SOME EXACT AND NONASYMPTOTIC ANALYSES OF DISCRETE GOODNESS-OF-FIT AND r-WAY CONTINGENCY TABLES

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ABSTRACT

Recently developed exact and nonasymptotic multinomial based methods for discrete goodness-of-fit and r-way contingency tables are described. These methods account for small expected frequencies which may or may not depend on parameters fit from observed data.

1 INTRODUCTION

Asymptotic analyses of discrete data goodness-of-fit and r-way contingency tables are specified by chi-squared distributions which depend solely on a single parameter (degrees of freedom) prescribed by (1) the number of cells, (2) a null hypothesis, and (3) the number of parameters fit with observed data. Conditions underlying these asymptotic analyses are seldom if ever satisfied for this broad class of problems. The exact and nonasymptotic analyses described here routinely depend on cell frequency probabilities which may or may not depend on parameters fit with observed data. The statistic associated with an exact analysis is the discrete probability distribution function (i.e., a Fisher's exact test). A statistic associated with a nonasymptotic analysis requires the statistic's exact mean, variance and skewness under the null hypothesis; this permits the use of a gamma (Pearson type III) distribution approximation which is completely characterized by its mean, variance and skewness. Approximate analyses based on simulation are excluded simply to avoid the obvious possibility of an additional type I statistical error due to the sampling process. In addition, exact higher order interaction analyses of 2^r contingency tables are described and comparisons between Pearson χ^2 and Goodman and Kruskal asymmetric tau statistics are discussed.

2 DISCRETE DATA GOODNESS-OF-FIT TESTS

Consider the random assignment of n objects to k cells where the probability that any of the n objects occurs in the ith cell is p_i $(i = 1, \cdots, k)$. Then the probability that o_i objects occur in the ith cell for $i = 1, \cdots, k$ is the multinomial probability given by

$$P(o_i \mid n, p_i) = P(o_1, \cdots, o_k \mid n, p_1, \cdots, p_k) = (n!/\prod_{i=1}^{k} o_i!) \prod_{i=1}^{k} p_i^{o_i}$$

where $\sum_{i=1}^{k} p_i = 1$ and $\sum_{i=1}^{k} o_i = n$.

2.1 Fisher's exact test

The P-value associated with Fisher's exact test is the sum of all distinct $P(o_i \mid n, p_i)$ values which are as small or smaller than the value of $P(o_i \mid n, p_i)$ associated with a set of observed cell frequencies. Since $p_1, \cdots, p_k$ are known positive values under H_0, $P(o_i \mid n, p_i)$ is a statistic. If a P-value is sufficiently small (e.g., less than 0.01), then it may be reasonable to reject the null hypothesis (H_0) that the known positive values $p_1, \cdots, p_k$ are true. Efficient programs exist for finding P-values when $k \leq 6$ (Mielke and Berry, 1993). The efficiency of these programs involves (1) a recursive procedure over the $\binom{n+k-1}{k-1}$ distinct configurations for $P(o_i \mid n, p_i)$, (2) initializing the recursive procedure at an arbitrary starting value such as 10^{-200}, and (3) a recursive procedure which depends only on elementary addition, subtraction, multiplication and division operations. The only reason for not obtaining P-values in this manner when $k \geq 7$ is that the amount of computational time becomes excessive. In addition, alternative exact tests may be obtained by summing up all values of $P(o_i \mid n, p_i)$ associated with a selected statistic (e.g., the Pearson χ^2 statistic) which are as or more extreme than the observed value of the selected statistic. However, the choice of $P(o_i \mid n, p_i)$ as the selected statistic seems far more intuitive for the present purpose than any other statistic.

2.2 Approximate tests

Approximate tests in the present nonasymptotic context require the possibility of obtaining the exact mean, variance and skewness of a selected statistic under H_0. For example, consider the broad class of statistics indexed by λ (Cressie and Read, 1984) given by

$$2I^\lambda = \frac{2}{\lambda(\lambda+1)}\sum_{i=1}^{k} o_i\left[\left(\frac{o_i}{e_i}\right)^\lambda - 1\right]$$

where $e_i = np_i$ is the expected value of o_i under H_0 and λ is a real number. While the limit of $2I^\lambda$ is the log-likelihood ratio test for the case when $\lambda \to 0$ and many other well-known tests are included for special cases of λ, only $\lambda = 1$ (where $2I^\lambda$ is the Pearson χ^2 statistic) yields a case where the exact mean, variance and skewness of $2I^\lambda$ are easily obtained under H_0. The exact mean, variance and skewness under H_0 of a statistic outside the previous class (Zelterman, 1987) are also easily obtained. Convenient representations of the Pearson χ^2 and Zelterman statistics (Mielke and Berry, 1988) are respectively given by

$$T = \sum_{i=1}^{k} o_i^2/e_i \quad \text{and} \quad S = \sum_{i=1}^{k} o_i(o_i-1)/e_i.$$

The exact mean, variance and skewness of T under H_0 are respectively given by

$$\mu_T = k+n-1,\ \sigma_T^2 = 2(k-1)+[3-(k-1)^2+\sum_{i=1}^{k} p_i^{-1}]/n, \quad \text{and} \quad \gamma_T = A\,/\,\sigma_T^3$$

where

$$A = 8(k-1)-[2n(3k-2)(3k+8)-2(k+3)(k^2+6k-4)-(22n-3k-22)\sum_{i=1}^{k} p_i^{-1} - \sum_{i=1}^{k} p_i^{-2}]\,/\,n^2.$$

The exact mean, variance and skewness of S under H_0) are respectively given by

$$\mu_S = n-1,\ \sigma_S^2 = 2(n-1)(k-1)\,/\,n, \quad \text{and} \quad \gamma_S = B\,/\,\sigma_S^3$$

where

$$B \;=\; 4(n-1)[2n(k-1) \;-\; 7k+6 \;-\; \sum_{i=1}^{k} p_i^{-1}] \,/\, n^2 .$$

If T_o and S_o denote observed values of T and S, then the respectively approximate P-values given by

$$P(T \geq T_o \;\mid\; H_0) \text{ and } P(S \geq S_o \;\mid\; H_0)$$

utilize the Pearson type III distribution for evaluation. Consequently a Pearson type III distribution P-value approximation is used in a program which depends on the exact mean, variance and skewness of either T or S under H_0 (Berry and Mielke, 1994). In particular, the standardized statistic for T (an analogous expression holds for S) given by

$$Z = (T - \mu_T)/\sigma_T \tag{2.5}$$

is presumed to follow the Pearson type III distribution (a specific gamma distribution) with the density function given by either

$$f(y) = \frac{(-2/\gamma)^{4/\gamma^2}}{\Gamma(4/\gamma^2)}[-(2+y\gamma)/\gamma]^{(4-\gamma^2)/\gamma^2} e^{-2(2+y\gamma)/\gamma^2}$$

where $-\infty < y < -2/\gamma$ and $\gamma < 0$ or

$$f(y) = \frac{(2/\gamma)^{4/\gamma^2}}{\Gamma(4/\gamma^2)}[(2+y\gamma)/\gamma]^{(4-\gamma^2)/\gamma^2} e^{-2(2+y\gamma)/\gamma^2}$$

where $-2/\gamma < y < \infty$ and $\gamma \geq 0$. It should be noted that

$$f(y) = (2\pi)^{-1/2} e^{-y^2/2}$$

if $\gamma = 0$ (i.e., the standard normal distribution). If $Z_o = (T_o - \mu_T)/\sigma_T$, then

$$P(T \geq T_o \mid H_0) \;\doteq\; \int_{Z_o}^{\infty} f(y)dy$$

denotes the approximate P-value which is evaluated numerically using say Simpson's rule over an appropriate finite interval (e.g., Z_o to $Z_o + 15$). The Pearson type III distribution is used to approximate the permutation distribution of Z because it is completely specified by $\gamma = \gamma_T$ (the exact skewness of T) and it includes the normal and chi-squared distributions as special cases (these distributions are asymptotic limits of the permutation distribution for some situations).

A choice between T and S is suggested by calculating the mean of each statistic under an alternative hypothesis (H_1). Suppose the probability that any of the n objects occurs in cell i is $r_i \geq 0\ (i = 1, \cdots, k)$ where $\sum_{i=1}^{k} r_i = 1$. Then the mean of T under H_1 is

$$E[T \mid H_1] = k + n - 1 + \sum_{i=1}^{k} (r_i - p_i)[1 + (n-1)(r_i - p_i)]p_i^{-1}$$

and the mean of S under H_1 is

$$E[S \mid H_1] = n - 1 + (n-1)\sum_{i=1}^{k} (r_i - p_i)^2 p_i^{-1}.$$

If H_0 and H_1 are specifically defined by $p_i = \epsilon/(k-1)$ and $r_i = 0$ for $i = 1, \cdots, k-1, p_k = 1-\epsilon$, and $r_k = 1$ where $0 < \epsilon < 1$, then $E[T \mid H_1] = n/(1-\epsilon) < n+k-1$ when $0 < \epsilon < (k-1)/(n+k-1)$ and $E[S \mid H_1] = (n-1)/(1-\epsilon) > n-1$. While $E[S \mid H_1] > \mu_S$ is always true, the unfortunate fact that $E[T \mid H_1] < \mu_T$ can occur provides a definite basis for choosing S instead of T.

The exact and approximate discrete goodness-of-fit tests considered here are inappropriate for continuous data because (1) the selection of k is subjective, and (2) the placement of $k-1$ partition values are subjective after k is selected. Thus alternative objectively defined goodness-of-fit tests should be used for applications involving continuous data.

3 CONTINGENCY TABLES

Consider an r-way contingency table consisting of $n_1 \times n_2 \times \cdots \times n_r$ cells where the observed frequency of the $(j_1, \cdots, j_r)$th cell is denoted by

$$o_{j_1, \cdots, j_r}$$

and the marginal frequency total associated with subscript j_i of category i is denoted by

$$< i >_{j_i} = \sum_{*\,|\,j_i} o_{j_1,\cdots,j_r}$$

for $j_i = 1,\cdots,n_i$, $i = 1,\cdots,r$, and $\sum_{*\,|\,j_i}$ is the sum over all cells with subscript j_i fixed. Then the frequency total of the entire r-way contingency table is

$$N = \sum_{j_i=1}^{n_i} < i >_{j_i}$$

for $i = 1,\cdots,r$. If

$$p_{j_1,\cdots,j_r} \geq 0$$

is the probability that any of the N total events occurs in the $(j_i,\cdots,j_r)$th cell, then the multinomial probability is given by

$$P\left(o_{j_1,\cdots,j_r}\right) = \left(N\,!\,/\prod_{i=1}^{r}\prod_{j_i=1}^{n_i} o_{j_1,\cdots,j_r}\,!\right)\left(\prod_{i=1}^{r}\prod_{j_i=1}^{n_i} p_{j_1,\cdots,j_r}{}^{o_{j_1,\cdots,j_r}}\right)$$

where $0^0 = 1$ is assumed. The assumed positive marginal probability associated with subscript j_i is given by

$$[i]_{j_i} = \sum_{*\,|\,j_i} p_{j_1,\cdots,j_r}$$

for $j_i = 1,\cdots,n_i$, $i = 1,\cdots,r$, and

$$\sum_{j_i=1}^{n_i}[i]_{j_i} = 1$$

for $i = 1,\cdots,r$. Then the marginal multinomial probability associated with category i is given by

$$P(< i >_{j_i}) = \left(N!\,/\prod_{j_i=1}^{n_i} < i >_{j_i}!\right)\left(\prod_{j_i=1}^{n_i}[i]_{j_i}^{<i>_{j_i}}\right)$$

for $i = 1,\cdots,r$. the null hypothesis (H_0) that the r categories are independent specifies that

$$p_{j_1,\cdots,j_r} = \prod_{i=1}^{r}[i]_{j_i} > 0$$

and the conditional distribution function of the r-way continency table under H_0 is given by

$$P(o_{j_1,\cdots,j_r} \mid <1>_{j_1},\cdots,<r>_{j_r},H_0) = \frac{P(o_{j_1,\cdots,j_r} \mid H_0)}{\prod_{i=1}^{r} P(<i>_{j_i})} .$$

Algebraic manipulation then yields the hypergeometric distribution function given by

$$P(o_{j_1,\cdots,j_r} \mid <1>_{j_1},\cdots,<r>_{j_r},H_0) = \frac{\prod_{i=1}^{r}\prod_{j_i=1}^{n_i} <i>_{j_i}!}{(N!)^{r-1}\prod_{i=1}^{r}\prod_{j_i=1}^{n_i} o_{j_1,\cdots,j_i}!}$$

which is independent of any unknown probabilities under H_0 (Mielke and Berry, 1988). Thus, the marginal totals, $<i>_{j_i}$, are sufficient statistics for the marginal multinomial probabilities, $[i]_{j_i}$, under H_0. This hypergeometric distribution function provides the basis for testing the independence of categories for any r-way contingency table.

3.1 Fisher's exact test

As indicated in Section 2.1, the use of the previously derived conditional distribution function for an r-way contingency table as the primary statistic is far more intuitive than alternative statistics (e.g., the Pearson χ^2 test statistic discussed in Section 3.2). Fisher's exact test's P-value is the probability of having a conditional distribution function equal to or smaller than the observed conditional function under H_0. The practical restriction suggested in Section 2.1 for implementing Fisher's exact test's computational algorithm is to use no more than five conditonal loops. Since the number of conditional loops associated with an r-way contingency table's computational algorithm is the degrees-of-freedom given by

$$\prod_{i=1}^{r} n_i + r - 1 - \sum_{i=1}^{r} n_i ,$$

efficient computational algorithms have been obtained for $2\times2, 3\times2, 4\times2, 5\times2, 6\times2, 3\times3$ and $2\times2\times2$ contingency tables (Mielke and Berry, 1992; Mielke, Berry and Zelterman, 1994; Zelterman, Chan and Mielke, 1995). The efficiency of these programs also depends on recursive procedures over all distinct table configurations which utilize an arbitrary starting value and elementary arithmetic operations as described in Section 2.1 for goodness-of-fit tests. When six or more conditional loops are involved, approximate tests are usually essential since an exact test's computing time may be unreasonable.

3.2 Approximate tests

The only r-way contingency table tests presented here are the Pearson χ^2 and Zelterman tests. As in Section 2.1, a test statistic's exact mean, variance and skewness under H_0 needed to implement the Pearson type III procedure for obtaining P-values are not available for many tests including the log-likelihood ratio test and most of the Cressie and Read (1984) class of tests. The present representations of the Pearson χ^2 and Zelterman test statistics (Mielke and Berry, 1988) are respectively given by

$$T = \sum_{j_1=1}^{n_1} \cdots \sum_{j_r=1}^{n_r} \left(o^2_{j_1,\cdots,j_r} \Big/ \prod_{i=1}^{r} < i >_{j_i} \right)$$

$$S = \sum_{j_1=1}^{n_1} \cdots \sum_{j_r=1}^{n_r} \left(o^{(2)}_{j_1,\cdots,j_r} \Big/ \prod_{i=1}^{r} < i >_{j_i} \right)$$

where

$$c^{(m)} = \prod_{i=1}^{m} (c + 1 - i)\,.$$

Note that $\chi^2 = N^{r-1}T - N$. The exact mean, variance and skewness of T under H_0 (μ_T, σ_T^2 and γ_T) are defined by $\mu_T = E(T)$, $\sigma_T^2 = E(T^2) - \mu_T^2$ and $\gamma_T = [E(T^3) - 3\sigma_T^2\mu_T - \mu_T^3] / \sigma_T^3$ and obtained from the first three moments about the origin under H_0 [$E(T)$, $E(T^2)$ and $E(T^3)$] given by

$$E(T) = \left\{ \prod_{i=1}^{r}(N - n_i) + (N-1)^{r-1} \prod_{i=1}^{r} n_i \right\} \Big/ (N^{(2)})^{r-1}\ ,$$

$$\begin{aligned} E(T^2) = \Bigg[& \prod_{i=1}^{r}(< i >_{4,1} + < i >_{4,2}) + 2N_{1,1}^{r-1} \left\{ 2 \prod_{i=1}^{r} < i >_{3,1} + \prod_{i=1}^{r}(< i >_{3,1} + < i >_{3,2}) \right\} \\ & + N_{2,1}^{r-1} \left\{ 6 \prod_{i=1}^{r} < i >_{2,1} + \prod_{i=1}^{r}(< i >_{2,1} + < i >_{2,2}) \right\} + N_{3,1}^{r-1} \prod_{i=1}^{r} < i >_{1,1} \Bigg] \Big/ N_{4,1}^{r-1}, \end{aligned}$$

$$
\begin{aligned}
E(T^3) \;=\; & \Bigg[\prod_{i=1}^{r}(<i>_{6,3} + 3<i>_{6,4} + <i>_{6,6}) + 3N_{1,2}^{r-1}\Bigg\{4\prod_{i=1}^{r}(<i>_{5,3} + <i>_{5,4}) \\
& \qquad + \prod_{i=1}^{r}(<i>_{5,3} + 2<i>_{5,4} + <i>_{5,5} + <i>_{5,6})\Bigg\} \\
& + N_{2,2}^{r-1}\Bigg\{32\prod_{i=1}^{r}<i>_{4,3} + 18\prod_{i=1}^{r}(<i>_{4,3} + <i>_{4,4}) + 12\prod_{i=1}^{r}(<i>_{4,3} + <i>_{4,5}) \\
& \qquad + 3\prod_{i=1}^{r}(<i>_{4,3} + <i>_{4,4} + 2<i>_{4,5} + <i>_{4,6})\Bigg\} \\
& + N_{3,2}^{r-1}\Bigg\{68\prod_{i=1}^{r}<i>_{3,3} + 3\prod_{i=1}^{r}(<i>_{3,3} + <i>_{3,4}) + 18\prod_{i=1}^{r}(<i>_{3,3} + <i>_{3,5}) \\
& \qquad + \prod_{i=1}^{r}(<i>_{3,3} + 3<i>_{3,5} + <i>_{3,6})\Bigg\} \\
& + N_{4,2}^{r-1}\Bigg\{28\prod_{i=1}^{r}<i>_{2,3} + 3\prod_{i=1}^{r}(<i>_{2,3} + <i>_{2,5})\Bigg\} + N_{5,2}^{r-1}\prod_{i=1}^{r}<i>_{1,3}\Bigg] \Big/ N_{6,2}^{r-1}, \\
& N_{m,1} = \prod_{i=1}^{m}(N+i-4) \quad (m=1,\cdots,4),\; N_{m,2} = \prod_{i=1}^{m}(N+i-6) \quad (m=1,\cdots,6),
\end{aligned}
$$

and, for $i = 1, \cdots, r$,

$$<i>_{m,1} = \sum_{j=1}^{n_i} <i>_j^{(m)} \;/\; <i>_j^2 \;\; (m=1,\cdots,4),\; <i>_{2,2} = n_i^{(2)},$$

$$<i>_{3,2} = (n_i-1)(N-n_i),\; <i>_{4,2} = \sum_{j=1}^{n_i}(<i>_j - 1)(N - <i>_j - n_i + 1),$$

$$<i>_{m,3} = \sum_{j=1}^{n_i} <i>_j^{(m)} \;/\; <i>_j^3 \quad (m=1,\cdots,6),$$

$$<i>_{m,4} = \sum_{j=1}^{n_i} <i>_j^{(m-2)}\,(N - <i>_j - n_i + 1) \,/\, <i>_j^2 \quad (m=3,\cdots,6),$$

$$<i>_{m,5} = (n_i-1)\sum_{j=1}^{n_i} <i>_j^{(m-1)} \;/\; <i>_j^2 \quad (m=2,\cdots,5),$$

$$<i>_{3,6} = n_i^{(3)},\;\; <i>_{4,6} = (n_i-1)(n_i-2)(N-n_i),$$

$$<i>_{5,6} = (n_i-2)\sum_{j=1}^{n_i}(<i>_j - 1)(N - <i>_j - n_i + 1),$$

$$<i>_{6,6} = \sum_{j=1}^{n_i}(<i>_j - 1)(N - <i>_j - n_i + 1)(N - 2<i>_j - n_i + 2).$$

In the same manner μ_T, σ_T^2 and γ_T are obtained, the corresponding values μ_S, σ_S^2 and γ_S are obtained from $E(S), E(S^2)$ and $E(S^3)$ given by

$$E(S) = \prod_{i=1}^{r}(N - n_i) \,/\, (N^{(2)})^{r-1},$$

$$E(S^2) = \left\{\prod_{i=1}^{r}(< i >_{4,1} + < i >_{4,2}) + 4N_{1,1}^{r-1} \prod_{i=1}^{r} < i >_{3,1} + 2N_{2,1}^{r-1} \prod_{i=1}^{r} < i >_{2,1}\right\} \Big/ N_{4,1}^{r-1},$$

$$E(S^3) = \Bigg[\prod_{i=1}^{r}(< i >_{6,3} + 3 < i >_{6,4} + < i >_{6,6}) + 12N_{1,2}^{r-1} \prod_{i=1}^{r}(< i >_{5,3} + < i >_{5,4})$$

$$+N_{2,2}^{r-1}\left\{6\prod_{i=1}^{r}(< i >_{4,3} + < i >_{4,4}) + 32\prod_{i=1}^{r} < i >_{4,3}\right\} + 32N_{3,2}^{r-1}\prod_{i=1}^{r} < i >_{3,3} + 4N_{4,2}^{r-1}\prod_{i=1}^{r} < i >_{2.3}\Bigg] \Big/ N_{6,2}^{r-1}.$$

If T_o and S_o are respectively the observed values of T and S, then the P-values of T and S are respectively given by

$$P(T \geq T_o \mid H_0) \quad \text{and} \quad P(S \geq S_o \mid H_0)$$

again utilize the Pearson type III procedure for evaluation purposes. A computer program based on these results (Berry and Mielke, 1989) yields P-values for both the Pearson χ^2 and Zelterman test statistics (T and S). The argument for choosing S over T near the end of Section 4.1 remains valid. The advantage of S over T is emphatic for large sparse r-way contingency tables (Zelterman, 1987).

3.3 Exact tests for interaction in 2^r tables

Let $p_{i_1 i_2 \cdots i_r}$ denote the probability of the $i_1 i_2 \cdots i_r th$ cell of a 2^r contingency table where index $i_j = 1$ or 2 for $j = 1, \cdots, r$. Marginal probability values are associated with the replacement of one or more indices by a $*$ (i.e., the indices are summed over 1 and 2). For a 2^2 table, the null hypothesis (H_0) that there is no interaction of order 1 is that

$$p_{11}p_{22} = p_{12}p_{21} \quad . \tag{6.1}$$

The H_0 that there is no interaction of order 1 in a 2^2 table is equivalent to the H_0 that the two classifications are independent. For a 2^3 table, the H_0 that there is no interaction of order 2 (Bartlett, 1935) is

$$p_{111}p_{221}p_{122}p_{212} = p_{112}p_{222}p_{121}p_{211} \tag{6.2}$$

and the three H_0's that there is no interaction of order 1 are

$$\begin{aligned} p_{11*}p_{22*} &= p_{12*}p_{21*} \, , \\ p_{1*1}p_{2*2} &= p_{1*2}p_{2*1} \, , \\ p_{*11}p_{*22} &= p_{*12}p_{*21} \, . \end{aligned} \tag{6.3}$$

In general, H_0 that there is no interaction of order $r-1$ in a 2^r table may be obtained recursively from H_0 that there is no interaction of order $r-2$ in a 2^{r-1} table in the following manner. The $1st$ ($2nd$) set of terms on the left side of H_0 that there is no interaction of order $r-1$ in a 2^r table are the left (right) side terms of H_0 that there is no interaction of order $r-2$ in a 2^{r-1} table where a 1 (2) is appended to the right side of each term's subscript. Similarly, the $1st$ ($2nd$) set of terms on the right side of H_0 that there is no interaction of order $r-1$ in a 2^r table are the left (right) side terms of H_0 that there is no interaction of order $r-2$ in a 2^{r-1} table where a 2 (1) is appended to the right side of each term's subscript. As an example, compare the stucture of H_0 that there is no interaction of order 2 in a 2^3 table in equation (6.2) with H_0 that there is no interaction of order 1 in a 2^2 table in equation (6.1). Thus, H_0 that there is no interaction of order 3 in a 2^4 table is obtained from H_0 that there is no interaction of order 2 in a 2^3 table in equation (6.2) and is given by

$$\begin{aligned} &p_{1111}p_{2211}p_{1221}p_{2121}p_{1122}p_{2222}p_{1212}p_{2112} = \\ &p_{1112}p_{2212}p_{1222}p_{2122}p_{1121}p_{2221}p_{1211}p_{2111} \, . \end{aligned} \tag{6.4}$$

The lower order $\binom{r}{j+1}$ H_0's that there is no interaction of order j in a 2^r table are obtained from H_0 that there is no interaction of order j in a 2^{j+1} table. This is accomplished in a manner analogous to constructing the three H_0's that there is no interaction of order 1 in a 2^3 table by inserting a $*$ in each of the $\binom{3}{2} = 3$ distinct positions indicated in (6.3). In general, $r-j-1$ $*$'s are inserted into the $\binom{r}{j+1}$ distinct positions associated with the 2^r table ($j = 1, 2, \cdots, r-2$). As an example, consider the 2^4 table. While the H_0 that there is no interaction of order 3 is given by (6.4), the $\binom{4}{2} = 6$ H_0's that there is no interaction of order 1 are

$$\begin{aligned}
p_{11**}p_{22**} &= p_{12**}p_{21**} \ , \\
p_{1*1*}p_{2*2*} &= p_{1*2*}p_{2*1*} \ , \\
p_{1**1}p_{2**2} &= p_{1**2}p_{2**1} \ , \\
p_{*11*}p_{*22*} &= p_{*12*}p_{*21*} \ , \\
p_{*1*1}p_{*2*2} &= p_{*1*2}p_{*2*1} \ , \\
p_{**11}p_{**22} &= p_{**12}p_{**21} \ ,
\end{aligned}$$

and the $\binom{4}{3} = 4$ H_0's that there is no interaction of order 2 are

$$\begin{aligned}
p_{111*}p_{221*}p_{122*}p_{212*} &= p_{112*}p_{222*}p_{121*}p_{211*} \ , \\
p_{11*1}p_{22*1}p_{12*2}p_{21*2} &= p_{11*2}p_{22*2}p_{12*1}p_{21*1} \ , \\
p_{1*11}p_{2*21}p_{1*22}p_{2*12} &= p_{1*12}p_{2*22}p_{1*21}p_{2*11} \ , \\
p_{*111}p_{*221}p_{*122}p_{*212} &= p_{*112}p_{*222}p_{*121}p_{*211} \ .
\end{aligned} \tag{6.5}$$

The number of distinct interactions associated with a 2^r table is $2^r - r - 1$, the degrees-of-freedom for testing H_0 that the r classifications are mutually independent. The computing time needed to implement $2^r - r - 1$ single loop exact tests for interaction is trivial relative to the computing time needed to implement a corresponding $2^r - r - 1$ loop exact test that the r classifications are mutually independent. Exact tests for the r classifications of a 2^r table being mutually independent have been constructed for r equal to 3 and 4 (Mielke, Berry and Zelterman, 1994; Zelterman, Chan and Mielke, 1995). In addition, exact tests for the $2^r - r - 1$ interactions of a 2^r table have also been constructed for r equal to 3 (Mielke and Berry, 1996). While extensions of exact tests for r classifications being mutually independent appear computationally unfeasible for 2^r tables in all but trivial examples when r is greater than 3, extensions of exact tests concerning interactions of 2^r tables beyond r equal to 3 are easily obtained and computationally feasible.

3.4 Relation between the Pearson χ^2 statistic and the Goodman and Kruskal asymmetric tau statistics

Since the Goodman and Kruskal (1954) asymmetric tau statistics (t_a and t_b) are defined for $r = 2$, let $o_{i,j} = o_{j_1,j_2}$, $n_1 = g$, $n_2 = h$, $G_i = < 1 >_i$ for $i = 1, \cdots, g$ and $H_j = < 2 >_j$ for $j = 1, \cdots, h$. Then the Goodman and Kruskal tau statistics are given by

$$t_a = \left(N \sum_{i=1}^{g} \sum_{j=1}^{h} \frac{o_{i,j}^2}{G_i} - \sum_{j=1}^{h} H_j^2 \right) \Bigg/ \left(N^2 - \sum_{j=1}^{h} H_j^2 \right)$$

and

$$t_b = \left(N \sum_{i=1}^{g} \sum_{j=1}^{h} \frac{o_{i,j}^2}{H_j} - \sum_{i=1}^{g} G_i^2 \right) \Bigg/ \left(N^2 - \sum_{i=1}^{g} G_i^2 \right) .$$

Also the Pearson χ^2 statistic is given by

$$\chi^2 = N \sum_{i=1}^{g} \sum_{j=1}^{h} \frac{o_{i,j}^2}{G_i H_j} - N^2 .$$

The above expressions for t_a, t_b and χ^2 imply that (i) t_a and χ^2 are equivalent if $H_j = N/h$ for $j = 1, \cdots, h$ and (ii) t_b and χ^2 are equivalent if $G_i = N/g$ for $i = 1, \cdots, g$. Therefore, t_a, t_b and χ^2 are equivalent if $G_i = N/g$ and $H_j = N/h$ for $i = 1, \cdots, g$ and $j = 1, \cdots, h$. It is also true that (iii) t_a and χ^2 are equivalent if $h = 2$ and (iv) t_b and χ^2 are equivalent if $g = 2$. Thus, t_a, t_b and χ^2 are equivalent if $g = h = 2$. In general, t_a, t_b and χ^2 are not equivalent. As obtained for χ^2 in Section 3.2, the exact mean, variance and skewness of t_a and t_b have also been obtained under their respective null hypotheses and used in nonasymptotic tests of significance based on the Pearson type III distribution (Berry and Mielke, 1985, 1986).

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