes no confusion as the probability space is that which supports both $\{A_n\}$ and D. $\square$

When $E[A_1] < 0$, $\pi(\{0\}) > 0$; hence, Lemma 1 establishes "divergence" for $r > r^*$ on a set of positive measure with respect to π.

COROLLARY 2. Let $\{X_n\}$ be the Lindley random walk in (1.1) with $E[A_1] < 0$ that satisfies (i) and (ii). Then (1.2) holds for all $x \geq 0$ when $r \leq \phi(r_0)^{-1}$ and fails for $x = 0$ when $r > \phi(r_0)^{-1}$.

PROOF. Equation I 6.78 in [5] identifies the form of $G_0(r)$ as

$$G_0(r) = 1 + (r-1) \exp\left[\sum_{k=1}^{\infty} \frac{r^k}{k} P(S_k > 0) \right], \tag{2.4}$$

where $S_k = A_1 + \ldots + A_k$ for $k \geq 1$. Hence, $G_0(r) < \infty$ if and only if $H(r) = \sum_k k^{-1} r^k P(S_k > 0) < \infty$. Theorem 1 of [4] identifies the radius of convergence of $H(r)$ as $\phi(r_0)^{-1}$.

To see that $H(r) < \infty$ when $r = \phi(r_0)^{-1}$, we apply the asymptotic expansion $P(S_k > 0) \sim M[\phi(r_0)]^k k^{-1/2}$ of [3] (M here is a finite constant). An appeal to Lemma 1 completes the proof. $\square$

REMARK: The deviations bound $P(S_k > 0) \leq \phi(\alpha)^k$ for $\alpha \geq 1$ shows that $G_0(r) < \infty$ for $r < \phi(r_0)^{-1}$ when used in (2.4). By Corollary 5.2 of [6], (1.2) holds for all $x \geq 0$ and $r < \phi(r_0)^{-1}$; hence, (ii) can be relaxed. The proof of Part b) of Lemma 1 shows that (1.2) fails at $x = 0$ for all $r > 1$ when $\phi(r) = \infty$

for all $r > 1$; hence, one must have $\phi(r) < \infty$ for some $r > 1$ to achieve geometric convergence. In cases where ϕ does not achieve its minimal value, one can argue as above and show that (1.2) holds for all $x \geq 0$ and $r < \inf\{\phi(r): r > 1\}^{-1}$; hence, (*ii*) can also be relaxed. In general, it is not clear whether (1.2) holds for $r = \inf\{\phi(r): r > 1\}^{-1}$ when (*i*) and/or (*ii*) do not hold. □

REMARK: For simulation purposes, one can take $x = 0$; however, a bound for the first constant multiplying the geometric decay rate in (1.2) is also needed to identify an n where the distribution of X_n is sufficiently close to π in a total variational sense. Following the arguments in [6], we obtain

$$\sup_B \left| P_0[X_n \in B] - \pi(B) \right| \leq C(r) r^{-n},$$

where $C(r) \leq G_\pi(r) \leq [G_0(r) - 1]/[r - 1]$. Combining this with (2.4) gives $C(r) \leq \exp(H(r))$. The bound $P(S_k > 0) \leq \phi(r_0)^k$ (assuming (*ii*) holds) and the identity $\sum_1^\infty k^{-1} x^k = -\ln(1 - x)$ for $0 \leq x < 1$ provide $C(r) \leq [1 - r\phi(r_0)]^{-1}$ for $r < \phi(r_0)^{-1}$ as required. □

Now let f: $[0, \infty) \to [0, \infty)$ be a general function. The following result establishes when the moment convergence in (1.3) takes place.

THEOREM 3. Let $\{X_n\}$ be the Lindley random walk in (1.1) with $E[A_1] < 0$ that satisfies (*i*) and (*ii*).

a) If $r < \phi(r_0)^{-1}$ and $f(x) \leq M r_0^{\,x}$ for all $x \geq 0$ and some $M < \infty$, then (1.3) holds for all $x \geq 0$.

b) If f is nondecreasing and $f(x + \Delta) - f(x) \geq M > 0$ for all $x \geq 0$ and some $M > 0$ and $\Delta > 0$, then (1.3) fails when $r > \phi(r_0)^{-1}$ and $x = 0$.

PROOF. To prove *a*), Theorems 3.1 and 5.1 of [6] show that it is sufficient to establish $E_x[r^{\tau_0}] \leq \kappa r_0^{\,x}$ for $r < \phi(r_0)^{-1}$ and some $\kappa < \infty$ (κ may depend on r). For this, we use

$$E_x[r^{\tau_0}] = 1 + (r - 1) \sum_{n=0}^{\infty} r^n P_x[\tau_0 > n],$$

the bound $P_x[\tau_0 > n] \leq P[x + A_1 + \ldots + A_n > 0] \leq r_0^{\,x} \phi(r_0)^n$, and $r_0^{\,x} \geq 1$ to obtain $E_x[r^{\tau_0}] \leq \kappa r_0^x$ for $r < \phi(r_0)^{-1}$ where $\kappa = r[1 - r\phi(r_0)]^{-1} < \infty$.

For *b*), we use the notation and arguments in the proof of Lemma 1 to get

$$\begin{aligned} \left| E[f(X_n)] - \pi(f) \right| &= \left| E\Big[f(X_n) \mathbb{1}_{[T > n]} \Big] - E\Big[f(X'_n) \mathbb{1}_{[T > n]} \Big] \right| \qquad (2.5) \\ &= E\Big[f(X'_n) \mathbb{1}_{[T > n]} \Big] - E\Big[f(X_n) \mathbb{1}_{[T > n]} \Big], \end{aligned}$$

where the last line in (2.5) follows from $f(X'_n) \geq f(X_n)$ for all n (by the nondecreasing f). Making the decomposition $\{T > n\} = \{T > n \cap \tau_0 > n\} \cup \{T > n \cap \tau_0 \leq n\}$ in (2.5) and using

$$E\Big[[f(X'_n) - f(X_n)]\mathbb{1}_{[T > n] \cap [\tau_0 \leq n]}\Big] \geq 0$$

gives

$$\big|E[f(X_n)] - \pi(f)\big| \geq E\Big[[f(X'_n) - f(X_n)]\mathbb{1}_{[T > n] \cap [\tau_0 > n]}\Big]. \tag{2.6}$$

Now if $\tau_0 > n$, then $X'_k = X_k + D$ for $1 \leq k \leq n$ and the event $T > n$ has also occurred. Hence, $\{T > n\} \cap \{\tau_0 > n\} = \{\tau_0 > n\}$ and (2.6) and the assumed properties of f provide

$$\big|E[f(X_n)] - \pi(f)\big| \geq E[[f(X_k + D) - f(X_k)]\mathbb{1}_{[\tau_0 > n]}] \geq M\pi((\Delta, \infty))P[\tau_0 > n]. \tag{2.7}$$

Multiplying both sides of (2.7) by r^n, taking a limit supremum, and using the fact that the radius of convergence of $G_0(r)$ is $\phi(r_0)^{-1}$ finishes the proof of b). □

It is clear that the assumptions on f in Part b) of Theorem 3 could be weakened with a more detailed analysis. However, we note that typical "moment" functions, such as the power class $f(x) = x^\alpha$, $\alpha \geq 1$, and exponential class $f(x) = \exp(\beta x)$, $\beta > 0$, satisfy these assumptions.

3. Examples.

EXAMPLE 3.1. Suppose that $A_1 = P - 1$ where P has a Poisson distribution with parameter $\lambda < 1$. Then (i) and (ii) hold, $\phi(r) = \exp\{-\lambda(1-r) - \ln(r)\} < \infty$ for all $r \geq 1$, $r_0 = \lambda^{-1}$, and (1.2) holds for all $x \geq 0$ if and only if $r \leq \phi(r_0)^{-1} = \lambda^{-1}e^{\lambda - 1}$. Theorem 3 shows that, for example, (1.3) holds for $f(y) = y^\alpha$, $\alpha \geq 1$, if $r < \lambda^{-1}e^{\lambda-1}$ and fails when $x = 0$ and $r > \lambda^{-1}e^{\lambda-1}$.

Now suppose that $A_1 = E - 1$ where E has the exponential density $\mu e^{-\mu y}$ for $y \geq 0$ with $\mu > 1$. Then (i) and (ii) hold, $\phi(r) = \mu[r(\mu - \ln(r))]^{-1}$ for $1 \leq r < e^\mu$, $r_0 = e^{\mu-1}$, and (1.2) holds for all $x \geq 0$ if and only if $r \leq \phi(r_0)^{-1} = \mu^{-1}e^{\mu-1}$. Again we have that, essentially, $E_x[X_n^\alpha]$ converges geometrically to its limit for all $\alpha \geq 1$ and $x \geq 0$ with best geometric rate $\mu^{-1}e^{\mu-1}$. □

EXAMPLE 3.2. Consider a GI/GI/1 queue where S_n is the service time of the nth customer and I_n is the interarrival time between the nth and $(n+1)$st customers; here, $\{S_n\}$ and $\{I_n\}$ are independent iid series of nonnegative random variables. The first customer arrives at time 0 and encounters a server with workload $x \geq 0$ before his/her service begins.

Let Q_n be the time the nth customer spends waiting for his/her service to commence (the virtual waiting time). Then $\{Q_n\}$ satisfies $Q_n = \max(Q_{n-1} + S_{n-1} - I_n, 0)$ for $n \geq 1$ with $Q_0 = x$ [2,5,7]; hence, (1.1) holds with $A_n = S_{n-1} - I_n$. If $E[S_0] < E[I_1]$ and (i) and (ii) hold with $\phi(r) = E[r^{S_0 - I_1}]$, then $\{Q_n\}$ has a limiting distribution π and the best geometric convergence rate is $\phi(r_0)^{-1}$ where $\phi'(r_0) = 0$; in general, r_0 must be obtained case by case. By Theorem 3, all moments $E_x[Q_n^\alpha]$, $\alpha \geq 1$, converge geometrically to their limits with "best" geometric rate $\phi(r_0)^{-1}$.

Convergence rates for other quantities in the queue can also be obtained from the virtual waiting time rates. For example, the total time the nth customer spends in the queue, denoted W_n, is $W_n = Q_n + S_n$. Let $\{Q_n\}$ and $\{Q'_n\}$ be trajectories of the virtual waiting time chain driven by the same $\{I_n\}$ and $\{S_n\}$ with $Q_0 = x$ and Q'_0 having the stationary virtual waiting time distribution. Then $T_Q = \inf\{n \geq 0\colon\ Q_n = Q'_n\} = T_W = \inf\{n \geq 0\colon\ W_n = W'_n\}$ and the coupling times for the virtual and total waiting times are identical (here, $\{W'_n\}$ is a stationary total waiting time chain constructed from $\{Q'_n\}$ in the obvious manner). Hence, (1.2) also holds for $r \leq \phi(r_0)^{-1}$ for $\{W_n\}$. □

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