An Overview of Asset-Price Models

Peter J. Brockwell

Department of Statistics, Colorado State University, Fort Collins, Colorado
pjbrock@stat.colostate.edu

Summary. Discrete-parameter time-series models for financial data have received, and continue to receive, a great deal of attention in the literature. Stochastic volatility models, ARCH and GARCH models and their many generalizations, designed to account for the so-called stylized features of financial time series, have been under development and refinement now for some thirty years. At the same time there has been a rapidly developing interest in continuous-time models, largely as a result of the very successful application of stochastic differential equation models to problems in finance, exemplified by the derivation of the Black-Scholes-Merton (BSM) option-pricing formula and its generalizations. In this overview we start with the BSM option-pricing model in which the asset price is represented by geometric Brownian motion. We then discuss the limitations of the model and survey the various models which have been proposed to provide more realistic representations of empirically observed asset prices. In particular, the observed non-Gaussian distributions of log returns and the appearance of sharp changes in log asset prices which are not consistent with Brownian motion paths have led to an upsurge of interest in Lévy processes and their applications to financial modelling.

1. Introduction

For approximately thirty years now, discrete-time models (including stochastic volatility, ARCH, GARCH and their many generalizations) have been developed to reflect the so-called *stylized features* of financial time series. These properties, which include tail heaviness, volatility clustering and serial dependence without correlation, cannot be captured with traditional linear time series models. If S_n denotes the price of a stock or other financial asset at time n, $n = 0, 1, 2, \ldots$, then the series of log returns, $\{\log S_n - \log S_{n-1}, n \in \mathbb{N}\}$, is typically represented by either a discrete-time stochastic volatility model or a GARCH process. These models have been studied intensively since their introduction and a variety of parameter estimation techniques have been developed. For an excellent review and comparison of these models see Shephard (1996). For a more recent account of GARCH processes see the article of Lindner (2008) in the current volume, and for stochastic volatility models see the article of Davis and Mikosch (2008). Apart from the need to develop models which capture the distinctive features of financial time series, much of the motivation for developing these models derives from the key role played by volatility in the pricing of options and the need to understand, quantify and forecast its evolution in time.

In mathematical finance, most of the theoretical developments in the pricing of contingent claims (or options) have been made in a continuous-time framework, thanks to the power of Itô calculus, Girsanov's theorem, martingale methods, and other tools associated with the analysis of stochastic differential equations. In fact these developments, which permit the analysis of quite complicated ('exotic') options, have also been a powerful stimulus for the popularization and development of stochastic calculus itself. The celebrated work of Black and Scholes (1973) and Merton (1973) was based on a geometric Brownian motion model for the asset price $S(t)$ at time t (see (1.1) below). Their results, besides winning the Nobel Economics Prize for Merton and Scholes in 1997 (unfortunately Black died before the award was made), inspired an explosion of interest, not only in the pricing of more complicated financial derivatives, but also in the development of new continuous-time models which, like the discrete-time ARCH, GARCH and stochastic volatility models, better reflect the observed properties of financial time series. This development has resulted in a variety of models which are the subject of the articles in this section of the Handbook.

In addition to their central role in option-pricing, time series models with continuous time parameter are particularly well-suited to modelling irregularly spaced data (see e.g. Jones (1985)). Lévy-driven continuous-time autoregressive moving average (CARMA) models play a role in continuous time analogous to that of ARMA models in discrete time, allowing a very flexible range of autocorrelations and marginal distributions, suitable in particular for the modelling of volatility as a continuous-time stationary series (see the article by Brockwell (2008) in this volume).

The use of continuous-time models in finance goes back to Bachelier (1900), who used Brownian motion to represent the prices $\{S(t), t \geq 0\}$ of a stock in the Paris Bourse. This model had the unfortunate feature of permitting negative stock prices, a shortcoming which was eliminated in the geometric Brownian motion model of Samuelson (1965), according to which $S(t)$ satisfies the Itô equation,

$$\mathrm{d}S(t) = \mu S(t)\,\mathrm{d}t + \sigma S(t)\,\mathrm{d}W(t) \text{ with } S(0) > 0. \tag{1.1}$$

In this equation $\{W(t), t \geq 0\}$ is standard Brownian motion defined on a complete probability space $(\Omega, \mathcal{F}, P)$ with filtration $\{\mathcal{F}_t\}$ where $\mathcal{F}_t$ is the sub-σ-algebra of $\mathcal{F}$ generated by $\{W(s), 0 \leq s \leq t\}$ and the null sets of $\mathcal{F}$. The solution of (1.1) satisfies

$$S(t) = S(0) \exp\left[(\mu - \sigma^2/2)t + \sigma W(t)\right], \tag{1.2}$$

so that the log asset price in this model is Brownian motion and the log return over the time-interval $(t, t+\Delta)$ is

$$\log \frac{S(t+\Delta)}{S(t)} = (\mu - \frac{1}{2}\sigma^2)\Delta + \sigma(W(t+\Delta) - W(t)).$$

For disjoint intervals of length Δ the log returns are therefore independent normally distributed random variables with mean $(\mu - \sigma^2/2)\Delta$ and variance $\sigma^2\Delta$. The normality of the log returns is a conclusion which can easily be checked against observed returns, and it is found that the deviations are substantial for time intervals of the order of a day or less, becoming less apparent as Δ increases. This is one of the reasons for developing the models described in later sections.

The parameter σ^2 in (1.1) is called the *volatility* parameter and its significance for option pricing was clearly demonstrated in the pricing by Black, Scholes and Merton of a European call option. Such an option, if sold at time 0, gives the buyer the right, but not the obligation, to buy one unit of the stock (with market price satisfying (1.1)) at the *strike time* T for the *strike price* K. At time T the option has the cash value $h(S(t)) = \max(S(T) - K, 0)$ since the option will be exercised only if $S(T) > K$, in which case the holder of the option can buy the stock at the price K and resell it instantly for $S(T)$. However it is not clear at time 0, since $S(T)$ is random, what price the buyer should pay for this privilege. Assuming (i) the existence of a risk-free asset with price process,

$$B(t) = B(0)\exp(rt), \ r > 0, \tag{1.3}$$

(ii) the ability to buy and sell arbitrary (positive or negative) amounts of the stock and the risk-free asset continuously with no transaction costs, and (iii) an arbitrage-free market (i.e., a market in which it is impossible to make a non-negative profit which is strictly positive with probability greater than zero), Black, Scholes and Merton showed that there is a unique *fair price* for the option in the sense that both higher and lower prices introduce demonstrable arbitrage opportunities. Details of the derivation can be found in most books dealing with mathematical finance (e.g. Campbell, Lo amd McKinlay (1996), Mikosch (1998), Steele (2001), Shreve (2003), Björk (2004) and Klebaner (2005)). In the following paragraphs we give a sketch of two arguments, following Mikosch (1998), leading to this fair price for the Black-Scholes-Merton (henceforth BSM) model.

In the first argument, we attempt to construct a self-financing portfolio, consisting at time t of a_t shares of the stock and b_t shares of the risk-free asset, where a_t and b_t are random variables measurable with respect to $\mathcal{F}_t$. We require the value of this portfolio at time t, namely

$$V(t) = a_t S(t) + b_t B(t), \tag{1.4}$$

to satisfy the self-financing condition,

$$\mathrm{d}V(t) = a_t \,\mathrm{d}S(t) + b_t \,\mathrm{d}B(t), \tag{1.5}$$

and to match the value of the option at time T, i.e.,

$$V(T) = h(S(T)) = \max(S(T) - K, 0). \tag{1.6}$$

If such an *investment strategy*, $\{(a_t, b_t), 0 \le t \le T\}$ can be found, then $V(0)$ must be the fair value of the option at the purchase time $t = 0$. A higher price for the option would allow the seller to pocket the difference δ and invest the amount $V(0)$ in such a way as to match the value of the option at time T. Then at time T, if $S(T) < K$ the option will not be exercised and the portfolio and the option will both have value zero. If $S(T) > K$ the seller sells the portfolio for $S(T) - K$, then buys one stock for $S(T)$ and receives K for it from the holder of the option. Since there is no loss involved in this transaction, the seller is left with a net profit of δ. The seller of the option therefore makes a non-negative profit which is strictly positive with non-zero probability, in violation of the no arbitrage assumption. Similarly a lower price than $V(0)$ would create an arbitrage opportunity for the buyer. In order to determine $V(t)$, a_t and b_t we look for a smooth function $v(t, x),\ t \in [0, T],\ x > 0$, such that

$$V(t) = v(t, S(t)),\ t \in [0, T], \tag{1.7}$$

satisfies the conditions (1.5) and (1.6). Equating the expressions for $V(t) - V(0)$ obtained by applying Itô calculus to (1.5) and (1.7), we find that $a_t = \frac{\partial v}{\partial x}(t, S(t))$, where the function v must satisfy the partial differential equation,

$$\frac{\partial v}{\partial t} + \frac{1}{2}\sigma^2 x^2 \frac{\partial^2 v}{\partial x^2} + r\frac{\partial v}{\partial x} = rv, \tag{1.8}$$

with boundary condition,

$$v(T, x) = h(x) = \max(x - K, 0), \tag{1.9}$$

which, with (1.8), uniquely determines the function v and hence $V(t)$, a_t and $b_t = (V(t) - a_t S(t))/B(t)$ for each $t \in [0, T]$. Thus we have arrived at an investment strategy $\{(a_t, b_t), 0 \le t \le T\}$ which satisfies (1.5) and (1.6) and which, under the assumed idealized trading conditions, can be implemented in practice. Since at time T this portfolio has the same value as the option, $V(0)$ must be the fair value of the option at time $t = 0$; otherwise an arbitrage opportunity would arise. The option is said to be *hedged* by the investment strategy $\{(a_t, b_t)\}$. A key feature of this solution (apparent from (1.8) and (1.9)) is that both the strategy and the fair price of the option are independent of μ, *depending on* S *only through the volatility parameter* σ^2.

A particularly elegant and powerful way of arriving at the solution of (1.8) and (1.9) is to use a second argument, based on the fact that for the BSM model there is a unique probability measure Q which is equivalent to the original probability measure P (i.e., it has exactly the same null sets) and which, when substituted for P in the probability space on which the stock prices are defined, causes the discounted price process $\{e^{-rt}S(t)\}$ to become a *martingale*, i.e., to satisfy the condition,

$$E_Q(e^{-rt}S(t)|\mathcal{F}_s) = e^{-rs}S(s) \text{ for all } s \le t, \tag{1.10}$$

where E_Q denotes expectation with respect to the new probability measure Q. The probability measure Q is called the *equivalent martingale measure* or EMM. It is unique for the BSM model, but for other models the questions of its existence and uniqueness become serious issues.

The martingale-based argument leading to the BSM pricing formula is as follows. Itô's formula applied to the discounted price process,

$$\tilde{S}(t) := e^{-rt}S(t),$$

gives

$$\frac{\mathrm{d}\tilde{S}(t)}{\tilde{S}(t)} = (\mu - r)\mathrm{d}t + \sigma \mathrm{d}W(t) = \sigma \mathrm{d}\tilde{W}(t), \tag{1.11}$$

where $\tilde{W}(t) := (\mu - r)t/\sigma + W(t)$. The solution of (1.11) satisfies

$$\tilde{S}(t) = \tilde{S}(0)e^{\sigma \tilde{W}(t) - \sigma^2 t/2},$$

which is an $\{\mathcal{F}_t\}$-martingale if $\{\tilde{W}(t), 0 \le t \le T\}$ is standard Brownian motion adapted to $\{\mathcal{F}_t\}$. However by Girsanov's theorem this is the case under the probability measure Q whose Radon-Nikodym derivative with respect to P is

$$\frac{\mathrm{d}Q}{\mathrm{d}P} = \exp\left(-\frac{\mu - r}{\sigma}W(T) - \frac{(\mu - r)^2}{2\sigma^2}T\right). \tag{1.12}$$

Assuming the existence of a portfolio (1.4) which satisfies the self-financing condition (1.5) and the boundary condition (1.6), the discounted portfolio value is

$$\tilde{V}(t) = e^{-rt}V(t). \tag{1.13}$$

Applying Itô's formula to this expression we obtain

$$\mathrm{d}\tilde{V}(t) = -r\tilde{V}(t)\mathrm{d}t + e^{-rt}\mathrm{d}V(t) = a_t \,\mathrm{d}\tilde{S}(t),$$

and hence, from (1.11),

$$\tilde{V}(t) = \tilde{V}(0) + \int_0^t a_s \mathrm{d}\tilde{S}(s) = V(0) + \sigma \int_0^t a_s \tilde{S}(s)\mathrm{d}\tilde{W}(s). \tag{1.14}$$

Since $a_t\tilde{S}(t) \in \mathcal{F}_t$ for each $t \in [0, T]$ and, under the probability measure Q, $\tilde{W}$ is Brownian motion adapted to $\{\mathcal{F}_t\}$, we conclude that $\tilde{V}$ is an $\{\mathcal{F}_t\}$-martingale. Hence

$$\tilde{V}(t) = E_Q[\tilde{V}(T)|\mathcal{F}_t], \quad t \in [0, T],$$

and

$$V(t) = e^{rt}\tilde{V}(t) = E_Q[e^{-r(T-t)}h(S(T))|\mathcal{F}_t], \tag{1.15}$$

where $h(S(T))$ is the value of the option at time T. For the European call option $h(S(T)) = \max(S(T) - K, 0)$ and a straightforward calculation using (1.15) gives, in the notation of (1.7),

$$v(t, x) = x\Phi(z_1) - Ke^{-r(T-t)}\Phi(z_2), \tag{1.16}$$

where Φ is the standard normal cumulative distribution function,

$$z_1 = \frac{\log(x/K) + (r + \sigma^2/2)(T - t)}{\sigma\sqrt{T - t}} \text{ and } z_2 = z_1 - \sigma\sqrt{T - t}.$$

The quantity $m = (\mu - r)/\sigma$ which appears in the Radon-Nikodym derivative $\mathrm{d}Q/\mathrm{d}P$ is called the *market price of risk* and represents the excess, in units of σ, of the instantaneous rate of return μ of the risky asset S over that of the risk-free asset B. If $m = 0$ then $Q = P$ and the model is said to be *risk-neutral*.

Although the model (1.1) has many shortcomings as a representation of asset prices, the remarkable achievement of Black, Scholes and Merton in using it to derive a unique arbitrage-free option price has inspired enormous interest and progress in the field of financial mathematics. As a result of their pioneering work, research in continuous-time financial models has blossomed, with much of it directed at the construction, estimation and analysis of more realistic continuous-time models for the evolution of stock prices, and the pricing of options based on such models. In the following sections we summarize the limitations of the BSM model and briefly discuss some of the models which have been developed to provide more realistic representations of the empirical data and to permit the analysis of more complicated contingent claims.

2. Shortcomings of the BSM Model

Under the model (1.1) the sample-paths of the log asset prices are those of Brownian motion. As already indicated, this implies that, for each fixed $\Delta > 0$, the log returns $\{\log S((n + 1)\Delta) - \log S(n\Delta), n = 0, 1, 2, \ldots\}$ are independent and identically distributed Gaussian random variables. However inspection of empirical log asset prices, especially at time intervals Δ of one day or less, reveals significant negative skewness of the distribution of these increments and kurtosis which is significantly higher than the value 3, appropriate for normally distributed random variables. In order to reflect these observations we need to consider models for which the marginal distribution of the increments are non-Gaussian.

Moreover, although the observed increments typically exhibit no significant sample correlations, their squares and absolute values usually have autocorrelation functions which are significantly different from zero, indicating the need for models in which the increments are not independent as expected under the BSM model.

The observed increments appear also not to be identically distributed. Their estimated variances change with time in an apparently random manner. Assuming the validity of the BSM model it is possible to estimate the parameter σ^2 per trading day on day n by computing the sum of squares

$$\hat{\sigma}_n^2 := \sum_{i=1}^{N} \left(\log S_n \left(\frac{i}{N} \right) - \log S_n \left(\frac{i-1}{N} \right) \right)^2, \tag{2.1}$$

where the summands are the squared increments of the log price over intervals obtained by breaking the day into intervals of length $1/N$ days with N large. The sequence $\hat{\sigma}_n^2$ is known as the *realized volatility* per day. It is found in practice to vary significantly from one day to the next. The sequence $\{\hat{\sigma}_n^2\}$ of realized volatilities exhibits clustering, i.e., periods of low values interrupted by bursts of large values, and has the appearance of a positively correlated stationary sequence, reinforcing the view that volatility is not constant as in the BSM model and suggesting the need for a model in which volatility is stochastic. Such observations are precisely those which led to the development in discrete time of stochastic volatility, ARCH and GARCH models, and suggest the need for analogous models with continuous time parameter.

If we were to assume the validity of the BSM model for stock prices and also to adopt the widely-held view that in real-world markets there is no arbitrage, then the quoted price of stock options should be exactly as computed in Section 1. This argument provides us with a consistency check on the BSM model. Given the time to maturity $T - t$, the strike price K and the quoted price $Q(t)$ for a European call option at time t, then from the risk-free interest rate r and the price at time t of the stock, equation (1.16) can be used to calculate the *implied volatility*, $\sigma_t^2(K, T)$ at time t. If the BSM model is appropriate the implied volatility should be independent of K and T, however it is found in practice to depend on both. If we plot $\sigma_t(\cdot, T)$ for fixed t and T the graph usually has the appearance of a smile, the so-called *volatility smile*. The non-constancy of implied volatility is another indicator of the need to improve on the BSM model.

Finally, if we compare daily stock prices with daily values of simulated Brownian motion having the corresponding estimated drift and volatility parameters, we find that the stock prices exhibit occasional jumps which are much larger than the daily increments of the Brownian paths. This suggests that a good model for stock prices and log stock prices should allow for the possibility of jumps.

3. A general framework for option pricing

The martingale argument of Section 1 was extended by Harrison and Pliska (1981), to the much more general model in which the processes $\{S(t)\}$ and

$\{B(t)\}$ are semimartingales and the claim at time T, instead of having the form $h(S(T))$, is a non-negative random variable $X \in \mathcal{F}_T$ with $EX < \infty$, where $\{\mathcal{F}_t, 0 \leq t \leq T\}$ is the filtration generated by $\{(S(t), B(t))\}$ and the P-null sets of $\mathcal{F}$. This allows for path-dependent claim functions such as $\max_{t\in[0,T]} S(t)$. In this more general setting however the existence and uniqueness of an equivalent martingale measure is not always guaranteed and pricing on the basis of no arbitrage is not always possible. In the definition (1.5) of a self-financing strategy $\{(a_t, b_t)\}$ the processes $\{a_t\}$ and $\{b_t\}$ are required to be predictable processes such that the integrated form of (1.5) is well-defined. The self-financing condition is then equivalent to the condition that the discounted price process, $\{V(t)/B(t)\}$, has the representation,

$$\frac{V(t)}{B(t)} = V(0) + \int_0^t a_u dZ(u), \tag{3.1}$$

where $Z(t)$ is the discounted stock price $S(t)/B(t)$.

For the remainder of this section we shall assume that there is (at least one) EMM Q, i.e., a probability measure on $\mathcal{F}$ with the same null-sets as P under which $Z(t)$ is a martingale. Under this assumption the model presents no arbitrage possibility. This result is often called the *first fundamental theorem of asset pricing*. With a weaker definition of arbitrage than the one we have given, the converse also holds (see Delbaen and Schachermayer (1994)).

Let $\mathcal{L}(Z)$ be the class of predictable processes H such that the process $\{\sqrt{\int_0^t H_u^2 d[Z,Z](u)}\}$ is locally integrable under Q, where $[Z,Z]$ is the quadratic variation process of Z.

An *admissible* strategy is defined to be a predictable self-financing strategy such that $a \in \mathcal{L}(Z)$ and $\{V(t)/B(t)\}$ is a non-negative Q-martingale.

A claim X at time T is said to be *attainable* if there is an admissible strategy $\{(a_t, b_t)\}$ such that $V(T) = X$. This means that there is an admissible strategy which replicates the value of the claim at time T and for which the corresponding discounted price process $V(t)/B(t)$ is a Q-martingale. Consequently, in order to avoid arbitrage, the fair value of the option at time $t < T$ must then be

$$V(t) = B(t)E_Q(X/B(T)|\mathcal{F}_t). \tag{3.2}$$

The following result enables us to identify the attainable claims. If X is an integrable claim (i.e., if $E_Q(X/B(T)) < \infty$), then X is attainable if and only if the process $M(t) := E_Q(X/B(T)|\mathcal{F}_t)$ has the representation,

$$M(t) = M(0) + \int_0^t H(u)dZ(u), \text{ for some } H \in \mathcal{L}(Z). \tag{3.3}$$

The model is said to be *complete* if every integrable claim X is attainable. But this is the same as saying that the discounted price process $Z(t)$ has the predictable representation property, i.e., that *every* martingale has a representation (3.3) for some $H \in \mathcal{L}(Z)$. A necessary and sufficient condition for this is that the equivalent martingale measure Q is unique. This is sometimes called the *second fundamental theorem of asset pricing*.

4. Some non-Gaussian models for asset prices

The preceding section provides a very general framework for the arbitrage-free pricing of a contingent claim X based on a single stock and a money-market account when there exists a unique equivalent martingale measure Q. The fair price at time zero, when the option is purchased, is, from (3.2),

$$V(0) = B(0)E_Q(X/B(T)). \tag{4.1}$$

The expectation in (4.1) cannot generally be calculated analytically, in spite of the elegant solution for the BSM model, however it does permit the estimation of $V(0)$ by Monte-Carlo simulation once the process $\{(S(t), B(t))\}$ has been specified. In this section we consider some of the models which have been proposed for $\{S(t)\}$, in order to address the limitations of the BSM model listed in Section 2.

The first of these is the diffusion model obtained by replacing the constant parameters μ, σ and r in (1.1) and (1.3) by functions $\mu(t, S(t))$, $\sigma(t, S(t))$ and $r(t, S(t))$. In this case the first argument used in Section 1 leads to the partial differential equation (1.8) and terminal condition (1.9) with μ, σ and r replaced by the corresponding functions of t and x. The fair value of the option is obtained as before from the solution $v(t, x)$ of this partial differential equation, and the weights a_t and b_t of the self-financing portfolio which replicates $h(X(T))$ can be expressed in terms of v and its derivatives. This family of diffusion models for the stock price S allows for a much greater variety of marginal distributions than the Gaussian marginals of the BSM model, however the sample paths of S are still continuous.

In order to account for the occasional sharp changes observed in the sample-paths of asset prices and at the same time to allow for observed log returns which are not Gaussian, a natural step is to replace the exponent in (1.2) by a Lévy process L, i.e., a process with homogeneous independent increments, continuous in probability, with càdlàg sample-paths and $L(0) = 0$. This leads to the so-called *exponential Lévy model*,

$$S(t) = S(0)\exp(L(t)),$$

whose log returns over time intervals of length 1 have the distribution of $L(1)$, which can be any infinitely divisible distribution. The simplest examples of Lévy processes are Brownian motion, which has continuous sample paths, and the Poisson process, which increases only by jumps of size one. In general a Lévy process can be expressed as the sum of a Brownian motion with drift and an independent pure-jump process. Pure jump Lévy processes and exponential Lévy processes are discussed in detail in the article by Eberlein (2008) in this volume, where examples of Lévy processes which have been found especially useful in financial modelling are also given. For more extensive treatments of Lévy processes, see the books of Applebaum (2004), Bertoin (1996), Protter (2004) and Sato (1999). Except in the Brownian motion and Poisson process cases, the exponential Lévy model is incomplete. There are many equivalent martingale measures and there is no unique arbitrage-based (or risk-neutral) price for an option. The problem of choosing an EMM in this situation, computing the corresponding price of a European option and matching it to prices quoted in the market is discussed in Schoutens (2003). A general account of option pricing, covering the general situation in which there is no unique EMM is contained in the article by Kallsen (2008) in this volume.

Lévy processes also play a key role in the stochastic volatility model of Barndorff-Nielsen and Shephard (2001a, 2001b) in which the volatility σ^2 is a stationary Ornstein–Uhlenbeck (O-U) process driven by a non-decreasing Lévy process L, i.e.,

$$\sigma^2(t) = \int_{-\infty}^{t} e^{-\lambda(t-y)} dL(\lambda y), \tag{4.2}$$

where $\lambda > 0$. The log asset price $G(t) = \ln P(t)$ satisfies an equation of the form

$$dG_t = (\mu - \frac{1}{2}\sigma^2(t))dt + \sigma(t)\, dW(t) + \rho dL(\lambda t), \tag{4.3}$$

where W is a Brownian motion independent of the Lévy process and ρ is a non-positive real parameter which accounts for the so-called *leverage effect*. The autocorrelation function of the process σ^2 is $\rho(h) = \exp(-\lambda|h|), h \in \mathbb{R}$, with $\lambda > 0$, but this class of functions can be extended by specifying the volatility to be a superposition of O-U processes as in Barndorff–Nielsen (2001), or a Lévy-driven CARMA (continuous-time ARMA) process as in Brockwell (2004). The BNS model defined by (4.2) and (4.3) is incomplete. There is a family of equivalent martingale measures, the structure of which was studied by Nicolato and Venardos (2003) who argue that it is sufficient to consider the subset of EMM's under which the log returns continue to be described by a BNS model. For such an EMM Q they show how to compute $E_Q(e^{-r(T-t)}h(X(T))|\mathcal{F}_t)$ for a European contract with claim $h(X(T))$. Using gamma and inverse gamma Ornstein-Uhlenbeck processes and estimating parameters by minimizing the mean-squared error between model and market option prices they find that both models perform well when applied to European call options on the S&P500 index, giving good matches between the observed and fitted volatility smiles. The book of Schoutens (2003) discusses option pricing also for a class of stochastic volatility models in which the stock price is the exponential of a stochastically time-changed Lévy process. Simulation methods and the pricing of exotic options are also discussed.

In view of the wide use of discrete-time ARCH and GARCH models for asset prices, a great deal of research has been devoted to the development of analogous continuous-time models. An early attempt to bridge the gap between discrete-time GARCH models and continuous-time models resulted in the GARCH(1,1) diffusion approximation of Nelson (1990). An outline of the argument used by Nelson is given by Lindner (2008) in this volume. See also Drost and Werker (1996) and Duan (1997). As in the continuous-time stochastic volatility models we model the logarithm of the asset price itself, i.e., $G(t) = \log S(t)$, rather than its increments as in discrete time. Nelson's diffusion limit for the log asset price and squared volatility is the unique solution $\{(G(t), \sigma^2(t)),\ t \geq 0\}$ of the equations,

$$\mathrm{d}G(t) = \sigma(t)\,\mathrm{d}W^{(1)}(t), \quad \mathrm{d}\sigma^2(t) = \theta(\gamma - \sigma^2(t)) + \rho\sigma^2(t)\,\mathrm{d}W^{(2)}(t), \tag{4.4}$$

with initial value $(G(0), \sigma^2(0))$, where $W^{(1)}$ and $W^{(2)}$ are independent standard Brownian motions and ω, λ and θ are parameters (see Lindner (2008) for details). This model for G differs fundamentally from the GARCH(1,1) model in that it is driven by two independent processes instead of one and the squared volatility evolves independently of $W^{(1)}$. The behaviour of this diffusion limit is therefore rather different from that of a GARCH process (see Lindner (2008)).

A different approach to constructing a continuous-time analogue of the GARCH(1,1), the COGARCH(1,1) process, was taken by Klüppelberg et al. (2004). The starting point was the explicit expression for the volatility of the discrete-time GARCH(1,1) process which can be computed recursively from the difference equations,

$$\sigma_n^2 = \alpha_0 + \beta_1 \sigma_{n-1}^2 + \alpha_1 e_{n-1}^2 \sigma_{n-1}^2, \tag{4.5}$$

where $\alpha_0 > 0$, $\alpha_1, \beta_1 \geq 0$, $\alpha_1 + \beta_1 \leq 1$ and $\{e_t,\ t = 1, 2, \ldots\}$ is an iid sequence with mean 0 and variance 1. This expression is written as an integral and the noise sequence replaced by the jumps of a Lévy process. Details of the construction are contained in Lindner (2008). For GARCH(p, q) processes of higher order, there is no analogue of the explicit expression for σ_n^2, however the process $\{\sigma_n^2\}$ can be regarded as a "self-exciting" ARMA$(q-1, p)$ process driven by the sequence $\{e_{n-1}^2 \sigma_{n-1}^2\}$. This can be clearly seen in equation (4.5) where $p = q = 1$. The COGARCH(p, q) process (with $p \leq q$) is obtained by replacing the self-exciting ARMA$(q, p-1)$ equation for σ_n^2 by a corresponding self-exciting continuous-time ARMA$(q, p-1)$ equation driven by a continuous time analogue of the sequence $\{e_{n-1}^2 \sigma_{n-1}^2\}$. Details can be found in Brockwell et al. (2006) and Brockwell (2008). COGARCH processes with a stationary volatility process have properties that are closely analogous to those of discrete-time GARCH processes. In particular if $G_t^{(r)}$ denotes the increment $G(t+r) - G(t)$ then, under conditions ensuring the finiteness of $E[G_t^{(r)}]^4$, $G_t^{(r)}$ has zero mean, $\{G_{t+h}^{(1)}), h = 0, 1, 2, \ldots\}$ is an uncorrelated sequence and the corresponding sequence of squared increments has the autocovariance function of an ARMA process, while the process $\{\sigma_t^2\}$ has the autocovariance function of a continuous-time ARMA process.

The COGARCH(1,1) process with stationary volatility has been shown to have many of the features of the discrete time GARCH(1,1) process. As shown in Klüppelberg et al. (2004, 2006), the COGARCH(1,1) process has uncorrelated increments, while the autocorrelation functions of the volatility σ^2 and of the squared increments of G decay exponentially. Further, the COGARCH(1,1) process has heavy tails and volatility clusters at high levels, see Klüppelberg et al. (2006) and Fasen et al. (2004). Cluster behaviour can also be achieved in the stochastic volatility model of Barndorff-Nielsen and Shephard if the driving Lévy process has regularly varying tails. For an overview of extremes of stochastic volatility models, see Fasen et al. (2005). The discrete-time EGARCH model of Nelson (1991) was introduced in order to account for the observation that negative shocks have a greater effect on volatility than positive ones. A continuous-time analogue of the EGARCH model is the ECOGARCH model of Haug and Czado (2007).

A unifying and large family of processes which includes several of those introduced in this section is the family of generalized Ornstein-Uhlenbeck (GOU) processes (Lindner and Maller (2005), Maller, Müller and Szimayer (2008)). A GOU process X is defined, in terms of a bivariate Lévy process (ξ, η) by

$$X_t = m(1 - e^{-\xi_t}) + e^{-\xi_t} \int_0^t e^{\xi_{s-}} d\eta_s + X_0 e^{-\xi_s}, t \geq 0, \tag{4.6}$$

where X_0 is independent of $\{(\xi_t, \eta_t), t \geq 0\}$. Among the processes in this family are the stochastic volatility model of Barndorff-Nielsen and Shephard, the COGARCH(1,1) process and the GARCH(1,1) limiting diffusion of Nelson. For some of the applications of this family in option pricing, insurance and risk theory see Maller et al. (2008). The extremal behaviour of stationary GOU processes is treated in this volume by Fasen (2008).

5. Further models

The asset-price models considered in the preceding sections constitute a small but important part of the multitude of continuous-time stochastic models currently of importance in mathematical finance. In this final section we highlight a few of the important classes of models and problems, the details of which cannot be included in this brief overview.

In order to account for dependence between the price processes of different assets, multiple-asset models are required. Shreve (2004) considers option pricing based on the model,

$$dS_i(t) = \alpha_i(t) S_i(t) \mathrm{d}t + S_i(t) \sum_{j=1}^{d} \sigma_{ij}(t) \mathrm{d}W_j(t), \; i = 1, \ldots, m,$$

where the vector $[\alpha_i]_{i=1,\ldots,m}$ and the volatility matrix $[\sigma_{ij}]_{i=1,\ldots,m;j=1,\ldots,d}$ are adapted processes and W_j, $j = 1, \ldots d$, are independent standard Brownian motions. Multivariate generalizations of the BNS model and of the

COGARCH(p, q) model have also been developed by Stelzer (2007) and Pigorsch and Stelzer (2007) respectively.

Another large class of models for which there is an extensive literature are those for bonds and interest rates. For an extensive treatment of these see the book of Björk (2004) and the article by Björk (2008) in this volume. For a Lévy based approach see also Eberlein and Rauble (1999).

The estimation of volatility itself from high frequency data presents many challenging problems. In equation (2.1) we introduced the notion of realized volatility and, in the context of the BSM model, this converges as $N \to \infty$ to the parameter σ^2 (per day). However in practice, factors such as within-day variation, discreteness of the price structure and the presence of jumps, complicate the choice of N and the interpretation of the realized volatility as defined by (2.1). An extensive discussion of realized volatility is contained in the article of Andersen (2008) in this volume. Realized volatility series are generally found to exhibit very slowly decaying autocorrelation functions, suggesting the use of long-memory models or continuous-time ARMA models with an autoregressive root close to zero in order to represent them (see, e.g. Todorov (2007)).

The general problem of parameter estimation for continuous-time models is complicated by the fact that observations are always made at discrete times. When the continuous-time process is Markovian and the transition probabilities can be computed it is possible to write down the likelihood of the observations and hence to carry out estimation by maximum likelihood. Except in very special cases however the transition probabilities have no simple explicit form and approximation of the likelihood or alternative methods must be used. The papers of Mykland and Aït-Sahalia (2008) and Phillips and Yu (2008) in this volume address these problems. See also the paper of Kelly et al.(2004).

Estimation for the BNS stochastic volatility model has been carried out by Roberts et al. (2004) and Gander and Stephens (2007) using Markov chain Monte-Carlo methods and estimation for COGARCH(1,1) models by Haug et al. (2007) using method of moments estimation.

There still remain many intriguing and challenging problems for the modelling of asset prices. The models described in this overview have provided a great deal of insight into the dynamics of price movements and the critical role of market volatility. They have also been of practical value in the pricing of options. Much remains to be discovered however, particularly with regard to the intra-day price movements and the factors affecting them. For the analysis of tick by tick (or ultra-high-frequency) data it is necessary to take into account both the discrete times at which transactions occur and the price changes at each transaction. The autoregressive conditional duration (ACD) model of Engle and Russell (1998) was constructed for this purpose. The analysis of high frequency data casts light on the trading mechanism and the detailed operation (or microstructure) of the market and remains a particularly active area of research. The book of Tsay (2005) contains a clear account, with applications, of such models.

Acknowledgments I am indebted to the National Science Foundation for support of this work under the grant DMS-0744058 and to Alexander Lindner for valuable comments on the manuscript.

References

1. Andersen, T. (2008). Realized volatility. *Handbook of Financial Time Series.* Springer-Verlag.
2. Applebaum, D. (2004). *Lévy Processes and Stochastic Calculus.* Cambridge University Press, Cambridge.
3. Bachelier, L. (1900). Théorie de la spéculation, *Annales de lÉcole Normale Supérieure*, **17**, 21–86.
4. Barndorff-Nielsen, O.E. (2001). Superposition of Ornstein-Uhlenbeck type processes. *Theory Probab. Appl.* **45**, 175–194.
5. Barndorff-Nielsen, O.E. and Shephard, N. (2001a). Non-Gaussian Ornstein–Uhlenbeck based models and some of their uses in financial economics (with discussion). *J. Roy. Statist. Soc. Ser. B* **63**, 167–241.
6. Barndorff-Nielsen, O.E. and Shephard, N. (2001b). Modelling by Lévy processes for financial econometrics. In *Lévy Processes - Theory and Applications* (ed. O.E. Barndorff-Nielsen, T. Mikosch and S. Resnick). 283–318, Birkhauser, Boston.
7. Bertoin, J. (1996). *Lévy Processes.* Cambridge University Press, Cambridge.
8. Björk, Thomas (2004). *Arbitrage Theory in Continuous Time* . Oxford University Press, Oxford.
9. Black, F. and Scholes, M. (1973). The pricing of options and corporate liabilities. *J. Political Economy*, **81**, 637–654.
10. Brockwell, P.J. (2004). Representations of continuous-time ARMA processes. *J. Appl. Probab.* **41A**, 375–382.
11. Brockwell, P.J. (2008). Lévy-driven continuous-time ARMA processes. *Handbook of Financial Time Series.* Springer-Verlag.
12. Brockwell, P.J., Chadraa, E., and Lindner, A., (2006). Continuous-time GARCH processes, *Ann. Appl. Prob.*, **16**, 790–826.
13. Campbell, J., Lo, A. and McKinlay, C. (1996). *The Econometrics of Financial Markets.* Princeton University Press, Princeton.
14. Davis, R.A. and Mikosch, T. (2008). Probabilistic properties of stochastic volatility models *Handbook of Financial Time Series.* Springer-Verlag.
15. Delbaen, F. and Schachermayer, W. (1994). A general version of the fundamental theorem of asset pricing. *Mathematische Annalen* **300**, 463–520.
16. Drost, F.C. and Werker, B.J.M. (1996). Closing the GARCH gap: continuous time GARCH modelling. *J. Econometrics* **74**, 31–57.
17. Duan, J.C. (1997). Augmented GARCH(p,q) process and its diffusion limit. *J. Econometrics* **79**, 97-127.
18. Eberlein, E. (2008). Jump-type Lévy processes. *Handbook of Financial Time Series.* Springer-Verlag.

19. Eberlein, E. and Raible, S. (1999). Term structure models driven by general Lévy processes. *Mathematical Finance* **9**, 31–53.
20. Engle, R.F. and Russell, J.R. (1998). Autoregressive conditional duration: a new model for irregularly spaced transaction data. *Econometrica* **66**, 1127–1162.
21. Fasen, V. (2008). Extremes of continuous-time processes. *Handbook of Financial Time Series*. Springer-Verlag.
22. Fasen, V., Klüppelberg, C. and Lindner, A. (2005). Extremal behavior of stochastic volatility models. In *Stochastic Finance*, eds. A. Shiryaev, M.R. Grossihno, P.E. Oliviera, M.L. Esquivel. Springer-Verlag, Heidelberg.
23. Gander, M.P.S. and Stephens, D.A. (2007). Simulation and inference for stochastic volatility models driven by Lévy processes. *Biometrika* **94**, 627–646.
24. Harrison, J.M. and Pliska, S.R. (1981). Martingales and stochastic integrals in the theory of continuous trading. *Stochastic Processes Appl.* **11**, 215–260.
25. Haug, S. and Czado, C. (2007). An exponential continuous-time GARCH process. *J. Appl. Prob.* **44**, 960–976.
26. Haug, S., Klüppelberg, C., Lindner, A. and Zapp, M. (2007). Method of moment estimation in the COGARCH(1,1) model. *The Econometrics Journal* **10**, 320–341.
27. Jones, R.H. (1985). Time series analysis with unequally spaced data. In *Time Series in the Time Domain*, Handbook of Statistics **5**, eds. E.J. Hannan, P.R. Krishnaiah and M.M. Rao. North Holland, Amsterdam, 157–178.
28. Kallsen, J. (2008). Option pricing. *Handbook of Financial Time Series*. Springer-Verlag.
29. Klebaner, Fima (2005). *Introduction to Stochastic Calculus with Applications*. Imperial College Press, London.
30. Klüppelberg, C., Lindner, A. and Maller, R. (2004). A continuous time GARCH process driven by a Lévy process: stationarity and second order behaviour. *J. Appl. Probab.* **41**, 601–622.
31. Klüppelberg, C., Lindner, A. and Maller, R.A. (2006). Continuous time volatility modelling: COGARCH versus Ornstein-Uhlenbeck models. In *From Stochastic Calculus to Mathematical Finance: The Shiryaev Festschrift*, eds Yuri Kabanov, R. Liptser and J. Stoyanov, Springer-Verlag, Heidelberg, 393–420.
32. Lindner, A. (2008). Continuous-time GARCH processes. *Handbook of Financial Time Series*. Springer-Verlag.
33. Lindner, A. and Maller, R.A. (2005). Lévy integrals and the stationarity of generalised Ornstein-Uhlenbeck processes. *Stoch. Proc. Appl.*, **115**, 1701–1722.
34. Merton, R. (1973). The theory of rational option pricing. *Bell J. Economics and Management Science*, **4**, 141–183.
35. Mikosch, Thomas (1998). *Elementary Stochastic Calculus with Finance in View*. World Scientific, Singapore.
36. Mykland, P. and Aït-Sahalia, Y. (2008). Diffusions. *Handbook of Financial Time Series*. Springer-Verlag.
37. Nelson, D. (1990). ARCH models as diffusion approximations. *J. Econometrics* **45**, 7–38.
38. Nicolato E. and Vernatos, E. (2003). Option pricing in stochastic volatility models of Ornstein-Uhlenbeck type. *Mathematical Finance* **13**, 445–466.
39. Phillips, P.C.B. and Yu, J. (2008). Maximum likelihood and Gaussian estimation of continuous time models in finance. *Handbook of Financial Time Series*. Springer-Verlag.

40. Pigorsch, C. and Stelzer, R. (2007). A Multivariate Generalization of the Ornstein-Uhlenbeck Stochastic Volatility Model. *http://www-m4.ma.tum.de/Papers/Stelzer/PigorschStelzerMultOU.pdf*
41. Protter, P.E. (2004). *Stochastic Integration and Differential Equations*. 2nd edition, Springer, New York.
42. Roberts, G., Papaspiliopoulos, O. and Dellaportas, P. (2004). Bayesian inference for non-Gaussian Ornstein-Uhlenbeck stochastic volatility processes. *J. R. Statist.Soc.B* **66**, 369–393.
43. Samuelson, P.A. (1965). Rational theory of warrant pricing, *Ind. Management Review*, **6**, 13–31.
44. Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions* . Cambridge University Press, Cambridge.
45. Schoutens, Wim (2003). *Lévy Processes in Finance*. John Wiley and Sons, Chichester.
46. Shephard, Neil (1996). Statistical aspects of ARCH and stochastic volatility. In *Time Series Models in Econometrics, Finance and Other Fields*, eds. D.R.Cox, D.V.Hinkley and O.E.Barndorff-Nielsen. Chapman and Hall, London, 1–67.
47. Shephard, Neil (2008). Stochastic volatility. *Handbook of Financial Time Series*. Springer-Verlag.
48. Shreve, Steven E. (2004). *Stochastic Calculus for Finance II: Continuous-time Models*. Springer-Verlag, New York.
49. Steele, J.M. (2001). *Stochastic Calculus and Financial Applications*. Springer-Verlag, New York.
50. Stelzer, R. (2007). Multivariate continuous time Lévy-driven GARCH processes. Preprint
51. Todorov, V. (2007). Econometric analysis of jump-driven stochastic volatility models. *www.kellogg.northwestern.edu/faculty/todorov/htm/papers/jdsv.pdf*
52. Tsay, Ruey S. (2005). *Analysis of Financial Time Series*, John Wiley and Sons, Hoboken.