A Recipe for Constructing Generalized Pivotal Quantities and Generalized Confidence Intervals

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Weerahandi (1993) introduced the idea of Generalized Pivotal Quantities and Generalized Confidence Intervals and derived confidence interval procedures for many problems where exact nontrivial frequentist intervals are unavailable. During the past few years generalized confidence intervals have been used by many authors to obtain useful inference procedures in nonstandard problems. Although Weerahandi (1993) provided a number of examples illustrating the application of GPQs, he did not provide a systematic approach for finding them. More recently, Chiang (2001) introduced an approach for constructing confidence intervals for functions of variance components in balanced mixed effects linear models. He called the approach the "method of surrogate variables." As Mathew and Iyer (2002) point out, Chiang's intervals are exactly the same as generalized confidence intervals. Nevertheless, Chiang did outline a systematic approach for constructing generalized pivotal quantities for functions of variance components in balanced mixed linear model problems. In this paper we put forward a general recipe for the construction of generalized pivotal quantities and generalized confidence intervals and illustrate its application through a number of examples. In particular, it is seen that Chiang's approach is a special case of our recipe.

1. Introduction

Tsui and Weerahandi (1989) introduced the concept of generalized P-values and generalized test variables and demonstrated how to use them to derive hypothesis testing procedures for situations where exact tests are unavailable. They noted that sometimes their approach resulted in the same tests as did a Bayesian approach with suitably selected priors. They further noted that, for the Behrens-Fisher problem, their approach yielded a solution identical with the solution put forward by Behrens and Fisher. See Behrens (1929) and Fisher (1939). Weerahandi (1993) extended the idea of generalized P-values to that of Generalized Pivotal Quantities (GPQ) and Generalized Confidence Intervals (GCI) and derived confidence interval procedures for many problems where exact nontrivial frequentist intervals are unavailable. Again, he observed that, for the Behrens-Fisher problem, the generalized confidence interval he derived turned out to be the same as the Behrens-Fisher interval. Additional results on GCIs may be found in Weerahandi's book (1995).

More recently, Chiang (2001) introduced an approach for constructing confidence intervals for functions of variance components in balanced mixed effects linear models. He called the approach the "method of surrogate variables." It turns out that Chiang's intervals are exactly the same as GCIs – see Mathew and Iyer (2002). In fact, using the terminology of generalized confidence intervals and generalized pivotal quantities, Weerahandi (1991, 1993, 1995), Zhou et al. (1994),

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and Khuri, Mathew, and Sinha (1998) had already discussed some of the variance components problems addressed by Chiang (2001). Nevertheless, it is important to note that Chiang did outline a systematic approach for constructing generalized pivots for functions of variance components in balanced mixed linear model problems. On the other hand, earlier authors simply presented GPQs for specific problems without outlining a systematic approach.

During the past few years, the idea of GCIs and generalized tests have been used by others to obtain useful inference procedures in nonstandard problems. See, for instance, Hamada and Weerahandi (2000), Chang and Huang (2000), and McNally et. al. (2001). Although Weerahandi (1993) provided a number of examples illustrating the application of GPQs, he did not provide a systematic approach for finding them. As a matter of fact, he says,

> "The problem of finding an appropriate pivotal quantity is a nontrivial task..."

He further states,

> "...Further research is necessary to develop simple methods of constructing generalized pivotals for classes of general problems, and this is beyond the scope of this article."

While Chiang (2001) did propose a systematic approach for constructing GPQs for functions of variance components in balanced mixed linear models, a general method for constructing GPQs is as yet unavailable in the literature and each particular problem appears to require some ingenuity in constructing an appropriate GPQ or a test variable. In this paper we provide a simple recipe for the construction of GPQs which can be used to obtain GCIs or generalized tests for a large class of practical problems. Most, if not all, published GCIs and generalized tests can be routinely constructed using this recipe. The utility of the general recipe is illustrated by constructing GPQs for problems that have not yet been considered by others either from the point of view of GCIs or that of surrogate variables.

Section 2 gives the statement of our main result along with a proof. Section 3 contains a collection of examples, some from the literature and some that have not yet appeared in published work. The final section contains some general comments on the subject of GCIs.

2. Main Result

We first introduce some notation. Let $\boldsymbol{D}$ denote the observable data vector whose distribution is indexed by a k-dimensional parameter $\boldsymbol{\xi} = (\xi_1, \cdots, \xi_k) \in \boldsymbol{\Omega} \subseteq R^k$. Let $\theta = h(\xi_1, \cdots, \xi_k)$ be a scalar function of $\boldsymbol{\xi}$ for which a confidence interval is required. Let $\boldsymbol{d}$ denote the observed value of $\boldsymbol{D}$. In subsequent discussions, $\boldsymbol{d}$ will be regarded as a vector of known constants, and any reference to the distribution of $\boldsymbol{D}$ will be to its unconditional distribution, ignoring the fact that $\boldsymbol{d}$, a realization of $\boldsymbol{D}$, is available.

We now state and prove our main result.

Proposition 1

Suppose conditions (a) and (b) given below hold.

(a) There exist functions $f_1, \ldots, f_k$, with $f_j : R^k \times R^k \to R$, such that, if we define $U_1, \ldots, U_k$ by $U_i = f_i(\boldsymbol{D}; \boldsymbol{\xi})$, for $i = 1, \ldots, k$, then $\boldsymbol{U} = (U_1, \ldots, U_k)^t$ has a joint distribution that is free of $\boldsymbol{\xi}$.

(b) For each $\boldsymbol{D}$, there exists a mapping $\boldsymbol{g}(\boldsymbol{D};\cdot): R^k \to R^k$

$$\boldsymbol{g}(\boldsymbol{D};\cdot) = (g_1(\boldsymbol{D};\cdot), \ldots, g_k(\boldsymbol{D};\cdot))$$

such that $((g_1(\boldsymbol{D};\boldsymbol{f}(\boldsymbol{D};\boldsymbol{\xi})), \ldots, g_k(\boldsymbol{D};\boldsymbol{f}(\boldsymbol{D};\boldsymbol{\xi})) = \boldsymbol{\xi}$, i.e., $\boldsymbol{g}(\boldsymbol{D};\boldsymbol{f}(\boldsymbol{D};\boldsymbol{\xi})) = \boldsymbol{\xi}$, where $\boldsymbol{f} = (f_1, \ldots, f_k)$. Define

$$\begin{aligned} R &= R(\boldsymbol{D};\boldsymbol{d},\boldsymbol{\xi}) = h(g_1(\boldsymbol{d};\boldsymbol{f}(\boldsymbol{D};\boldsymbol{\xi})), \ldots, g_k(\boldsymbol{d};\boldsymbol{f}(\boldsymbol{D};\boldsymbol{\xi}))) \\ &= h(g_1(\boldsymbol{d};\boldsymbol{U}), \ldots, g_k(\boldsymbol{d};\boldsymbol{U})) \end{aligned}$$

Then the following hold.

(1) R is a Generalized Pivotal Quantity for $\theta = h(\boldsymbol{\xi})$.

(2) $R_{\alpha/2} \leq \theta \leq R_{1-\alpha/2}$ is an equal-tailed 2-sided GCI for θ. (One-sided generalized confidence bounds are obtained in an obvious manner).

(3) $T = T(\boldsymbol{D};\boldsymbol{d},\boldsymbol{\xi}) = h(\boldsymbol{\xi}) - R = \theta - R$, is a Generalized Test Variable for testing $H_0 : \theta \leq \theta_0$ versus $H_0 : \theta > \theta_0$, with the corresponding generalized P-value given by

$$GPV = Pr\left[T(\boldsymbol{D};\boldsymbol{d},\boldsymbol{\xi}) \geq 0 \mid \theta = \theta_0\right]$$

Proof: The distribution of R in the above proposition is completely determined by the joint distribution of $\boldsymbol{U} = (U_1, \ldots, U_k)$ and the constants $\boldsymbol{d}$ (observed data), and so is free of $\boldsymbol{\xi}$. Furthermore, substituting the observed data $\boldsymbol{d}$ for $\boldsymbol{D}$ in R gives

$$R = h(g_1(\boldsymbol{d};\boldsymbol{f}(\boldsymbol{d};\boldsymbol{\xi})), \ldots, g_k(\boldsymbol{d};\boldsymbol{f}(\boldsymbol{d};\boldsymbol{\xi}))) = h(\boldsymbol{g}(\boldsymbol{d};\boldsymbol{f}(\boldsymbol{d};\boldsymbol{\xi}))) = h(\boldsymbol{\xi}) = \theta.$$

This establishes (1).

The distribution of T, by construction, depends on $\boldsymbol{\xi}$ only through θ. Also, the observed value t of T is 0, and so is completely free of $\boldsymbol{\xi}$. Finally, $Pr\left[T(\boldsymbol{D};\boldsymbol{d},\boldsymbol{\xi}) \geq 0\right] = Pr\left[R(\boldsymbol{D};\boldsymbol{d},\boldsymbol{\xi}) \leq \theta\right]$ is clearly nondecreasing in θ. Thus, all three of Weerahandi's (1987) conditions for T to be a generalized test variable are satisfied. The rest of Proposition 1 is now obvious.

Remark 1. Generally speaking, the required percentiles of R cannot be obtained in closed form, but may be estimated using Monte-carlo methods or analytical approximations.

Remark 2. In many applications, the random variables $U_1, \ldots, U_k$ are mutually independent. For instance, the independence of the U_i holds in the cases considered by Chiang (2001). However, the stated procedure for obtaining generalized confidence intervals or tests is valid even when the U_i are not independent.

3. Applications of the Main Result

We illustrate the application of Proposition 1 by considering several examples. The first example demonstrates that the recipe will sometimes lead to the usual frequentist solution.

Example 1. The one-sample t-test

Suppose X and V are independent random variables with $X \sim N(\mu, c^2\sigma^2)$ and $\nu V/\sigma^2 \sim \chi^2_\nu$ where μ and σ are unknown constants ($\sigma > 0$). The pair (X, V) may be viewed as the sample mean and the sample variance (complete, sufficient statistics) from an i.i.d $N(\mu, \sigma^2)$ sample of size $n = \nu + 1$. Write $\xi_1 = \mu$, and $\xi_2 = \sigma$ in the proposition. Consider the functions f_1, f_2 defined by

$$\begin{aligned} U_1 &= f_1(X, V; \mu, \sigma) = \frac{X - \mu}{c\,\sigma} \\ U_2 &= f_2(X, V; \mu, \sigma) = \frac{\nu\, V}{\sigma^2}. \end{aligned}$$

Clearly, U_1 and U_2 are independent. Thus, the joint distribution of (U_1, U_2) is free of any model parameters. Inverting the functions f_1 and f_2, we get

$$\begin{aligned} \xi_1 &= \mu = g_1(X, V; U_1, U_2) = X - cU_1\sqrt{\frac{f\,V}{U_2}} \\ \xi_2 &= \sigma = g_2(X, V; U_1, U_2) = \sqrt{\frac{f\,V}{U_2}} \end{aligned}$$

We now derive a generalized $(1-\alpha)$ confidence interval for μ. Taking $h(\xi_1, \xi_2) = \mu$ in Proposition 1, we obtain the generalized pivotal quantity R given by

$$\begin{aligned} R &= h(g_1(x, v; U_1, U_2), g_2(x, v; U_1, U_2)) = g_1(x, v; U_1, U_2) \\ &= x - cU_1\sqrt{\frac{f\,v}{U_2}} = \left(x - \frac{X - \mu}{\sqrt{V/v}}\right) \end{aligned}$$

where x and v are the observed values of X and V, respectively. A $(1-\alpha)$ generalized confidence interval for μ is given by $[L, U]$ where $L = R_{\alpha/2}$ is the $\alpha/2$ quantile and $U = R_{1-\alpha/2}$ is the $1-\alpha/2$ quantile of the distribution of R. In this simple situation the distribution of R is easily determined. In fact,

$$\frac{x - R}{c\sqrt{v}} = \frac{U_1}{\sqrt{U_2/\nu}}$$

which has a t-distribution with ν degrees of freedom. Therefore,

$$Pr\left[t_{\alpha/2} \le \frac{x - R}{c\sqrt{v}} \le t_{1-\alpha/2}\right] = 1 - \alpha,$$

from which we get

$$Pr[x - ct_{1-\alpha/2}\sqrt{v} \le R \le x - ct_{\alpha/2}\sqrt{v}] = 1 - \alpha.$$

Hence, $R_{\alpha/2} = x - ct_{1-\alpha/2}\sqrt{v} = L$ and $R_{1-\alpha/2} = x - ct_{\alpha/2}\sqrt{v} = U$. As a result, the $(1-\alpha)$ generalized confidence interval for μ reduces to the standard $(1-\alpha)$ confidence interval for μ.

Suppose we wish to test $H_0 : \mu \leq \mu_0$ versus the alternative $H_a : \mu > \mu_0$. According to the proposition, a generalized test variable for testing the above pair of hypotheses is given by

$$\begin{aligned} T(\boldsymbol{D}, \boldsymbol{d}, \boldsymbol{\xi}) &= \theta - R = \mu - \left(x - cU_1 \sqrt{\frac{\nu\, v}{U_2}} \right) \\ &= \mu - \left(x - \frac{X - \mu}{\sqrt{V/v}} \right) \end{aligned}$$

The generalized P-value is given by

$$\begin{aligned} p &= Pr[T(\boldsymbol{D}, \boldsymbol{d}, \boldsymbol{\xi} | \theta = \theta_0) > 0] \\ &= Pr[T(\boldsymbol{D}, \boldsymbol{d}, \boldsymbol{\xi} | \mu = \mu_0) > 0] \\ &= Pr[x - cU_1 \sqrt{\frac{\nu\, v}{U_2}} < 0] \\ &= Pr\left[\frac{U_1}{\sqrt{U_2/\nu}} > \frac{x - \mu_0}{c\sqrt{v}} \right] \\ &= 1 - G\left(\frac{x - \mu_0}{c\sqrt{v}} \right) \end{aligned}$$

where G is the cdf of a Student's-t random variable with ν degrees of freedom. Thus we see that the generalized P-value here coincides with the usual P-value using a t-test.

Example 2. The ratio of two normal means

The problem of determining confidence bounds for the ratio of means of two normal distributions has received a great deal of attention in the literature. Some of the early work on this problem may be found in Fieller (1940, 1944) and M. Creasy (1954). 'Fieller's method' is used nowadays in a variety of problems involving confidence bounds for ratios of parameters.

Suppose $(X_1, Y_1), \ldots, (X_n, Y_n)$ is an i.i.d sample from a bivariate normal distribution with mean $\boldsymbol{\mu}$ and covariance matrix $\boldsymbol{\Sigma}$, where

$$\boldsymbol{\mu} = \begin{pmatrix} \mu_x \\ \mu_y \end{pmatrix} \text{ and } \boldsymbol{\Sigma} = \begin{pmatrix} \sigma_{xx} & \sigma_{xy} \\ \sigma_{xy} & \sigma_{yy} \end{pmatrix}$$

Let

$$\boldsymbol{M} = \begin{pmatrix} \bar{x} \\ \bar{y} \end{pmatrix}$$

denote the sample mean vector. The case where $\sigma_{xy} = 0$ and $\sigma_{xx} = \sigma_{yy} = \sigma^2$ ($\sigma > 0$ unknown) is one of the cases considered by Creasy (1954) where she derived a fiducial interval for the parameter $\theta = \mu_x/\mu_y$. For this case, we will use Proposition 1 and derive a generalized confidence interval for θ. The reader may verify that the solution obtained here is the same as the fiducial solution derived by Creasy (1954).

We use the notation $\xi_1 = \mu_x$, $\xi_2 = \mu_y$, and $\xi_3 = \sigma$. Then $\theta = h(\xi_1, \xi_2, \xi_3) = \xi_1/\xi_2 = \mu_x/\mu_y$. Define the following statistics:

$$\bar{X} = \frac{1}{n}\sum_{i=1}^{n} X_i\,, \quad \bar{Y} = \frac{1}{n}\sum_{i=1}^{n} Y_i\,, \quad V = \frac{\sum_{i=1}^{n}(X_i - \bar{X})^2 + \sum_{i=1}^{n}(Y_i - \bar{Y})^2}{2(n-1)}\,.$$

It can be easily verified that $\{\bar{X},\ \bar{Y},\ V\}$ is a set of complete, sufficient statistics for $\boldsymbol{\xi}$. We let $\nu = 2(n-1)$, $\boldsymbol{D} = (\bar{X}, \bar{Y}, V)$, and define

$$\begin{aligned} U_1 &= f_1(\boldsymbol{D}; \boldsymbol{\xi}) = \frac{\bar{X} - \mu_x}{\sigma/\sqrt{n}} \\ U_2 &= f_2(\boldsymbol{D}; \boldsymbol{\xi}) = \frac{\bar{Y} - \mu_y}{\sigma/\sqrt{n}} \\ U_3 &= f_3(\boldsymbol{D}; \boldsymbol{\xi}) = \frac{\nu\, V}{\sigma^2} \end{aligned}$$

Then U_1, U_2, U_3 are jointly independent and have the following distributions.

$$U_1 \sim N(0,1), \qquad U_2 \sim N(0,1), \qquad U_3 \sim \chi^2_\nu.$$

We can express $\boldsymbol{\xi} = (\mu_x, \mu_y, \sigma)$ in terms of $\boldsymbol{D}$ and $\boldsymbol{U}$ as follows.

$$\begin{aligned} \mu_x &= \bar{X} - U_1\sqrt{\frac{\nu\, V}{n\, U_3}} = g_1(\boldsymbol{D}; \boldsymbol{U}) \\ \mu_y &= \bar{Y} - U_2\sqrt{\frac{\nu\, V}{n\, U_3}} = g_2(\boldsymbol{D}; \boldsymbol{U}) \\ \sigma &= \sqrt{\frac{\nu\, V}{U_3}} = g_3(\boldsymbol{D}; \boldsymbol{U}) \end{aligned}$$

Let $\boldsymbol{d} = (\bar{x}, \bar{y}, v)$ be the observed value of $\boldsymbol{D}$. A generalized pivotal quantity for θ is given by

$$\begin{aligned} R &= h(g_1(\boldsymbol{d}; \boldsymbol{U}), g_2(\boldsymbol{d}; \boldsymbol{U}), g_3(\boldsymbol{d}; \boldsymbol{U})) = g_1(\boldsymbol{d}; \boldsymbol{U})/g_2(\boldsymbol{d}; \boldsymbol{U}) \\ &= \frac{\bar{x} - U_1\sqrt{\dfrac{\nu\, v}{n\, U_3}}}{\bar{y} - U_2\sqrt{\dfrac{\nu\, v}{n\, U_3}}} = \frac{\bar{x} - (\bar{X} - \mu_x)\sqrt{\dfrac{v}{V}}}{\bar{y} - (\bar{Y} - \mu_y)\sqrt{\dfrac{v}{V}}} \end{aligned}$$

The distribution of R may be obtained by observing that, conditional on V, R is distributed as the ratio of two independent normal random variables, with the numerator mean equal to $\bar{x}$, the denominator mean equal to $\bar{y}$, and both the numerator and the denominator having a common variance equal to $\dfrac{\sigma^2\, v}{n\, V}$. The distribution of a ratio of normal random variables has been studied by Geary (1930), Fieller (1932), and by Nicholson (1941). Let us denote the conditional density of R, given $V = v_0$, by $p_{R|V}(r|v_0; \bar{x}, \bar{y}, v)$. The conditional density $p_{R|V}$ depends not only on v_0, but also on the quantities following the semicolon ';', namely, $\bar{x}$, $\bar{y}$, and v. Finally, the unconditional density of R, denoted by p_R, is given by

$$p_R(r; \bar{x}, \bar{y}, v) = \int_0^\infty p_{R|V}(r|v_0; \bar{x}, \bar{y}, v)\chi^2_\nu(v_0)dv_0.$$

Here $\chi^2_\nu(w)$ represents the pdf of a chi-squared random variable with ν degrees of freedom evaluated at the point w. While it is possible to use numerical integration methods or series approximation methods to obtain the percentiles of the distribution of R, they are perhaps most conveniently estimated using Monte-carlo methods. The reader can verify that the bounds for ξ proposed by Creasy (1954) are precisely those obtained by considering the percentiles of R.

Our next illustration is the well known Behrens-Fisher problem.

Example 3. The Behrens-Fisher problem

Suppose $X \sim N(\mu_1, c_1^2\sigma_1^2)$, $Y \sim N(\mu_2, c_2^2\sigma_2^2)$, $\nu_1 V_1/\sigma_1^2 \sim \chi^2(\nu_1)$, and $\nu_2 V_2/\sigma_2^2 \sim \chi^2(\nu_2)$. Here c_1, c_2, ν_1, ν_2 are known constants ($c_1 > 0$, $c_2 > 0$) and $\mu_1, \mu_2, \sigma_1, \sigma_2$ are unknown parameters. The statistics X, Y, V_1, V_2 form a set of complete sufficient statistics for $\boldsymbol{\xi} = (\mu_1, \mu_2, \sigma_1, \sigma_2)$. The problem is to obtain confidence bounds for the difference $\theta = \mu_1 - \mu_2$.

It is now common knowledge that a nontrivial exact confidence interval, in the frequentist sense, is unavailable for this problem. See, for instance, Linnik (1968). A solution to this problem was put forward by Behrens (1929) and later, Fisher (1939) showed that Behren's procedure could be derived very simply using the fiducial argument. The fiducial distribution of $\xi = \mu_1 - \mu_2$ is now referred to as the Behrens-Fisher distribution. Critical values for this distribution have been tabulated by Sukhathme (1958). The quantiles of this distribution lead to the lower and upper confidence bounds for $\mu_1 - \mu_2$.

Many other approximate confidence interval procedures for this problem have been discussed in the literature – see, for instance, Welch (1947), Satterthwaite (1942, 1946), Cochran (1964), and Graybill and Wang (1980). Weerahandi (1993) derived a generalized confidence interval for $\mu_1 - \mu_2$ and showed that it is the same as the Behrens-Fisher solution. His procedure is obtained by starting with the statistics $X - Y$, V_1, and V_2. Weerahandi jusifies this based on invariance arguments.

Let $\boldsymbol{D} = (X - Y, V_1, V_2)$, and $\boldsymbol{\xi} = (\mu_1 - \mu_2, \sigma_1, \sigma_2) = (\theta, \sigma_1, \sigma_2)$. Then define,

$$\begin{aligned}
U_1 &= f_1(\boldsymbol{D}; \boldsymbol{\xi}) = \frac{(X - Y) - \theta}{\sqrt{c_1^2\sigma_1^2 + c_2^2\sigma_2^2}} \\
U_2 &= \frac{\nu_1 V_1}{\sigma_1^2} \\
U_3 &= \frac{\nu_2 V_2}{\sigma_2^2}.
\end{aligned}$$

From this we obtain,

$$\begin{aligned}
\xi_1 &= \theta = \mu_1 - \mu_2 = g_1(\boldsymbol{D}; \boldsymbol{U}) \\
&= (X - Y) - U_1\sqrt{c_1^2\frac{\nu_1 V_1}{U_2} + c_2^2\frac{\nu_2 V_2}{U_3}} \\
\xi_2 &= \sigma_1 = g_2(\boldsymbol{D}; \boldsymbol{U}) = \sqrt{\frac{\nu_1 V_1}{U_2}} \\
\xi_3 &= \sigma_2 = g_3(\boldsymbol{D}; \boldsymbol{U}) = \sqrt{\frac{\nu_2 V_2}{U_3}}.
\end{aligned}$$

With $\theta = h(\boldsymbol{\xi}) = \theta_1$, the generalized pivot is

$$\begin{aligned} R_1 &= (x-y) - U_1\sqrt{c_1^2\frac{\nu_1 v_1}{U_2} + c_2^2\frac{\nu_2 v_2}{U_3}} \\ &= (x-y) - \frac{(X-Y)-\xi}{\sqrt{c_1^2\sigma_1^2 + c_2^2\sigma_2^2}}\sqrt{\frac{c_1^2\sigma_1^2 v_1}{V_1} + \frac{c_2^2\sigma_2^2 v_2}{V_2}} \\ &= (x-y) - R \end{aligned}$$

where x, y, v_1, v_2 are the observed values of X, Y, V_1, V_2, respectively, and

$$R = U_1\sqrt{c_1^2\frac{\nu_1 v_1}{U_2} + c_2^2\frac{\nu_2 v_2}{U_3}} = \frac{(X-Y)-\xi}{\sqrt{c_1^2\sigma_1^2 + c_2^2\sigma_2^2}}\sqrt{\frac{c_1^2\sigma_1^2 v_1}{V_1} + \frac{c_2^2\sigma_2^2 v_2}{V_2}}.$$

The quantity R is the same as the R in equation (3.4) of Weerahandi (1993). Let R_γ denote the γ-percentile of R. A $1-\alpha$ generalized confidence interval for $\mu_1 - \mu_2$ is $[L, U]$, where

$$L = (x-y) - R_{1-\alpha/2} \text{ and } U = (x-y) - R_{\alpha/2}. \tag{1}$$

We now show that an identical GCI is obtained by applying the Proposition to the vector $\boldsymbol{D} = (\bar{X}, \bar{Y}, V_1, V_2)$ rather than the vector $(\bar{X} - \bar{Y}, V_1, V_2)$. Let $\boldsymbol{\xi} = (\mu_1, \mu_2, \sigma_1, \sigma_2)$. Further define,

$$U_1 = \frac{X - \mu_1}{c_1\sigma_1},\ U_2 = \frac{Y-\mu_2}{c_2\sigma_2},\ U_3 = \frac{\nu_1 V_1}{\sigma_1^2},\ U_4 = \frac{\nu_2 V_2}{\sigma_2^2}.$$

Clearly, U_1, U_2, U_3, U_4 are jointly independent and have the following respective distributions.

$$U_1 \sim N(0,1),\ U_2 \sim N(0,1),\ U_3 \sim \chi^2(\nu_1),\ \text{and } U_4 \sim \chi^2(\nu_2).$$

Expressing $\boldsymbol{\xi}$ in terms of $\boldsymbol{D}$ and $\boldsymbol{U}$ we get,

$$\mu_1 = g_1(\boldsymbol{D};\boldsymbol{U}) = X - c_1 U_1\sqrt{\frac{\nu_1 V_1}{U_3}}, \qquad \mu_2 = g_2(\boldsymbol{D};\boldsymbol{U}) = Y - c_2 U_2\sqrt{\frac{\nu_2 V_2}{U_4}}$$

$$\sigma_1 = g_3(\boldsymbol{D};\boldsymbol{U}) = \sqrt{\frac{\nu_1 V_1}{U_3}}, \qquad \text{and} \qquad \sigma_2 = g_4(\boldsymbol{D};\boldsymbol{U}) = \sqrt{\frac{\nu_2 V_2}{U_4}}.$$

Since $\theta = h(\boldsymbol{\xi}) = \mu_1 - \mu_2$, a generalized pivotal quantity R_2 for θ is given by

$$\begin{aligned} R_2 &= \left(x - c_1 U_1\sqrt{\frac{\nu_1 v_1}{U_3}}\right) - \left(y - c_2 U_2\sqrt{\frac{\nu_2 v_2}{U_4}}\right) \\ &= (x-y) - \left(c_1 U_1\sqrt{\frac{\nu_1 v_1}{U_3}} - c_2 U_2\sqrt{\frac{\nu_2 v_2}{U_4}}\right) \\ &= (x-y) - \left((X-\mu_1)\sqrt{\frac{v_1}{V_1}} - (Y-\mu_2)\sqrt{\frac{v_2}{V_2}}\right) \\ &= (x-y) - R' \end{aligned}$$

where

$$\begin{aligned} R' &= c_1 U_1 \sqrt{\frac{\nu_1 v_1}{U_3}} - c_2 U_2 \sqrt{\frac{\nu_2 v_2}{U_4}} \\ &= (X - \mu_1)\sqrt{\frac{v_1}{V_1}} - (Y - \mu_2)\sqrt{\frac{v_2}{V_2}}. \end{aligned}$$

Then the distribution of R' is free of any model parameters. Let R'_γ denote the γ-percentile of the distribution of R'. Then we have

$$Pr[(x-y) - R'_{1-\alpha/2} \le R_2] = \alpha/2$$

and

$$Pr[R_2 \le (x-y) - R'_{\alpha/2}] = \alpha/2.$$

Therefore, a $(1-\alpha)$ generalized confidence interval for $\mu_1 - \mu_2$ is given by $[L, U]$, where

$$L = (x-y) - R'_{1-\alpha/2} \text{ and } U = (x-y) - R'_{\alpha/2}. \tag{2}$$

Note that the distribution of R' is that of a linear combination of two independent Student's-t random variables.

The quantity R' defined in Weerahandi (1993, page 902) is the same as R' defined above. Weerahandi (1993) recognizes R' as a generalized pivotal quantity for $\mu_1 - \mu_2$, but states that it is not invariant under location changes of data, and does not consider it any further. On closer examination, one can see that Weerahandi's generalized pivotal quantity R' (Weerahandi, 1993, page 902) leads to same confidence interval for $\mu_1 - \mu_2$ as does the one based on his R (Weerahandi, 1993, equation 3.4). This is seen by observing that the conditional distribution of R given V_1, V_2 is normal with mean $x - y$ and variance $\dfrac{c_1^2\sigma_1^2 v_1}{V_1} + \dfrac{c_2^2\sigma_2^2 v_2}{V_2}$. This is also the conditional distribution of R' given V_1 and V_2. Hence, the interval given in (1) is identical to that given in (2).

We provide additional examples to illustrate the generality of the method of Proposition 1 in obtaining confidence intervals in standard and nonstandard problems.

Example 4. Difference in Two Exponential Means.

This example is discussed in Weerahandi (1993). In his notation, let $X_1, \ldots, X_M$ and $Y_1, \ldots, Y_N$ be two independent sets of iid lifetime random variables having exponential distributions with means μ_x and μ_y, respectively. Further suppose that these lifetimes are censored after observing the first m and the first n failures, respectively. Thus, the observed lifetimes are $X_{(1)}, \ldots, X_{(m)}$ and $Y_{(1)}, \ldots, Y_{(n)}$, respectively, where, as usual, $X_{(i)}$ represents the i^{th} order statistic of X, etc. We are interested in obtaining a confidence interval for $\xi = \mu_x - \mu_y$.

Let $X = \sum_{i=1}^m X_{(i)} + (M-m)X_{(m)}$ and $Y = \sum_{i=1}^n Y_{(i)} + (N-n)Y_{(n)}$. It is well known (e.g., Lawless (1982)) that X and Y are sufficient statistics for μ_x and μ_y. Additionally, X and Y are independent with

$$X \sim G(m, 1/\mu_x) \text{ and } Y \sim G(n, 1/\mu_y)$$

where $G(\alpha, \lambda)$ represents a gamma distribution with pdf

$$\frac{\lambda^\alpha}{\Gamma(\alpha)} t^{\alpha-1} e^{-\lambda t} \qquad (t > 0).$$

We use x and y to denote the observed values of X and Y, respectively.

In the notation of Proposition 1, we let $\boldsymbol{\xi} = (\xi_1, \xi_2) = (\mu_x, \mu_y)$, $\boldsymbol{D} = (X, Y)$, and $\boldsymbol{d} = (x, y)$. Let,

$$U_1 = \frac{X}{\mu_x}, \qquad U_2 = \frac{Y}{\mu_y}.$$

Then, $U_1 \sim G(m, 1)$ and $U_2 \sim G(n, 1)$. We have,

$$\begin{aligned} \mu_x &= \frac{X}{U_1} \\ \mu_y &= \frac{Y}{U_2}. \end{aligned}$$

Following Proposition 1, a generalized pivot for $\mu_1 - \mu_2$ is given by

$$R = \frac{x}{U_1} - \frac{y}{U_2} = \frac{x\mu_x}{X} - \frac{y\mu_y}{Y}. \tag{3}$$

If we define $W_1 = 2X/\mu_x$, $W_2 = 2Y/\mu_y$, and

$$H = \frac{1}{W_1} - \frac{y}{x}\frac{1}{W_2},$$

then we recognize H as the generalized pivot of Weerahandi (1993, eqn (3.2)). Note that the generalized pivot R defined in (3) is related to H via

$$H = 2xR.$$

Therefore, both H and R will yield the same generalized confidence interval for $\mu_1 - \mu_2$. Application of Proposition 1 thus yields the same generalized confidence interval as the one derived by Weerahandi, but does not appear to need any invariance discussions of Weerahandi (1993).

Example 5. Balanced One-Way Random Effects Model.

This example is discussed in Weerahandi (1991) in the context of generalized test variables and generalized P-values and also in Weerahandi (1993) in the context of generalized confidence intervals. Here we will demonstrate that his results can be derived by a direct application of Proposition 1.

Following Weerahandi's notation, let

$$X_{ij} = \mu + \alpha_i + e_{ij}, \qquad i = 1, \ldots, a; \;\; j = 1, \ldots, n$$

where x_{ij} is the j^{th} observation corresponding to the i^{th} random effect α_i. It is further assumed that $\alpha_i \sim N(0, \sigma_\alpha^2)$ and $e_{ij} \sim N(0, \sigma_e^2)$, and $\{\alpha_i\}$ and $\{e_{ij}\}$ are all jointly independent. Define

$$\bar{X} = \frac{1}{na}\sum_{i=1}^{a}\sum_{j=1}^{n}, \;\; SSw = \sum_{i=1}^{a}\sum_{j=1}^{n}(X_{ij} - \bar{X}_{i*})^2, \;\; SSb = n\sum_{i=1}^{a}(\bar{X}_{i*} - \bar{X}_{**})^2$$

where

$$\bar{X}_{i*} = \frac{1}{n}\sum_{j=1}^{n} X_{ij} \quad \text{and} \quad \bar{X}_{**} = \frac{1}{a}\sum_{i=1}^{a}\bar{X}_{i*}.$$

Here SSw is the usual "within sum of squares" and SSb is the "between sum of squares" in the analysis of variance. It is well known that

$$U_1 = \frac{SSw}{\sigma_e^2} \sim \chi^2_{a(n-1)}, \quad U_2 = \frac{SSb}{\sigma_e^2 + n\sigma_\alpha^2} \sim \chi^2_{a-1}, \quad U_3 = \frac{\bar{X} - \mu}{\sqrt{\dfrac{n\sigma_\alpha^2 + \sigma_e^2}{na}}} \sim N(0,1),$$

and U_1, U_2, U_3 are mutually independent. Suppose we are interested in constructing a test of

$$H_0 : \sigma_\alpha^2 \leq \delta \text{ versus } H_a : \sigma_\alpha^2 > \delta$$

using Proposition 1. We let

$$\boldsymbol{\xi} = (\sigma_e^2, \sigma_\alpha^2, \mu), \quad \boldsymbol{D} = (SSw, SSb, \bar{X}), \quad \text{and} \quad \boldsymbol{d} = (ssw, ssb, \bar{x})$$

where $ssw, ssb, \bar{x}$ are the observed values of $SSw, SSb, \bar{X}$, respectively. Further let $\boldsymbol{U} = (U_1, U_2, U_3)$. The parameter of interest is $\theta = h(\boldsymbol{\xi}) = \xi_2 = \sigma_\alpha^2$. Since

$$\sigma_\alpha^2 = \frac{SSb}{nU_2} - \frac{SSw}{nU_1},$$

a GPQ for σ_α^2 is given by

$$\begin{aligned} R &= \frac{ssb}{nU_2} - \frac{ssw}{nU_1} \\ &= \frac{ssb}{nSSb}(\sigma_e^2 + n\sigma_\alpha^2) - \frac{ssw}{nSSw}\sigma_e^2. \end{aligned}$$

Thus a confidence interval for σ_α^2 is given by $[L, U]$ where $L = R_{\alpha/2}$ and $U = R_{1-\alpha/2}$. The percentiles of R may be calculated using numerical methods or estimated using monte-carlo methods. We omit the details. Weerahandi (1993, page 904) mentions a generalized pivotal quantity for σ_α^2 which is equal to the absolute value of R given above. The reader may verify that $|R|$ satisfies the conditions needed for it to be a generalized pivotal. Weerahandi suggests using $|R|$ rather than R as a GPQ presumably because this assures the nonegativity of the confidence bounds.

By Proposition 1, a generalized test variable for σ_α^2 is given by $T = \sigma_\alpha^2 - R$. When the observed quantities $\bar{x}$, ssw, and ssb are substituted for $\bar{X}$, SSw, and SSb in T, it becomes zero. Hence the generalized P-value for the test is

$$\begin{aligned} P &= Pr\left[T > 0 | \sigma_\alpha^2 = \delta\right] = Pr\left[\delta - \frac{ssb}{nSSb}(\sigma_e^2 + n\sigma_\alpha^2) - \frac{ssw}{nSSw}\sigma_e^2 > 0\right] \\ &= Pr\left[\frac{ssb}{nU_2} - \frac{ssw}{nU_1} < \delta\right] = Pr\left[U_2 > \frac{ssb}{n\delta + ssw/U_1}\right] \\ &= 1 - E\left\{G\left(\frac{ssb}{n\delta + ssw/U_1}\right)\right\} \end{aligned} \qquad (4)$$

where G is the cdf of a χ^2_{a-1} random variable and the expectation is taken with respect to the distribution of U_1 which is $\chi^2_{a(n-1)}$. Equation (4) above is identical to Weerahandi's (1991) equation (2.4), demonstrating that the generalized test function obtained using Proposition 1 is equivalent to Weerahandi's test function. Weerahandi (1991) points out that the above generalized test can also be derived as a generalized Bayes procedure.

Application of Proposition 1 to the construction of generalized confidence intervals and generalized tests for fixed effects as well as variance components in balanced, saturated, higher-way mixed models is now obvious once we recognize that unique sums of squares exist whose distributions are scaled chi-squared distributions where the scale parameters are known functions of variance components and these sums of squares, together with the estimators of the fixed effects, provide a set of mutually independent, complete-sufficient statistics with known distributional properties. It is also possible to extend the approach of Proposition 1 to unbalanced and unsaturated mixed models, but this will be discussed elsewhere.

Example 6. Proportion of conforming parts

Consider a manufacturing process for producing mechanical parts. For the parts to be considered acceptable, suppose a certain measurement is required to be within a prespecified range, say between A units and B units. When the process is in control, suppose the parts distribution has a mean μ and standard deviation σ, both unknown. A random sample of n parts is selected and the measured values are $Y_1, \ldots, Y_n$. Based on these data it is required to estimate the proportion θ of the measurement distribution that is between A and B (conforming parts) and obtain a confidence interval for it.

Denote the sample mean by $\bar{Y}$ and the sample variance by V. Then $\bar{Y}$ and V are complete and sufficient for μ and σ. We can apply Proposition 1 to develop a generalized confidence interval for θ as follows. Let $\boldsymbol{\xi} = (\xi_1, \xi_2) = (\mu, \sigma)$, $\boldsymbol{D} = (\bar{Y}, V)$, $\boldsymbol{d} = (\bar{y}, v)$, where $\bar{y}$ and v are the observed values of $\bar{Y}$ and V, respectively. Define

$$U_1 = \frac{\bar{X} - \mu}{\sqrt{\sigma^2/n}} \quad \text{and} \quad U_2 = \frac{(n-1)V}{\sigma^2}.$$

Then U_1 and U_2 are independent with $U_1 \sim N(0,1)$ and $U_2 \sim \chi^2_{n-1}$. Note that

$$\theta = h(\mu, \sigma) = \Phi\left(\frac{B-\mu}{\sigma}\right) - \Phi\left(\frac{A-\mu}{\sigma}\right).$$

Observing that

$$\mu = \bar{X} - U_1\sqrt{\frac{(n-1)V}{nU_2}}$$

we get, by Proposition 1, a GPQ for θ as

$$R = \Phi\left(\frac{B - \left(\bar{x} - (\bar{X}-\mu)\sqrt{\dfrac{v}{V}}\right)}{\sqrt{\dfrac{v\sigma^2}{V}}}\right) - \Phi\left(\frac{A - \left(\bar{x} - (\bar{X}-\mu)\sqrt{\dfrac{v}{V}}\right)}{\sqrt{\dfrac{v\sigma^2}{V}}}\right).$$

A $(1-\alpha)$ confidence interval for θ, the proportion of conforming parts, is given by $[L, U]$, where $L = R_{\alpha/2}$ and $U = R_{1-\alpha/2}$. Wang and Lam (1996) offer a different confidence interval procedure for θ.

In problems where complete, sufficient statistics are available, they may be taken as the data vector $\boldsymbol{D}$ based on sufficienty arguments. In this case, there is a 1-1 correspondence between the parameter vector $\boldsymbol{\xi}$ and the data vector $\boldsymbol{D}$. Also, in many practical situations, pivotal quantities $\boldsymbol{U}$

associated with the complete sufficient statistics are well known or are easily derived. When the set of minimal sufficient statistics is not complete, it is still possible to appropriately apply our recipe. The following example illustrates this situation.

Example 7. Common mean of two normal populations – Fixed Effects Model

Suppose $X_1, \dots, X_m$ are iid $N(\mu, \sigma_x^2)$ and $Y_1, \dots, Y_n$ are iid $N(\mu, \sigma_y^2)$. It is of interest to construct a confidence interval for μ. Let

$$\bar{X} = \frac{\sum_{i=1}^m X_i}{m}, \quad \bar{Y} = \frac{\sum_{i=1}^n Y_i}{n}, \quad SSx = \sum_{i=1}^m (X_i - \bar{X})^2, \quad \text{and} \quad SSy = \sum_{i=1}^n (Y_i - \bar{Y})^2.$$

It is well known that $\bar{X}, \bar{Y}, SSX, SSY$ together form a set of minimal sufficient statistics but they are not complete.

Many authors have addressed the problem of point estimation as well as confidence interval estimation for μ in this setting, for two normal populations as well as for more than two normal populations, using frequentist approaches based on pivotal quantities. See, for instance, Jordan and Krishnamoorthy (1996) and Yu et. al (1999). Here we show how to derive a generalized pivotal quantity for μ based on which a generalized confidence interval may be calculated.

For any given vector $\boldsymbol{c} = (c_x, c_y)$ different from the zero vector, we define the quantity $M_{\boldsymbol{c}}$ by

$$M_{\boldsymbol{c}} = \frac{c_x\bar{X} + c_y\bar{Y}}{c_x + c_y}.$$

Of particular interest will be the quantity $M_{\boldsymbol{w}}$ where $\boldsymbol{w} = (w_x, w_y)$, with $w_x = 1/\sigma_x^2$ and $w_y = 1/\sigma_y^2$. We write $\tau^2 = 1/(w_x + w_y)$. It follows that

$$M_{\boldsymbol{w}} \sim N(\mu, \tau^2).$$

Let $\boldsymbol{\xi} = (\mu, \sigma_x^2, \sigma_y^2)$ be the vector of model parameters. Consider the following functions of the data and the parameters.

$$U_1 = \frac{SSx}{\sigma_x^2}, \qquad U_2 = \frac{SSy}{\sigma_y^2}, \qquad \text{and} \qquad U_3 = \frac{\bar{Y}_{\boldsymbol{w}} - \mu}{\tau}$$

It is easy to verify that U_1, U_2, U_3 are mutually independent, and that $U_1 \sim \chi^2_{m-1}$, $U_2 \sim \chi^2_{n-1}$ and $U_3 \sim N(0, 1)$. Define

$$W_x = \frac{ssx}{U_1}, \qquad W_y = \frac{ssy}{U_2}, \qquad \text{and} \qquad \boldsymbol{W} = (W_x, W_y),$$

where ssx and ssy denote the observed values of SSx and SSy respectively. Likewise, let $\bar{x}$ and $\bar{y}$ denote the observed values of $\bar{X}$ and $\bar{Y}$, respectively. Note that the distribution of the random vector $\boldsymbol{W}$ is not dependent on $\boldsymbol{\xi}$. We denote the vector $(\bar{X}, \bar{Y}, SSx, SSy)$ by $\boldsymbol{D}$ and the corresponding observed value $(\bar{x}, \bar{y}, ssx, ssy)$ by $\boldsymbol{d}$. Define a generalized pivotal quantity $R(\boldsymbol{D}; \boldsymbol{d}, \boldsymbol{\xi})$ by

$$R(\boldsymbol{D}; \boldsymbol{d}, \boldsymbol{\xi}) = \bar{y}_{\boldsymbol{W}} - \frac{\dfrac{\bar{Y}_{\boldsymbol{w}} - \mu}{\tau}}{\sqrt{\dfrac{ssx\sigma_x^2}{SSx} + \dfrac{ssy\sigma_y^2}{SSy}}} = \bar{y}_{\boldsymbol{W}} - \frac{U_3}{W_x + W_y}.$$

A $(1-\alpha)$ two-sided, equal-tailed, generalized confidence interval for μ is given by $[R_{\alpha/2}, R_{1-\alpha/2}]$, where R_γ denotes the γ-percentile of the distribution of $R(\boldsymbol{D};\boldsymbol{d},\boldsymbol{\xi})$. In applications, the percentiles of $R(\boldsymbol{D};\boldsymbol{d},\boldsymbol{\xi})$ may be estimated using a Monte-carlo approach.

Example 8. One-sided tolerance bound for a normal distribution

Suppose that $X_1,\ldots,X_n$ are iid $N(\mu,\sigma^2)$. A one-sided β-content γ-confidence upper tolerance bound of the distribution $N(\mu,\sigma^2)$ is a statistic U such that, with $100\gamma\%$ confidence, we can assert that a proportion β or more of the $N(\mu,\sigma^2)$ population values are less than U. Thus, U is actually a one-sided upper confidence bound for the percentile $\mu+z_\beta\sigma$ of $N(\mu,\sigma^2)$. Here z_β stands for the β-percentile of the standard normal distribution. A solution to this problem is well known and is based on the noncentral-t distribution. Here we derive a generalized one-sided confidence bound for $\mu+Z_\beta\sigma$ and show that it is in fact identical to the classical tolerance interval solution.

Define

$$Z=\frac{\bar{X}-\mu}{\sigma/\sqrt{n}} \quad \text{and} \quad U=\frac{SSx}{\sigma^2}.$$

Observe that $Z\sim N(0,1)$ and $U\sim\chi^2_{n-1}$. Solving for μ and σ in terms of $Z,U,\bar{X},SSx$ gives

$$\sigma=\sqrt{\frac{SSx}{U}} \quad \text{and} \quad \mu=\bar{X}-Z\sqrt{\frac{SSx}{nU}}.$$

A GPQ for $\theta=\mu+z_\gamma\sigma$ is then given by

$$R=\bar{x}-Z\sqrt{\frac{ssx}{nU}}+z_\gamma\sqrt{\frac{ssx}{U}}=\bar{x}-(\bar{X}-\mu)\sqrt{\frac{ssx}{SSx}}+z_\gamma\sigma\sqrt{\frac{ssx}{SSx}}.$$

It is an easy exercise to verify that the resulting GCI for θ (γ-content, $1-\alpha$ confidence, upper tolerance bound) turns out to be identical to the classical noncentral-t based bound.

Example 9. Maximum of a Collection of Normal Means

Consider k normal populations $N(\mu_i,\sigma^2), i=1,\ldots,k$ having a common variance σ^2. Let μ_{max} denote the maximum of the k means. It is of interest to obtain an interval estimate for μ_{max}. Generalized confidence intervals for this problem and related problems have been discussed by Chang and Huang (2000). Here we show how these might be derived using Proposition 1.

For each $i=1,\ldots,k$, let $X_{ij}, j=1,\ldots,n_i$ be a random sample from $N(\mu_i,\sigma^2)$. Further, let $\bar{X}_i, i=1,\ldots,k$, denote the corresponding sample means and SSE denote the pooled sum of squares for error based on all the data. It then follows that

$$\bar{X}_i\sim N(\mu_i,\sigma^2/n_i) \quad \text{for} \quad i=1,\ldots,k, \quad \text{and} \quad \frac{SSE}{\sigma^2}\sim\chi^2_\nu$$

where $\nu=\sum_{i=1}^k(n_i-1)$. Let $\boldsymbol{D}$ denote the vector $(\bar{X}_1,\ldots,\bar{X}_k,SSE)$ and $\boldsymbol{d}=(\bar{x}_1,\ldots,\bar{x}_k,sse)$ denote the corresponding observed value of $\boldsymbol{D}$. We will regard $\boldsymbol{d}$ as a vector of known constants. Furthermore, let $\boldsymbol{\xi}=(\mu_1,\ldots,\mu_k,\sigma^2)$ denote the vector of model parameters. We now define a generalized pivotal quantity $R(\boldsymbol{D};\boldsymbol{d},\boldsymbol{\xi})$ by

$$\begin{aligned} R(\boldsymbol{D};\boldsymbol{d},\boldsymbol{\xi}) &= max\left\{ \bar{y}_1 - \frac{\dfrac{\bar{Y}_1-\mu_1}{\sigma/\sqrt{n_1}}}{\sqrt{\dfrac{sse\,\sigma^2}{SSE}}}, \ldots, \bar{y}_k - \frac{\dfrac{\bar{Y}_k-\mu_k}{\sigma/\sqrt{n_k}}}{\sqrt{\dfrac{sse\,\sigma^2}{SSE}}} \right\} \\ &= max\left\{ \bar{y}_1 - Z_1\sqrt{\frac{U}{sse}}, \ldots, \bar{y}_k - Z_k\sqrt{\frac{U}{sse}} \right\}. \end{aligned}$$

Then a $1-\alpha$ two-sided, equal-tailed, generalized confidence interval for μ_{max} is the interval $[L, U]$ where $L = R_{\alpha/2}$ and $U = R_{1-\alpha/2}$. This is exactly what Chang and Huang (2000) propose.

4. Final Remarks

In this paper we have provided a general recipe for the construction of generalized test variables and generalized pivotal quantities introduced by Weerahandi. Although no general theory appears to exist that can provide a mathematical justification for the use of generalized inference, overwhelming empirical evidence is available from a number of published simulation studies on the performance of generalized P-value based tests and generalized confidence intervals in specific applications, all of which indicate that these procedures generally perform extremely well in the frequentist sense. In some instances they outperform available procedures and in other instances they provide useful procedures where no satisfactory procedures are available. Until theoretical results on the performance of these procedures become available, one is forced to assess this via Monte-carlo studies on a case by case basis.

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