On Multidimensional Fiducial Generalized Confidence Intervals

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Abstract

Generalized Pivotal Quantities and Generalized Confidence Intervals have proved to be useful tools for making inferences in many practical problems. Although generalized confidence intervals are not guaranteed to have exact frequentist coverage, a number of published and unpublished simulation studies suggest that the coverage probabilities of such intervals are sufficiently close to their nominal value to be useful in practice. In this paper we extend a result of Hannig et al. (2005) to allow for multidimensional parameter structure.

Key words: Generalized Pivots, Fiducial inference, Structural Inference, Conditional Inference, Asymptotic properties, Multivariate inference.

Tsui and Weerahandi (1989) introduced the concept of generalized P-values and generalized test variables which are useful for developing hypothesis tests in situations where traditional frequentist approaches do not provide useful solutions. Subsequently, Weerahandi (1993) introduced the concept of a generalized pivotal quantity (GPQ) for a scalar parameter θ, using which one can construct an interval estimator for θ in situations where standard pivotal quantity based approaches may not be applicable. He referred to such intervals as generalized confidence intervals (GCI). Since then, several GCIs have been constructed in many practical problems. See for instance, Weerahandi (1995), Hamada and Weerahandi (2000), Chang and Huang (2000), McNally et. al. (2001), Burdick and Park (2003), Burdick et. al. (2004, 2005), Mathew and Krishnamoorthy (2004), Krishnamoorthy and Mathew (2004), Krishnamoorthy and Lu (2003), and Iyer et. al. (2004). These intervals do not always have exact frequentist coverage. Nevertheless, results of simulation studies reported in the literature appear to support the claim that coverage probabilities of GCIs are sufficiently close to their stated value that they are in fact useful procedures in practical problems. In spite of the large number of successful applications of GCIs reported in the literature, it is surprising that there are no published results that discuss either small sample properties of GCIs or their asymptotic behavior. In a recent paper Hannig et al. (2005) show that under some general conditions generalized confidence intervals for scalar parameters have a proper frequentist coverage at least

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asymptotically. It is the aim of this present paper to extend their result to generalized confidence intervals for multidimensional parameter.

We will first review a definition of a Fiducial Generalized Pivotal Quantity.

Definition 1. A GPQ $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$ for a parameter θ is called a *Fiducial Generalized Pivotal Quantity* (FGPQ) if it satisfies the following two conditions.

(FGPQ1) The conditional distribution of $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$, conditional on $\mathbb{S} = \mathbf{s}$, is free of ξ.

(FGPQ2) For every allowable $\mathbf{s} \in \mathbb{R}^k$, $\mathcal{R}_\theta(\mathbf{s}, \mathbf{s}, \xi) = \theta$.

Let us consider a parametric statistical problem where we observe $X_1, \dots, X_n$ whose joint distribution belongs to some family of continuous distributions parameterized by $\xi \in \Xi \subset \mathbb{R}^p$. Let $\mathbb{S} = (S_1, \dots, S_k)$ denote a statistic based on the X_i's. In theory we can consider an independent copy of $X_1^\star, \dots, X_n^\star$ and denote the statistic based on $X_i^\star$'s by $\mathbb{S}^\star$. Finally, suppose a vector-valued function $\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi) = (\mathcal{R}_{\theta,1}(\mathbb{S}, \mathbb{S}^\star, \xi), \dots, \mathcal{R}_{\theta,d}(\mathbb{S}, \mathbb{S}^\star, \xi))$ is available that is a FGPQ for a parameter $\theta = \pi(\xi) \in \mathbb{R}^d,\ d \le k$.

In what follows we need the following notation,

Definition 2. Open sets A_n converge to an open set A, i.e., $A_n \to A$, if $(\lim A_n)^\circ = A$. Here $\lim A_n = B$ exists if $I_{A_n} \to I_B$, I_A is the indicator function of A and B° is the interior of B.

Additionally assume

Assumptions A. 1. Assume that there exists $\mathbf{t}(\xi) \in \mathbb{R}^k$ such that

$$\sqrt{n}\,(S_1^\star - t_1(\xi), \dots, S_k^\star - t_k(\xi)) \xrightarrow{\mathcal{D}} \mathbf{N} = (N_1, \dots, N_k)^\top,$$

where $\mathbf{N}$ has a non-degenerate distribution multivariate normal distribution.

2. Assuming existence and continuity of second partial derivatives with respect to $\mathbf{s}^\star$ of $\mathcal{R}_{\theta,l}(\mathbf{s}, \mathbf{s}^\star, \xi)$ we have the following one-term Taylor expansion with a remainder term:

$$\mathcal{R}_{\theta,l}(\mathbf{s}, \mathbb{S}^\star, \xi) = g_{0,l,n}(\mathbf{s}, \xi) + \sum_{j=1}^{k} g_{1,l,j,n}(\mathbf{s}, \xi)\left(S_j^\star - t_j(\xi)\right) + R_{l,n}(\mathbf{s}, \mathbb{S}^\star, \xi). \tag{1}$$

Here

$$g_{0,l,n}(\mathbf{s}, \xi) = \mathcal{R}_{\theta,l}(\mathbf{s}, \mathbf{t}(\xi), \xi), \quad g_{1,l,j,n}(\mathbf{s}, \xi) = \frac{\partial}{\partial s_j^\star}\mathcal{R}_{\theta,l}(\mathbf{s}, \mathbf{s}^\star, \xi)\bigg|_{\mathbf{s}^\star = \mathbf{t}(\xi)}$$

Suppose $A \subset \mathbb{R}^k$ is an open set containing $\mathbf{t}(\xi)$ with the following properties:

(a) The functions $g_{1,l,j,n}(\mathbf{s}, \xi)$ converge uniformly in $\mathbf{s} \in A$ to a function $g_{1,l,j}(\mathbf{s}, \xi)$ continuous at $\mathbf{s} = \mathbf{t}(\xi)$.

(b) The matrix

$$J(\mathbf{s}) = \begin{pmatrix} g_{1,1,1}(\mathbf{s}, \xi), & \dots, & g_{1,1,k}(\mathbf{s}, \xi) \\ \vdots & \ddots & \vdots \\ g_{1,d,1}(\mathbf{s}, \xi), & \dots, & g_{1,d,k}(\mathbf{s}, \xi) \end{pmatrix}$$

is of rank d for all $\mathbf{s} \in A$.

(c) For each $l = 1, \dots, d$, the remainder $\sqrt{n}R_{l,n}(\mathbf{s}, \mathbb{S}^\star, \xi) \xrightarrow{P_\xi} 0$ uniformly in $\mathbf{s}$ on the open neighborhood A of $\mathbf{t}(\xi)$.

3. We consider a collection of open regions $C(\mathbf{X}, \gamma) \subset \mathbb{R}^d$ with $\lambda(\partial C(\mathbf{X}, \gamma)) = 0$, i.e., the boundary has zero Lebesgue measure, indexed by continuous random variables $\mathbf{X}$ and $\gamma \in (0, 1)$ satisfying:

   (a) $P(\mathbf{X} \in C(\mathbf{X}, \gamma)) = \gamma$.

   (b) $C(a\mathbf{X} + b, \gamma) = aC(\mathbf{X}, \gamma) + b$.

   (c) If $\mathbf{X}$ has non-degenerate normal distribution, $\mathbf{X}_n \xrightarrow{\mathcal{D}} \mathbf{X}$ and $\gamma_n \to \gamma$ then $C(\mathbf{X}_n, \gamma_n) \to C(\mathbf{X}, \gamma)$.

Recall that $R_\theta(\mathbb{S}, \mathbb{S}^\star, \xi)$ has a distribution that is independent of the parameters. This allows us to state the following theorem:

Theorem 1. *Suppose Assumptions A hold and $\gamma_n \to \gamma \in (0, 1)$. Then*

$$\lim_{n\to\infty} P_\xi \left(\theta = \mathcal{R}_\theta(\mathbb{S}, \mathbb{S}, \xi) \in C_n(R_\theta(\mathbb{S}, \mathbb{S}^\star, \xi), \gamma_n)\right) = \gamma.$$

In particular $C_n(R_\theta(\mathbb{S}, \mathbb{S}^\star, \xi), \gamma)$ is a confidence region for θ with asymptotic coverage probability equal to γ.

Remark 1. If $d = 1$ an example of $C(X, \gamma) = (-\infty, q(X, \gamma))$, where $q(X, \gamma)$ is the γ-quantile of the distribution of X.

If $d > 1$ we can consider many different regions. As an example we can consider cubical equal tail region.

Proof. Define

$$\mathbf{H}(\mathbf{s}) = J(\mathbf{s}) \cdot \mathbf{N}. \tag{2}$$

Assumptions A parts 1 and 2b imply that for all $\mathbf{s} \in A$, $\mathbf{H}(\mathbf{s})$ a is non-degenerate normal random variable. In particular the support of $\mathbf{H}(\mathbf{s})$ is $\mathbb{R}^d$.

First observe that assumption 2c implies that for all $\mathbf{s} \in A$

$$\sqrt{n}R_n(\mathbf{s}, \mathbb{S}^\star, \xi) \xrightarrow{P_\xi} 0 \tag{3}$$

Next observe that Slutsky's theorem, Assumption A 1, and equations (1), (2), (3) imply

$$\sqrt{n}(\mathcal{R}_\theta(\mathbf{s}, \mathbb{S}^\star, \xi) - g_{0,n}(\mathbf{s}, \xi)). \xrightarrow{\mathcal{D}} \mathbf{H}(\mathbf{s}). \tag{4}$$

By assumption 3a, $P_\xi(\mathbf{H}(\mathbf{s}) \in C(\mathbf{H}(\mathbf{s}), \gamma) = \gamma$. Therefore by assumption 3c and (4)

$$C\left(\sqrt{n}(\mathcal{R}_\theta(\mathbf{s}, \mathbb{S}^\star, \xi) - g_{0,n}(\mathbf{s}, \xi)), \gamma_n\right) \to C(\mathbf{H}(\mathbf{s}), \gamma). \tag{5}$$

Moreover, the continuity of the functions $g_{1,l,j}$ imply that $C(H(\mathbf{s}), \gamma)$ is continuous at $\mathbf{s} = \mathbf{t}(\xi)$. Thus by assumptions 2a, 3c and Slutsky's theorem and we get

$$C(\sqrt{n}(\mathcal{R}_\theta(\mathbb{S}, \mathbb{S}^\star, \xi) - g_{0,n}(\mathbb{S}, \xi)), \gamma_n) \to C(\mathbf{H}(\mathbf{t}(\xi)), \gamma).$$

Notice that Assumption 2c also implies $\sqrt{n}R_n(\mathbb{S},\mathbb{S},\xi) \xrightarrow{P_\xi} 0$. The same arguments as above give

$$\sqrt{n}(\mathcal{R}_\theta(\mathbb{S},\mathbb{S},\xi) - g_{0,n}(\mathbb{S},\xi)) \xrightarrow{\mathcal{D}} \mathbf{H}(\mathbf{t}(\xi))$$

For simplicity denote $\mathbf{G}_n = \sqrt{n}(\mathcal{R}_\theta(\mathbb{S},\mathbb{S},\xi) - g_{0,n}(\mathbb{S},\xi))$, $\mathbf{G} = \mathbf{H}(\mathbf{t}(\xi))$, $B_n = C(\mathcal{R}_\theta(\mathbb{S},\mathbb{S}^\star,\gamma_n)$ and $B = C\left(\mathbf{H}(\mathbf{t}(\xi)),\gamma\right)$. As noted before we have

$$\mathbf{G}_n \xrightarrow{\mathcal{D}} \mathbf{G} \quad \text{and} \quad B_n \to B.$$

Denote $C_m = \bigcup_{k=m}^\infty B_k \setminus (\bigcap_{k=m}^\infty B_k)^\circ$. Notice that by assumption 3c we have $C_n \downarrow C$, where $P(G \in C) = 0$. Moreover if $m \le n$, $B_n \triangle B \subset C_m$.

Fix an $\varepsilon > 0$. Continuity of probability implies that there is m_1 such that $P(G \in C_{m_1}) < \varepsilon$. Consequently there is m_2 such that for all $n > m_2$, $P(G_n \in C_{m_2}) < \varepsilon$. This implies that for $n > \max(m_1, m_2)$

$$|P(G_n \in B_n) - P(G_n \in B)| \le P(G_n \in C_{m_1}) < \varepsilon.$$

Finally notice that

$$|P(G_n \in B_n) - P(G \in B)| \le |P(G_n \in B_n) - P(G_n \in b)| + |P(G \in B_n) - P(G \in B)|.$$

Thus by the definition of convergence in distribution we observe that

$$P_\xi\left(\mathcal{R}_\theta(\mathbb{S},\mathbb{S},\xi) \in C(\mathcal{R}_\theta(\mathbb{S},\mathbb{S}^\star,\gamma_n)\right) \to P_\xi\left(\mathbf{H}\left(\mathbf{t}(\xi)\right) \in C\left(\mathbf{H}(\mathbf{t}(\xi)),\gamma\right)\right) = \gamma.$$

This concludes the proof of the theorem. □

Remark 2. The various conditions stated in Assumption A could be weakened. For example we do not have to assume that the limiting random variable **H** is normal. The assumption3c and the proof of Theorem 1 would then have to be modified accordingly.

As an example of a possible modification we state the general conditions in the $d = 1$ case. The statement of the Theorem 1 remains valid under Assumptions B with $C(X,\gamma) = (-\infty, q(X,\gamma))$. Here $q(X,\gamma)$ is γ's quantile of X.. We omit the proof because it is essentially the same as above.

Assumptions B. (a) Let $\mathbb{S} \xrightarrow{P_\xi} \mathbf{t}(\xi) = (t_1(\xi),\dots,t_k(\xi))$ as $n \to \infty$. Using a Taylor series approximation we can write, for some $1 \le l \le k$,

$$g_n(\mathbf{s},\mathbb{S}^\star,\xi) = g_{0,n}(\mathbf{s},\xi) + \sum_{j=1}^{l} g_{1,j,n}(\mathbf{s},\xi) h_j\left(S_j^\star - t_j(\xi)\right) + R_n(\mathbf{s},\mathbb{S}^\star,\xi).$$

Assume that, in some open neighborhood A of $\mathbf{t}(\xi)$, the functions $g_{1,j,n}(\mathbf{s},\xi)$ converge uniformly in $\mathbf{s}$ to a function $g_{1,j}(\mathbf{s},\xi)$ continuous at $\mathbf{s} = \mathbf{t}(\xi)$.

(b) Furthermore, assume there is a nondecreasing sequence $\{a_n\}$ of nonnegative real numbers and a random vector $\mathbf{H} = (H_1,\dots,H_l)$ satisfying,

$$a_n\left(h_1(S_1^\star - t_1(\xi)),\dots,h_l(S_l^\star - t_l(\xi))\right) \xrightarrow{\mathcal{D}} (H_1,\dots,H_l) \quad \text{and}$$

$$a_n R_n(\mathbf{s},\mathbb{S}^\star,\xi) \xrightarrow{P_\xi} 0 \quad \text{uniformly in } \mathbf{s} \text{ on the open neighborhood } A \text{ of } \mathbf{t}(\xi).$$

(c) Denote

$$H(\mathbf{s}) = \sum_{j=1}^{l} g_{1,j}(\mathbf{s}, \xi) H_j$$

and assume that for all $\mathbf{s}$ in the open neighborhood A of $\mathbf{t}(\xi)$ the distribution function of $H(\mathbf{s})$ is continuous and strictly increasing at the α^{th} quantile.

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