Short Communication
The Geometric Convergence Rate of a Lindley Random Walk

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Abstract

Let \( \{X_n\} \) be the Lindley random walk on \([0,\infty)\) defined by \( X_n = \max\{X_{n-1} + A_n, 0\} \) for \( n \geq 1 \) with \( X_0 = x \geq 0 \). Here, \( \{A_n\} \) is a sequence of independent and identically distributed random variables. When \( E[A_1] < 0 \) and \( E[r^{A_1}] < \infty \) for some \( r > 1 \), \( \{X_n\} \) converges at a geometric rate in total variation to an invariant distribution \( \pi \); that is, there exists \( r > 1 \) such that

\[
\lim_{n \to \infty} r^n \sup_B |P_{x} [X_n \in B] - \pi(B)| = 0
\]

for every initial state \( x \geq 0 \). In this communication, we supply a short proof and some extensions of a result initially due to Veraverbeke and Teugels (1975 and 1976): the largest \( r \) satisfying the above relationship is \( \phi(r_0)^{-1} \) where \( \phi(r) = E[r^{A_1}] \) and \( r_0 > 1 \) satisfies \( \phi'(r_0) = 1 \).

MARKOV CHAIN; GEOMETRIC CONVERGENCE; TOTAL VARIATION; QUEUES.

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1. Introduction. Consider the Lindley random walk \( \{X_n\} \) driven by an independent and identically distributed (iid) random sequence \( \{A_n\} \):

\[
X_n = \max\{X_{n-1} + A_n, 0\} \quad \text{for } n \geq 1,
\]

with a possibly random initial state \( X_0 \geq 0 \). The recursion in (1.1) governs customer waiting times in queues [2,5] and arises in discrete storage modeling [5].

Equation (1.1) is also useful for simulating a probability distribution \( \pi \) on \([0, \infty)\) that satisfies the Wiener-Hopf equation

\[
\pi([0,x]) = \int_{[0,\infty)} F(z-y)\pi(dy) \quad \text{for } x \geq 0,
\]

where \( F \) is a known cumulative distribution function supported on \((-\infty, \infty)\) with a strictly negative mean. For this, one generates an iid \( \{A_n\} \) with distribution function \( F \), selects an initial state \( X_0 = x \), and then recurses with (1.1) to generate \( \{X_n\} \). The convergence rate results presented below can be used to find a natural number \( n \) where the distribution of \( X_n \) is sufficiently close to \( \pi \).

It is well known that \( \{X_n\} \) is a Markov chain that has a unique invariant probability distribution \( \pi \) if and only if \( E[A_1] < 0 \) [2,7] which we henceforth assume. To avoid degeneracy in \( \{X_n\} \), we assume that \( A_1 \) is nondegenerate with \( P(A_1 > 0) > 0 \). Let \( \phi(r) = E[r^{A_1}] \) and suppose that

(i) There exists \( r_0 > 1 \) with \( \phi(r_0) < \infty \) satisfying \( \phi'(r_0) = 0 \); and

(ii) If \( A_1 \) is lattice, then \( P(A_1 = 0) > 0 \).

A classic result, proven in [8] and [9], states that when \( X_0 = 0 \) and (i) and (ii) hold, \( P[X_n \leq x] \) converges geometrically fast to \( \pi([0,x]) \) uniformly in \( x \geq 0 \). The result also identifies the exact geometric convergence rate as \( \phi(r_0)^{-1} \). We note that \( r_0 \) is unique and \( \phi(r_0) < 1 \); these properties follow from \( E[A_1] < 0 \), the convexity of \( \phi \), and (i). Our objective here is to give a short proof of this result based on sample path orderings; we also upgrade the converge mode to total variation and consider the case where \( X_0 > 0 \). Specifically, we show that when \( E[A_1] < 0 \) and (i) and (ii) hold,

\[
\lim_{n \to \infty} r^n \sup_A \big| P_x[X_n \in A] - \pi(A) \big| = 0
\]

for every \( x \geq 0 \) if and only if \( r \leq \phi(r_0)^{-1} \). In (1.2), the notation \( P_x \) indicates that \( X_0 \equiv x \). In
addition, we will examine geometric convergence of process moments; that is, for a function $f$, we investigate the values of $r > 1$ where

$$
\lim_{n \to \infty} r^n |E_x[f(X_n)] - \pi(f)| = 0
$$

(1.3)

for all $x \geq 0$. In (1.3), $\pi(f)$ denotes the $f$th moment of $\pi$:

$$
\pi(f) = \int_{[0,\infty)} f(x)\pi(dx).
$$

By stationarity, $\pi(f) = E_x[f(X_n)]$ for all $n \geq 1$; in the above, $E_x$ and $E_\pi$ denote expectation when $X_0 = x$ and $X_0$ has distribution $\pi$ respectively.

2. Results. From (1.1), if $\{X_n\}$ and $\{X'_n\}$ are trajectories of the chain driven by the same $\{A_n\}$ with $X_0 \geq X'_0$, then $X_n \geq X'_n$ for all $n \geq 1$; hence $\{X_n\}$ is pathwise ordered and the results of [6] apply. Let

$$
\tau_0 = \inf\{n > 0: X_n = 0\} \quad \text{and} \quad G_a(r) = E_a[r^{\tau_0}].
$$

From $P(A_1 > 0) > 0$ and the Markov property of $\{X_n\}$, it is easy to see that $G_a(r)$ has the same radius of convergence in $r$, say $r^\ast$, for each $x \geq 0$.

**Lemma 1.** Let $\{X_n\}$ be the Lindley random walk in (1.1) with $E[A_1] < 0$ that satisfies (i) and (ii).

a) If $r$ is such that $G_a(r) < \infty$, then (1.2) holds for all $x \geq 0$.

b) Equation (1.2) fails when $x = 0$ and $r > r^\ast$.

**Proof:** Statement a) is Corollary 5.2 of [6]. To prove b), the argument in Theorem 6.2 of [6] must be modified as Equation (6.1) of [6] does not hold. Towards this, consider two copies of the chain, say $\{X_n\}$ and $\{X'_n\}$, driven by the same $\{A_n\}$, but with the different (possibly) initial conditions $X_0 = 0$ and $X'_0 = D$, where $D$ is random with distribution $\pi$ and is independent of $\{A_n\}$. Since $\{X'_n\}$ is stationary, $\pi(B) = P[X'_n \in B]$ for all $n \geq 1$ and all measurable $B$. Hence,

$$
\sup_B |P_0[X_n \in B] - \pi(B)| \geq |P[X_n = 0] - P[X'_n = 0]|
$$

(2.1)

Now define the coupling time $T = \inf\{n \geq 0: X_n = X'_n\}$. Since $X_n \leq X'_n$ for all $n$, $X_n = X'_n$ whenever $T \leq n$; hence, $P[X_n = 0 \cap T \leq n] = P[X'_n = 0 \cap T \leq n]$ and (2.1) is

$$
\sup_B |P_0[X_n \in B] - \pi(B)| \geq |P[X_n = 0 \cap T > n] - P[X'_n = 0 \cap T > n]|.
$$

(2.2)

Further, if $X'_n = 0$, then $X_n = 0$ and $T \leq n$; hence, the right hand side of (2.2) is $P[X_n = 0 \cap T > n]$. Suppose that $\tau_0 = \inf\{k > 0: X_k = 0\} = n$ and $S_n = A_1 + \ldots + A_n \in [-\Delta, 0]$ for some fixed $\Delta \geq 0$.
Then $X_n = 0$; furthermore, if $D > \Delta$, then $S' = D + A_1 + \ldots + A_n > 0$, $\tau_0 = \inf\{k > 0: X_k = 0\} > n$, and $T > n$. Thus, (2.2) gives

$$
\sup_B \left| P_0[X_n \in B] - \pi(B) \right| \geq P[\tau_0 = n \land S_n \geq -\Delta \land D > \Delta] \geq \pi((\Delta, \infty)) P[S_n \geq -\Delta \mid \tau_0 = n] P[\tau_0 = n],
$$

where the last line in (2.3) follows from the independence of $D$ and $\{A_n\}$.

When (i) and (ii) hold, [1] shows that $\lim_{n \to \infty} P[S_n \geq -\Delta \mid \tau_0 = n] = G(\Delta)$ for all $\Delta \geq 0$ where $G$ is a proper cumulative distribution function; hence, there is a $\Delta < \infty$ such that $G(\Delta) > 0$. Since $P(A_1 > 0)$, the support set of $\pi$ is $[0, \infty)$ and $\pi((\Delta, \infty)) > 0$ for all $\Delta > 0$. Selecting such a $\Delta > 0$, multiplying (2.3) by $r^n$, and taking a limit supremum gives

$$
\lim_{n \to \infty} \sup_B r^n \sup_B \left| P_0[X_n \in B] - \pi(B) \right| \geq \pi((\Delta, \infty)) G(\Delta) \lim_{n \to \infty} \sup_r r^n P[\tau_0 = n] = \infty
$$

when $r > r^*$; this proves b). We remark that the subscripts on $P$ have been suppressed; this causes no confusion as the probability space is that which supports both $\{A_n\}$ and $D$. \hfill \Box

When $E[A_1] < 0$, $\pi(0) > 0$; hence, Lemma 1 establishes "divergence" for $r > r^*$ on a set of positive measure with respect to $\pi$.

**Corollary 2.** Let $\{X_n\}$ be the Lindley random walk in (1.1) with $E[A_1] < 0$ that satisfies (i) and (ii). Then (1.2) holds for all $x \geq 0$ when $r \leq \phi(r_0)^{-1}$ and fails for $x = 0$ when $r > \phi(r_0)^{-1}$.

**Proof.** Equation I 6.78 in [5] identifies the form of $G_0(r)$ as

$$
G_0(r) = 1 + (r - 1) \exp \left[ \sum_{k=1}^{\infty} \frac{r^k}{k} P(S_k > 0) \right],
$$

where $S_k = A_1 + \ldots + A_k$ for $k \geq 1$. Hence, $G_0(r) < \infty$ if and only if $H(r) = \sum k^{-1} r^k P(S_k > 0) < \infty$. Theorem 1 of [4] identifies the radius of convergence of $H(r)$ as $\phi(r_0)^{-1}$.

To see that $H(r) < \infty$ when $r = \phi(r_0)^{-1}$, we apply the asymptotic expansion $P(S_k > 0) \sim M[\phi(r_0)]^{-1/2}$ of [3] ($M$ here is a finite constant). An appeal to Lemma 1 completes the proof. \hfill \Box

**Remark:** The deviations bound $P(S_k > 0) \leq \phi(\alpha)^k$ for $\alpha \geq 1$ shows that $G_0(r) < \infty$ for $r < \phi(r_0)^{-1}$ when used in (2.4). By Corollary 5.2 of [6], (1.2) holds for all $x \geq 0$ and $r < \phi(r_0)^{-1}$; hence, (ii) can be relaxed. The proof of Part b) of Lemma 1 shows that (1.2) fails at $x = 0$ for all $r > 1$ when $\phi(r) = \infty$.
for all $r > 1$; hence, one must have $\phi(r) < \infty$ for some $r > 1$ to achieve geometric convergence. In cases where $\phi$ does not achieve its minimal value, one can argue as above and show that (1.2) holds for all $x \geq 0$ and $r < \inf\{\phi(r); r > 1\}^{-1}$; hence, (ii) can also be relaxed. In general, it is not clear whether (1.2) holds for $r = \inf\{\phi(r); r > 1\}^{-1}$ when (i) and/or (ii) do not hold.

**REMARK:** For simulation purposes, one can take $x = 0$; however, a bound for the first constant multiplying the geometric decay rate in (1.2) is also needed to identify an $n$ where the distribution of $X_n$ is sufficiently close to $\pi$ in a total variational sense. Following the arguments in [6], we obtain

$$
\sup_B \left| P_0[ X_n \in B ] - \pi(B) \right| \leq C(r)r^{-n},
$$

where $C(r) \leq G_\pi(r) \leq [G_0(r) - 1]/[r - 1]$. Combining this with (2.4) gives $C(r) \leq \exp(H(r))$. The bound $P(S_k > 0) \leq \phi(r_0)^k$ (assuming (ii) holds) and the identity $\sum_1^\infty k^{-1}x^k = -\ln(1 - x)$ for $0 \leq x < 1$ provide $C(r) \leq [1 - r\phi(r_0)]^{-1}$ for $r < \phi(r_0)^{-1}$ as required.

Now let $f : [0, \infty) \rightarrow [0, \infty)$ be a general function. The following result establishes when the moment convergence in (1.3) takes place.

**THEOREM 3.** Let $\{X_n\}$ be the Lindley random walk in (1.1) with $E[A_1] < 0$ that satisfies (i) and (ii).

a) If $r < \phi(r_0)^{-1}$ and $f(x) \leq Mx^\gamma$ for all $x \geq 0$ and some $M < \infty$, then (1.3) holds for all $x \geq 0$.

b) If $f$ is nondecreasing and $f(x + \Delta) - f(x) \geq M > 0$ for all $x \geq 0$ and some $M > 0$ and $\Delta > 0$, then (1.3) fails when $r > \phi(r_0)^{-1}$ and $x = 0$.

**PROOF.** To prove a), Theorems 3.1 and 5.1 of [6] show that it is sufficient to establish $E_x[ r_0^{r_0} ] \leq \kappa r_0^x$ for $r < \phi(r_0)^{-1}$ and some $\kappa < \infty$ ($\kappa$ may depend on $r$). For this, we use

$$
E_x[ r_0^{r_0} ] = 1 + (r-1) \sum_{n=0}^\infty r^n P_x[ \tau_0 > n ],
$$

the bound $P_x[ \tau_0 > n ] \leq P[ x + A_1 + \ldots + A_n > 0 ] \leq r_0^x \phi(r_0)^n$, and $r_0^x \geq 1$ to obtain $E_x[ r_0^{r_0} ] \leq \kappa r_0^x$ for $r < \phi(r_0)^{-1}$ where $\kappa = r[1 - r\phi(r_0)]^{-1} < \infty$.

For b), we use the notation and arguments in the proof of Lemma 1 to get

$$
| E[f(X_n)] - \pi(f) | = | E[ f(X_n) I_{[T > n]} ] - E[ f(X_n') I_{[T > n]} ] | \leq E[ f(X_n') I_{[T > n]} ] - E[ f(X_n) I_{[T > n]} ] (2.5)
$$

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where the last line in (2.5) follows from \( f(X_n) \geq f(X_n') \) for all \( n \) (by the nondecreasing \( f \)). Making the decomposition \( \{T > n\} = \{T > n \cap \tau_0 > n\} \cup \{T > n \cap \tau_0 \leq n\} \) in (2.5) and using
\[
E\left[ |f(X_n) - f(X_n')| \mathbb{1}_{\{T > n \cap \tau_0 \leq n\}} \right] \geq 0
\]
gives
\[
|E[f(X_n)] - \pi(f)| \geq E\left[ |f(X_n) - f(X_n')| \mathbb{1}_{\{T > n \cap \tau_0 > n\}} \right] \geq 0
\] (2.6)

Now if \( \tau_0 > n \), then \( X'_k = X_k + D \) for \( 1 \leq k \leq n \) and the event \( T > n \) has also occurred. Hence, \( \{T > n\} \cap \{\tau_0 > n\} = \{\tau_0 > n\} \) and (2.6) and the assumed properties of \( f \) provide
\[
|E[f(X_n)] - \pi(f)| \geq E\left[ |f(X_k + D) - f(X_k)| \mathbb{1}_{\{\tau_0 > n\}} \right] \geq \mu \pi((\Delta, \infty)) \mathbb{P}[\tau_0 > n].
\] (2.7)

Multiplying both sides of (2.7) by \( r^n \), taking a limit supremum, and using the fact that the radius of convergence of \( G_0(r) \) is \( \phi(r_0)^{-1} \) finishes the proof of (b).

It is clear that the assumptions on \( f \) in Part \( b) \) of Theorem 3 could be weakened with a more detailed analysis. However, we note that typical "moment" functions, such as the power class \( f(x) = x^\alpha, \alpha \geq 1 \), and exponential class \( f(x) = \exp(\beta x), \beta > 0 \), satisfy these assumptions.

3. Examples.

**Example 3.1.** Suppose that \( A_1 = P - 1 \) where \( P \) has a Poisson distribution with parameter \( \lambda < 1 \). Then (i) and (ii) hold, \( \phi(r) = \exp\{-\lambda(1 - r) - \ln(r)\} < \infty \) for all \( r \geq 1 \), \( r_0 = \lambda^{-1} \), and (1.2) holds for all \( x \geq 0 \) if and only if \( r \leq \phi(r_0)^{-1} = \lambda^{-1} e^{\lambda^{-1}} \). Theorem 3 shows that, for example, (1.3) holds for \( f(y) = y^\alpha, \alpha \geq 1 \), if \( r < \lambda^{-1} e^{\lambda^{-1}} \) and fails when \( x = 0 \) and \( r > \lambda^{-1} e^{\lambda^{-1}} \).

Now suppose that \( A_1 = E - 1 \) where \( E \) has the exponential density \( \mu e^{\mu y} \) for \( y \geq 0 \) with \( \mu > 1 \). Then (i) and (ii) hold, \( \phi(r) = \mu[r(\mu - \ln(r))]^{-1} \) for \( 1 \leq r < e^\mu \), \( r_0 = e^{\mu^{-1}} \), and (1.2) holds for all \( x \geq 0 \) if and only if \( r \leq \phi(r_0)^{-1} = \mu^{-1} e^{\mu^{-1}} \). Again we have that, essentially, \( E_x[X_n^\alpha] \) converges geometrically to its limit for all \( \alpha \geq 1 \) and \( x \geq 0 \) with best geometric rate \( \mu^{-1} e^{\mu^{-1}} \).

**Example 3.2.** Consider a GI/GI/1 queue where \( S_n \) is the service time of the \( n \)th customer and \( I_n \) is the interarrival time between the \( n \)th and \( (n + 1) \)st customers; here, \( \{S_n\} \) and \( \{I_n\} \) are independent iid series of nonnegative random variables. The first customer arrives at time 0 and encounters a server
with workload $x \geq 0$ before his/her service begins.

Let $Q_n$ be the time the $n$th customer spends waiting for his/her service to commence (the virtual waiting time). Then $\{Q_n\}$ satisfies $Q_n = \max(Q_{n-1} + S_{n-1} - I_n, 0)$ for $n \geq 1$ with $Q_0 = x$ [2,5,7]; hence, (1.1) holds with $A_n = S_{n-1} - I_n$. If $E[S_0] < E[I_1]$ and (i) and (ii) hold with $\phi(r) = E[r^{S_0-I_1}]$, then $\{Q_n\}$ has a limiting distribution $\pi$ and the best geometric convergence rate is $\phi(r_0)^{-1}$ where $\phi'(r_0) = 0$; in general, $r_0$ must be obtained case by case. By Theorem 3, all moments $E_x(Q_n^\alpha)$, $\alpha \geq 1$, converge geometrically to their limits with “best” geometric rate $\phi(r_0)^{-1}$.

Convergence rates for other quantities in the queue can also be obtained from the virtual waiting time rates. For example, the total time the $n$th customer spends in the queue, denoted $W_n$, is $W_n = Q_n + S_n$. Let $\{Q_n\}$ and $\{Q_n'\}$ be trajectories of the virtual waiting time chain driven by the same $\{I_n\}$ and $\{S_n\}$ with $Q_0 = x$ and $Q'_0$ having the stationary virtual waiting time distribution. Then $T_Q = \inf\{n \geq 0: Q_n = Q_n'\} = T_W = \inf\{n \geq 0: W_n = W'_n\}$ and the coupling times for the virtual and total waiting times are identical (here, $\{W'_n\}$ is a stationary total waiting time chain constructed from $\{Q'_n\}$ in the obvious manner). Hence, (1.2) also holds for $r \leq \phi(r_0)^{-1}$ for $\{W_n\}$.

References


