Comparison between the Chernoff and factorial moment bounds for discrete random variables

Philippe Naveau

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Abstract

The purpose of this note is to establish the superiority of the factorial moment bound over Chernoff's bound for non negative integer-valued random variables.

Introduction Classically, three candidates are used as an upper bound to the survivor probability $P(X \geq t)$, $t > 0$, of a non-negative discrete random variable X with distribution function F_X.

One of them is the famous *Chernoff's bound* defined by

$$\mathcal{C}(t) = \inf_{\theta \geq 0} M_X(\theta) \exp(-\theta t)$$

where $M_X(\theta) = \int e^{\theta x} dF_X(x)$ is the moment generating function. We will compare the accuracy of $\mathcal{C}(t)$ with those of the *moment bound* and the *factorial moment bound* defined respectively (assuming the moments are well defined) by

$$\mathcal{M}(t) = \inf_{n \geq 0} \frac{E(X^n)}{t^n}$$

and

$$\mathcal{F}(t) = \inf_{0 \leq n \leq t-1} \frac{E(X(X-1)...(X-n))}{t(t-1)...(t-n)}$$

for n non negative integer.

These three bounds are derived from the Markov's inequality. The factorial bound works only for integer-valued random variables.

Philips and Nelson [1] showed that for positive random variables, the moment bound is better than the Chernoff's. They also suggest that for integer-valued random variable, the factorial bound is more efficient than the Chernoff's, and the goal of this note is to prove that their conjecture is correct.

For non negative discrete variables the factorial moment bound is thus an improvement compared to the Chernoff's bound. A remaining problem is the comparison between the moment and factorial bounds. In practical cases, the first one is often much complicated to compute than the second one. Examples in specific cases are given in details in [1].

Comparison of the bounds The following theorem establishes the relationship on the bounds, as conjectured in [1].

Theorem For any non negative discrete random variable, the following inequalities hold for every positive t

$$P(X \geq t) \leq \mathcal{F}(t) \leq \mathcal{C}(t).$$

*Philippe Naveau is a PhD student at the Department of Statistics of Colorado State University, Fort-Collins CO 80524. The author thanks Richard Tweedie for helpful comments on this note.

Proof The first inequality is a direct application of the Markov's inequality (see [2]).

Now, suppose that t and α are positive numbers. The two sequences u_i and v_i , which depend on α are defined by

$$\begin{aligned} u_0 &= v_0 = 1 \\ u_i &= \frac{\alpha^i}{i!} E(X(X-1)...(X-i+1)) \\ v_i &= \frac{\alpha^i}{i!} t(t-1)...(t-i+1) \end{aligned}$$

for $i = 1, 2, ...$

By the equality $(1+\alpha)^t = \Sigma v_i$ and the existence of $\mathcal{F}(t)$, it follows that both summations Σu_i and Σv_i are convergent for some $\alpha > 0$. Since $(1+\alpha)^t > 0$, it is also possible to define $\mathcal{I}(.)$ by

$$\mathcal{I}(t) = \inf_{\alpha \geq 0} \frac{E((1+\alpha)^X)}{(1+\alpha)^t} = \inf_{\alpha \geq 0} \frac{\Sigma u_i}{\Sigma v_i}.$$

The sequence u_i is non negative since $h_n(x) = x(x-1)...(x-n)$ is a positive function when x takes integer value, the sequence v_i is negative when $i \geq t+1$. Hence, for every positive integer i

$$v_i \inf_{0 \leq n \leq t-1} \frac{E(X(X-1)...(X-n))}{t(t-1)...(t-n)} \leq u_i.$$

Summing each side of the last inequality gives $\mathcal{F}(t)\Sigma v_i \leq \Sigma u_i$. That implies $\mathcal{F}(t) \leq \mathcal{I}(t)$.

The last step is to prove that $\mathcal{I}(t) = \mathcal{C}(t)$. This is immediate using the new non negative number α define by $1 + \alpha = e^\theta$. Hence, we note that

$$\begin{aligned} \mathcal{F}(t) &= \inf_{\theta \geq 0} E(\exp(\theta X)) \exp(-\theta t) \\ &= \inf_{\alpha \geq 0} \frac{E((1+\alpha)^X)}{(1+\alpha)^t} \\ &= \mathcal{I}(t). \end{aligned}$$

That gives the required result.

Note When a discrete variable can be negative, the inequalities in the theorem can be replaced by

$$\begin{aligned} P(|X| \geq t) &\leq \inf_{0 \leq n < t} \frac{E(|X|(|X|-1)...(|X|-n))}{t(t-1)...(t-n)} \\ &\leq \inf_{\alpha \geq 0} \frac{E((1+\alpha)^{|X|})}{(1+\alpha)^t}. \end{aligned}$$

References [1] Philips T. and Nelson R. (1995) *The moment bound is tighter than Chernoff's bound for positive tail probabilities*, the American Statistician, Vol. 49, number 2, may 1995.

[2] Bucklew, J.A. (1990), *Large deviation techniques in decision, simulation and estimation*, New York: John Wiley.