Exact Confidence Intervals for a Variance Ratio
(or Heritability) in a Mixed Linear Model

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SUMMARY

A family of procedures is given to construct confidence intervals for the heritability coefficient in a mixed linear model. The procedures are applicable for constructing confidence intervals for a ratio of variance components in a mixed linear model having two sources of variation. If the random effects are correlated, the procedure is valid even when there are zero degrees of freedom for error. The resulting intervals are evaluated in terms of bias and expected length. A sufficient condition for local unbiasedness is given and a numerical procedure is discussed for computing expected lengths. The investigator may select the best confidence interval procedure from the family of procedures based on these criteria. Computer software for obtaining the ‘best’ interval is available from the authors.

1 Introduction

Mixed linear models are frequently used in applications such as plant and animal breeding. In these fields of study the components of variation are often perceived as having either a genetic or environmental (non-genetic) origin. Mixed linear models having two variance components are often used with the random effect corresponding to the genetic source and the error corresponding to the environmental source. In many cases it is assumed that the genetic and environmental effects are independent. However, the observational units are not independent of one another if they possess common genetic material and the covariance between two observations will usually involve the genetic component of the overall variance. Additionally, the random effects themselves may be correlated with one another depending on the genetic relationship between the corresponding observational units. Usual analysis of variance procedures do not directly apply to such situations.

Key words: Confidence intervals; Mixed linear model; Variance components; Unbiasedness; Expected length.
In mixed linear models inferences concerning ratios of variance components are often of primary importance. If $\sigma_a^2$ is the 'additive' component of genetic variance and $\sigma_e^2$ is the 'environmental' variance component, then the quantity $\rho = \sigma_a^2/(\sigma_a^2 + \sigma_e^2)$ is the proportion of total variation due to genetic effects. $\rho$ is usually referred to as the heritability of the trait under study. This is an important parameter in many biological applications and much has been written about statistical inference procedures for $\rho$ under various model assumptions. The ratio of variance components has also been widely discussed in mixed linear model literature. If $\gamma = \sigma_a^2/\sigma_e^2$, then $\rho = \gamma/(1 + \gamma)$ so that heritability is an increasing function of the ratio of variance components. It follows that inferences for one parameter can readily be converted to inferences for the other parameter.

Point estimation of a ratio of variance components has been addressed through the use of ANOVA, MINQUE, ML, and REML techniques (Hartley and Rao (1967), Harville (1977), and Searle, Casella and McCulloch (1991)). LaMotte and McWhorter (1978), LaMotte, McWhorter and Prasad (1988), Westfall (1989), and Lin and Harville (1991) discussed hypothesis tests for ratios of variance components. Wald (1940) developed a confidence interval for a ratio of variance components in an unbalanced one-way random effects model. Spjøtvoll (1968) examined confidence intervals and Thomsen (1975) discussed hypotheses testing procedures for variance ratios in an unbalanced two-way random effects model.

Wald's (1940) procedure for a confidence interval for $\gamma$ is applicable for sire models. Harville and Fenech (1985) extend Wald's (1940) approach to mixed linear models with two variance components under the assumption that the degrees of freedom for error are positive. Although this assumption is generally met when considering sire models, it is often not met when considering a full animal model.

Procedures are needed which do not require such a restriction on the error degrees of freedom. This fact is illustrated by the animal breeding example in Section 4. It turns out that the tests for $\gamma$ suggested by LaMotte and McWhorter (1978) can be inverted to obtain confidence intervals and these procedures apply even when the error degrees of freedom are zero. Lin and Harville make this observation in their 1991 paper where they proposed
alternative confidence regions for $\gamma$ based on locally most powerful tests and some 'extended' Neymann-Pearson tests for $\gamma$.

These alternative procedures of Lin and Harville (1991) are computationally intensive whereas the procedures based on the LaMotte and McWhorter (1978) tests are relatively simple. However, to our knowledge, none of these procedures have been systematically investigated nor their operating characteristics well understood. For instance, conditions are not known under which the extended Neyman-Pearson confidence regions of Lin and Harville (1991) will actually be intervals. Also, LaMotte and McWhorter (1978) propose a family of pivotal quantities on which to base a test about $\gamma$ but no guidance is provided as to which of the several pivotal quantities is appropriate for use in any given situation.

Confidence intervals arising from the LaMotte and McWhorter (1978) tests of $\gamma$ (hereafter referred to as LM confidence intervals) are attractive because of their computational simplicity. Moreover, in simulation studies comparing their procedures to Neyman-Pearson results of simple versus simple hypotheses tests, LaMotte, McWhorter and Prasad (1988) found that neither class of procedures dominated the other. Although this was a limited simulation study, the results suggest that the LM confidence intervals are worthy of further consideration.

The inferences in this paper focus on $\rho$ due to its interpretation in plant and animal breeding studies. We investigate the bias and expected length properties of LM confidence interval procedures. For equal-tailed confidence intervals for $\rho$ we identify a locally unbiased confidence interval when one exists. We also discuss numerical procedures for evaluating the expected lengths of each member of the class of LM confidence intervals. The usefulness of the bias and expected length information is made clear by examining in detail the examples presented in this paper. In doing so, we identify certain members of the LM class of confidence intervals that have satisfactory bias and expected length properties.

The paper is organized as follows. Section 2 sets up the mixed model framework considered in this paper. In Section 3 we first present results concerning the bias of the LM confidence interval procedures and then discuss the calculation of their expected lengths. To
illustrate these results we present an animal breeding example in Section 4. In addition, a simple unbalanced mixed linear model example is presented to show that the confidence interval corresponding to Wald's (1940) interval is usually biased and does not necessarily have uniformly minimum expected length. A summary and additional discussions are presented in Section 5. The proofs of various propositions are given in the Appendix.

2 Mixed Linear Model Notation and Terminology

The mixed linear model under consideration is

$$Y = X\beta + Zu + e,$$  \hspace{1cm} (1)

where $Y$ is a $n \times 1$ vector of observable random variables, $\beta$ is a $p \times 1$ vector of unknown parameters, and $u$ and $e$ are vectors of unobservable random variables of size $m \times 1$ and $n \times 1$, respectively. The incidence matrices $X$ and $Z$ are known and without loss of generality, $\text{rank}(X) = p$. It is assumed that $u$ and $e$ are independent where $u \sim \text{MVN}(0, \sigma_u^2 A)$ and $e \sim \text{MVN}(0, \sigma_e^2 I_n)$. In animal breeding contexts, the known matrix $A$ is referred to as the relationship matrix since it describes the degree to which the $u$'s are related. In that scenario, if the elements $u_1$ and $u_2$ of $u$ are the (additive) genetic effects corresponding to a parent and offspring, respectively, then $\text{Cov}(u_1, u_2) = \sigma_u^2 / 2$ (see Falconer, 1989, p.150).

It follows that $Y \sim \text{MVN}(X\beta, \sigma_u^2 I_n + \sigma_e^2 ZAZ')$. It is interesting to note that if $Z = I_n$, then every unobservable random variable $u_i$ has an associated observable random variable $Y_i$. This particular model is referred to as a full animal model.

Using the notation given earlier, $\gamma = \sigma_u^2 / \sigma_e^2$. In the usual manner, we take $\sigma_e^2 \geq 0, \sigma_u^2 > 0$ so that $0 \leq \gamma < \infty$ and $0 \leq \rho < 1$. Let $H$ be a $n \times (n-p)$ matrix whose columns span the space orthogonal to the columns of $X$ and satisfy $H'X = I_{n-p}$. Then $H'Y \sim \text{MVN}(0, \sigma_u^2 I_{n-p} + \sigma_e^2 H'ZAZ'H)$. Let $0 \leq \Delta_1 < ... < \Delta_d$ be the distinct eigenvalues of $H'ZAZ'H$ having multiplicities $r_1, ..., r_d$, respectively. There exists an $(n-p) \times (n-p)$ orthogonal matrix $P$ such that $P'(H'ZAZ'H)P = \text{Diag}(\Delta_1, ..., \Delta_1, ..., \Delta_d, ..., \Delta_d)$ where each $\Delta_i$ is repeated $r_i$ times, $i = 1, ..., d$. It follows that $H'ZAZ'H = \sum_{i=1}^{d} \Delta_i P_i P_i'$ where $P =...$
[\mathbf{P}_1, \ldots, \mathbf{P}_d] and each matrix \( \mathbf{P}_i \) corresponding to \( \Delta_i \) is of size \((n - p) \times r_i\). For \( i = 1, \ldots, d \),
\[
\mathbf{P}_i' \mathbf{H} \mathbf{Y} \sim \mathcal{MVN}(0, (\sigma_e^2 + \sigma^2 \Delta_i) I_{r_i}).
\]
So \( \mathbf{Y}' (\mathbf{H} \mathbf{P}_i' \mathbf{H}) \mathbf{Y} = Q_i \sim (\sigma_e^2 + \sigma^2 \Delta_i) \chi^2(r_i), i = 1, \ldots, d \). Olsen, Seely and Birkes (1976) remark that \( \Delta_i, r_i \), and \( Q_i \) \((i = 1, \ldots, d)\) do not depend upon the choice of \( \mathbf{H} \). By construction, the quadratic forms \( Q_1, \ldots, Q_d \) are independent. In addition, they are a set of minimal sufficient statistics associated with the reduced linear model void of the fixed effect. Rewriting the distribution of \( Q_i \) in terms of \( \rho \) and the nuisance parameter \( \sigma_e^2 \), we have that \( Q_i \sim \sigma_e^2 (1 + \Delta_i \rho / (1 - \rho)) \chi^2(r_i), i = 1, \ldots, d \). These quadratic forms play a central role in the construction of the LM confidence intervals.

3 Confidence Intervals for \( \rho \)

3.1 Pivotal Quantities

As observed by LaMotte and McWhorter (1978), a class of pivotal quantities for \( \rho \) is obtained by considering a ratio of appropriate linear combinations of the \( Q \)'s. In particular,

\[
\frac{\frac{\sum_{i \in \text{Set}_2} Q_i}{1 + \rho(\Delta_i - 1)}}{\frac{\sum_{i \in \text{Set}_1} Q_i}{1 + \rho(\Delta_i - 1)}} \sim F\left( \frac{\sum_{i \in \text{Set}_2} r_i, \sum_{i \in \text{Set}_1} r_i}{} \right)
\]

(2)

where \( \text{Set}_1 \) and \( \text{Set}_2 \) are nonempty nonintersecting subsets of \( \{1, 2, \ldots, d\} \). A test of \( H_0 : \rho = \rho_0 \) versus \( H_a : \rho \neq \rho_0 \) can be based on these pivotal quantities. By inverting such a test one can obtain exact confidence regions for \( \rho \). In general, these regions may not be intervals, but a union of disjoint intervals. However, if the pivotal quantities in (2) are monotone functions of \( \rho \), the regions will in fact be intervals. The following proposition gives a necessary and sufficient condition for this to be the case.

**Proposition 1.** The pivotal quantities in (2) are monotone decreasing functions of \( \rho \) if and only if \( \text{Set}_1 = \{1, \ldots, k\} \) and \( \text{Set}_2 = \{k + 1, \ldots, d\} \) for some \( k \) such that \( 1 \leq k \leq d - 1 \).

**Proof** See Appendix.
Of course, monotone increasing functions of $\rho$ are obtained by interchanging $Set_1$ and $Set_2$. Without loss of generality, we will restrict our attention to the monotone decreasing cases only. Using this approach, Proposition 1 reduces the number of viable pivotal quantities to those of the form

$$\frac{\sum_{i=k+1}^{d} \frac{Q_i}{1+\rho(\Delta_i-1)/\sum_{i=k+1}^{d} r_i}}{\sum_{i=1}^{k} \frac{Q_i}{1+\rho(\Delta_i-1)/\sum_{i=1}^{k} r_i}} \sim F\left(\sum_{i=k+1}^{d} r_i, \sum_{i=1}^{k} r_i\right). \tag{3}$$

In fact, these are the pivotal quantities suggested by LaMotte and McWhorter (1978) and referred to by Lin and Harville (1991). However, the issue of monotonicity was not addressed in these papers. Since there are $d$ distinct eigenvalues, there are $d-1$ possible pivotal quantities that are monotone decreasing functions in $\rho$. These quantities can be inverted numerically to obtain confidence intervals for $\rho$. Notationally, a 100(1-$\alpha$)% confidence interval for $\rho$ is given by the set

$$\rho \in [0, 1] : F_{\alpha_1} \leq \frac{\sum_{i=k+1}^{d} \frac{Q_i}{1+\rho(\Delta_i-1)/\sum_{i=k+1}^{d} r_i}}{\sum_{i=1}^{k} \frac{Q_i}{1+\rho(\Delta_i-1)/\sum_{i=1}^{k} r_i}} \leq F_{1-\alpha_2} \tag{4}$$

where $\alpha_1 + \alpha_2 = \alpha$ and $F_{\alpha_1}, F_{1-\alpha_2}$ are the $\alpha_1, 1-\alpha_2$ percentiles of the $F$ distribution having numerator and denominator degrees of freedom $\sum_{i=k+1}^{d} r_i, \sum_{i=1}^{k} r_i$, respectively. Let $L$ denote the infimum of this set and $U$ the supremum. Then $P\{L \leq \rho \leq U\} = 1 - \alpha$.

### 3.2 Unbiasedness and Expected Length of Confidence Intervals

Although LaMotte and McWhorter (1978) proposed the family of pivotal quantities in (3) from which exact confidence intervals for $\rho$ can be obtained, to our knowledge, no systematic investigation of the properties of these interval procedures has been made. Consequently, no guidance is available to the user as to which member of this family should be used for a given application. In this section we investigate the bias and expected length properties of the LM confidence interval procedures. We derive a sufficient condition for a member of the LM family to be locally unbiased. In addition, we discuss how Pratt's (1961) result relating expected length of a confidence interval to its accuracy function can be used, in
conjunction with numerical procedures for computing cumulative distribution functions of linear combinations of independent \( \chi^2 \) random variables, to obtain the expected length of any member of the LM family. This information will allow the user to make an intelligent choice of an interval procedure for a given application.

Traditionally, confidence intervals have been judged by their coverage probabilities and expected lengths. The confidence intervals considered here are exact, so coverage probabilities are always equal to \( 1 - \alpha \). Since there are many intervals possessing the same coverage probability, an unbiased interval procedure may be preferred. Alternatively, the interval having the shortest expected length may be considered the most desirable. In many applications, however, there does not exist a confidence interval having uniformly minimum expected length. In such situations, prior knowledge of the likely value of \( \rho \), if available, can guide the user in making a sensible selection of a confidence interval procedure. In this section we first consider the bias of LM confidence intervals and then discuss expected length.

A confidence interval for \( \rho \) given by \((L, U)\) is unbiased if \( P \{ L \leq \rho_T \leq U \} \geq P \{ L \leq \rho \leq U \} \) where \( \rho_T \) is the true value of the parameter and \( \rho \in [0, 1) \). In other words, a confidence interval is unbiased if the probability it covers the true parameter value is no less than the probability it covers any false parameter value. It follows that if \( \frac{\partial}{\partial \rho} \{ P \{ L \leq \rho \leq U \} \} \big|_{\rho=\rho_T} = 0 \) and \( \frac{\partial^2}{\partial \rho^2} \{ P \{ L \leq \rho \leq U \} \} < 0 \) for \( \rho \in [0, 1) \), then \((L, U)\) is unbiased. The interval \((L, U)\) is locally unbiased if \( \frac{\partial}{\partial \rho} \{ P \{ L \leq \rho \leq U \} \} \big|_{\rho=\rho_T} = 0 \) and \( \frac{\partial^2}{\partial \rho^2} \{ P \{ L \leq \rho \leq U \} \} \big|_{\rho=\rho_T} < 0 \). The following proposition provides a closed form expression for the first derivative of \( P \{ L \leq \rho \leq U \} \) with respect to \( \rho \) evaluated at \( \rho = \rho_T \), for any LM confidence interval.

**Proposition 2.** For LM confidence intervals,
\[
\frac{\partial}{\partial \rho} \{ P \{ L \leq \rho \leq U \} \} \big|_{\rho=\rho_T} = \frac{\Gamma(\frac{1}{2}(S_1+S_2))}{\Gamma(\frac{1}{2}S_1)\Gamma(\frac{1}{2}S_2)} S_1^{1/2} S_2^{1/2} \left( \frac{\frac{S_2}{\alpha_{1-\alpha}}}{(S_1+S_2)^{(S_1+S_2)/2}} - \frac{\frac{S_2}{\alpha_1^{1/2}}}{(S_1+F_{1-\alpha_1} S_2)^{(S_1+S_2)/2}} \right) \]
\[
\times \left[ \frac{1}{S_2} \sum_{i=k+1}^{d} \frac{r_i(\Delta_i-1)}{(1+\rho_T(\Delta_i-1))} - \frac{1}{S_1} \sum_{i=1}^{k} \frac{r_i(\Delta_i-1)}{(1+\rho_T(\Delta_i-1))} \right],
\]
where \( S_2 = \sum_{i=k+1}^{d} r_i \) and \( S_1 = \sum_{i=1}^{k} r_i \).
Proof See Appendix.

Corollary 1. If \( \alpha_1 = \alpha_2 \) and \( S_1 = S_2 \), then \( \frac{\partial}{\partial \rho} \{ P[L \leq \rho \leq U]\}_{\rho = \rho_T} = 0 \) for the LM confidence intervals.

Proposition 3. If \( \alpha_1 = \alpha_2 \) and \( S_1 = S_2 \), then \( \frac{\partial^2}{\partial \rho^2} \{ P[L \leq \rho \leq U]\}_{\rho = \rho_T} < 0 \) for the LM confidence intervals.

Proof See Appendix.

Corollary 1 and Proposition 3 together imply the following.

Theorem 1. A LM confidence interval for \( \rho \) given by \( (L, U) \) is locally unbiased if it is equal tailed, i.e., \( \alpha_1 = \alpha_2 \), and the corresponding pivotal quantity has its numerator degrees of freedom equal to the denominator degrees of freedom, i.e., \( \sum_{i=k+1}^{d} r_i = \sum_{i=1}^{k} r_i \).

It is clear that confidence intervals obtained by inverting the pivotal quantity in (3) depend on the quadratic forms \( Q_1, \ldots, Q_d \). For \( Q = (Q_1, \ldots, Q_d) \), the expected length of a confidence interval \( (L, U) = (L(Q), U(Q)) \) is

\[
E[U(Q) - L(Q)] = \int \cdots \int [U(q) - L(q)] f_Q(q)dq
\]

where \( f_Q \) is the \( d \)-dimensional probability density function of \( Q \). An alternative expression for expected length is

\[
E[U(Q)] - E[L(Q)] = \int v f_V(v)dv - \int w f_W(w)dw
\]

where \( V = U(Q) \), \( W = L(Q) \), and \( f_V, f_W \) are the one dimensional probability density functions of \( V, W \), respectively. To compute the expected length, one must evaluate the \( d \)-dimensional integral in (5) or have knowledge of the distributions of \( V \) and \( W \) in (6). For
large values of \( d \), implementing (5) may be computationally infeasible. Furthermore, using (6) appears intractable since the distribution of the endpoints of the confidence interval are not known.

Pratt (1961) derived a relationship between the expected length of a confidence interval and its accuracy function. In short, the expected length of a two-sided confidence interval is equal to the integral of the accuracy function. The accuracy function is defined to be \( A(\rho|\rho_T) = P_{\rho_T}[L \leq \rho \leq U] \) so that \( A(\rho_T|\rho_T) = 1 - \alpha \) and \( A(\rho|\rho_T) \) is the probability of false coverage of \( \rho \) when \( \rho \neq \rho_T \). From Pratt (1961),

\[
E_{\rho_T}[U(Q) - L(Q)] = \int_{\rho \neq \rho_T} P_{\rho_T}[L \leq \rho \leq U] \, d\rho.
\]  

(7)

The subscripts in (7) accentuate the fact that the probability and expectation are functions of \( \rho_T \). Using notation involving the quadratic forms \( Q_1, ..., Q_d \), the expression for expected length becomes

\[
E_{\rho_T}[U(Q) - L(Q)] = \int_0^1 P_{\rho_T} \left[ F_{\alpha_1} \leq \frac{\sum_{i=k+1}^{d} \frac{Q_i}{1+\rho(\Delta_i-1)} / \sum_{i=k+1}^{d} r_i}{\sum_{i=1}^{b} \frac{Q_i}{1+\rho(\Delta_i-1)} / \sum_{i=1}^{b} r_i} \leq F_{1-\alpha_2} \right] \, d\rho
\]  

(8)

which can be computed numerically since the integrand in (8) can be written as the difference in probabilities of linear combinations of independent \( \chi^2 \) random variables. Based on the work of Imhoff (1961), Davies (1980) developed an algorithm that computes the cumulative distribution function of a linear combination of independent \( \chi^2 \) random variables. Our program to compute expected lengths of LM confidence intervals uses Davies' FORTRAN routine which is also known as algorithm AS 155 from Applied Statistics.

Inverting the pivotal quantity in (3) may result in values of \( \rho \) that are outside the parameter space since the pivotal quantity is a monotone decreasing function of \( \rho \) for \(-1/(\Delta_d - 1) < \rho < 1\). However, the limits of integration in (8) are from 0 to 1. It follows that \( E_{\rho_T}[U(Q) - L(Q)] \leq 1 \) and the confidence intervals considered in this paper are truncated so that they only contain values in the parameter space. The coverage probabilities, of course, are unaffected.
4 Examples

In this section we examine two completely different scenarios. The first data set results in a full animal model and there are zero degrees of freedom for error. The second example may be interpreted as a sire model in which case the degrees of freedom for error are positive. These examples illustrate the variety of outcomes that may occur when employing unbiasedness and/or expected length as criteria for good confidence intervals.

4.1 Full Animal Model

Data were obtained on one hundred and seventy one yearling bulls from a Red Angus seed stock herd in Montana (Evans et al. (1995)). A trait of interest was the loin eye (i.e., ribeye) muscle area measured in square inches. Ultrasound techniques were used to procure these measurements. The fixed effect was age of dam which had been originally recorded as belonging to one of eight categories: 2 years, 3 years, 4 years, 5-9 years, 10 years, 11 years, 12 years, and 13 or more years. Since there were only a few observations associated with dams greater or equal 10 years of age, our analysis used five categories for age of dam: 2 years, 3 years, 4 years, 5-9 years, and 10 or more years. The random effects are the animal’s (additive) genetic effect and error.

In terms of the mixed linear model equation,

\[ Y = X\beta + Zu + e, \]

where \( Y \) is a 171 × 1 vector of observable random variables, \( X \) is a 171 × 5 incidence matrix, \( \beta \) is a 5 × 1 vector of unknown parameters, \( Z = I_{171} \), and \( u \) and \( e \) are vectors of unobservable random variables of size 171 × 1. The relationship matrix \( A \) was determined using a recursive method given in Henderson (1976). It uses knowledge of the animal’s sire, dam, and grandparents. Note that some animals are inbred so that it is possible that \( Var(u_i) > \sigma^2_a \). For instance, it turns out that \( Var(u_1) = 1.03125\sigma^2_a \).

The number of distinct eigenvalues of \( H'ZAZ'H \) is \( d = 165 \). Eigenvalues range in magnitude from \( \Delta_1 = 0.56569 \) to \( \Delta_{165} = 8.65925 \). Except for \( \Delta_{61} = 0.67188 \) having \( r_{61} = 2 \), all
eigenvalues have a multiplicity of one. There are one hundred and sixty four possible pivotal quantities of the form (3) that can used to construct confidence intervals for $\rho$. Taking $\alpha_1 = \alpha_2 = 0.05$, the pivotal quantity that results in an equal-tailed unbiased 90% confidence interval corresponds to $k = 82$. In this case, $\sum_{i=1}^{82} r_i = \sum_{i=83}^{165} r_i = 83$.

Figure 1 shows the accuracy curves for selected intervals when $\rho_T = 0.5$. It is interesting to note that while $k = 82$ corresponds to the equal-tailed unbiased interval, $k = 100, 125, 150$, and 155 correspond to intervals that have smaller tail probabilities of false coverage. Thus, when $\rho_T = 0.5$, it follows that the intervals corresponding to $k = 100, 125, 150$, and 155 will have shorter expected lengths than the unbiased interval. Figure 2 depicts the expected length for the selected confidence intervals as a function of $\rho_T$. Over the entire parameter space, the expected lengths for $k = 150$ and 155 correspond to those confidence intervals having short expected lengths. The unbiased interval does not perform well in this case.

The actual confidence intervals were computed using the one hundred and seventy one loin eye measurements. In many cases, inverting the pivotal quantities in (3) result in confidence intervals whose endpoints fall outside of the parameter space. That is, $F_{\alpha_1} \leq \frac{\sum_{i=k+1}^{d} \frac{Q_i}{\sum_{i=k}^{d} \frac{Q_i}{r_i}} / \sum_{i=k+1}^{d} \frac{Q_i}{r_i}} {\sum_{i=1}^{d} \frac{Q_i}{\sum_{i=k+1}^{d} \frac{Q_i}{r_i}}} \leq F_{1-\alpha_2} \text{ for } \rho \in [0, 1)$. In particular, the 90% confidence interval corresponding to the pivotal quantity having $k = 82$ covers the entire parameter space. Of the one hundred and sixty four LM confidence intervals, the shortest interval is (0.00, 0.42). This confidence interval corresponds to the pivotal quantity having $k = 153$. It should be noted that employing expected length as a criterion for good confidence intervals does not guarantee the selected interval from a single realization has minimal length.

### 4.2 Sire Model

Consider the unbalanced mixed linear model

$$Y_{ijk} = \mu_i + u_j + e_{ijk},$$  \hspace{1cm} (10)

where $Y_{ijk}$ is the $k^{th}$ observation of the $i^{th}$ fixed effect and $j^{th}$ random effect combination. In this example, $i = 1, 2, j = 1, \ldots, 4$, and $k = 1$ or $2$ depending on the $(i,j)$ combination. Table 1 depicts the scenario for this simple data set having six observations.
Table 1: Simple Data Set

<table>
<thead>
<tr>
<th>Fixed Effect</th>
<th>Random Effect</th>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$Y_{111}$</td>
<td>$Y_{121}$</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td></td>
<td>$Y_{112}$</td>
<td>-</td>
<td>-</td>
<td>-</td>
</tr>
<tr>
<td>2</td>
<td>$Y_{211}$</td>
<td>-</td>
<td>$Y_{231}$</td>
<td>$Y_{241}$</td>
</tr>
</tbody>
</table>
It is assumed that $u_i$ and $e_{ij}$ are independent and the relationship matrix $A = I_d$. The number of distinct eigenvalues of $H'ZA'ZH$ is $d = 4$ where $\Delta_1 = 0.0, \Delta_2 = 0.47948, \Delta_3 = 1.0$, and $\Delta_4 = 1.85385$. Each eigenvalue has multiplicity one so that $n - p = \sum_{i=1}^{4} r_i = 4$. Suppose we are interested in obtaining a 90% confidence interval for $\rho$.

There are three possible pivotal quantities of the form (3) that can used to construct confidence intervals for $\rho$. Taking $\alpha_1 = \alpha_2 = 0.05$, the only pivotal quantity that results in an unbiased interval is $k = 2$. In this case, $\sum_{i=1}^{2} r_i = \sum_{i=3}^{4} r_i = 2$. Figure 3 depicts the expected length for the three 90% confidence intervals as a function of $\rho_T$. The interval with $k = 3$ is clearly not acceptable. The interval having $k = 1$ corresponds to the Wald (1940) interval in the unbalanced one-way random effects model. This interval has minimum expected length for values of $\rho_T$ less than 0.68 and values of $\rho_T$ greater than 0.98. For $0.68 \leq \rho_T \leq 0.98$, the unbiased confidence interval possesses the shortest expected length.

Employing the usual analysis of variance procedure results in one degree of freedom for error. That is, the analysis of variance table which partitions sources of variation into fixed effect, random effect given the fixed effect, and error would use the pivotal quantity corresponding to $k = 1$.

The parameter space of $\rho$ depends on the particular application under consideration. For instance, in an industrial setting, $Y_{ijk}$ may be the length of the $k^{th}$ part produced by the $j^{th}$ worker using the $i^{th}$ machine. In this case, $0 \leq \rho < 1$. On the other hand, the random effects may be the (additive) genetic effects of four unrelated bulls and the fixed effects are two herds in which the offspring of the bulls reside. Since the observations are made on the progeny of the bulls, this is an example of a sire model. In this case, $0 \leq \rho < 1/4$.

5 Discussion and Conclusions

In general, there are a number of LM pivotal quantities that may be used to obtain confidence intervals for $\rho$. In this paper we have examined bias and expected length properties with the intention of providing guidance to users for selecting a member of this family in any particular application. A sufficient condition for local unbiasedness was derived. We conjecture that
the equal-tailed LM interval with corresponding pivotal quantity having equal numerator and denominator degrees of freedom is in fact globally unbiased. However, we have not been able to prove this yet, nor have we found a counterexample. In many instances the balance of degrees of freedom does not exist. In such cases, pivotal quantities that have roughly equal numerator and denominator degrees of freedom result in equal-tailed confidence intervals having small bias.

We also demonstrated that it is possible to numerically compute expected lengths for LM confidence intervals for moderately sized data sets. Software for performing the relevant calculations is available from the authors via e-mail. The e-mail address is brent@stat.colostate.edu. The usefulness of the bias and/or expected length information in selecting a procedure for a given application is made clear by the animal breeding example. Unbiased intervals do not necessarily correspond to intervals having short expected lengths.

The confidence intervals discussed in this paper were limited to those having equal-tails. From the equation in Proposition 2, one may note that for given values of $S_1$ and $S_2$, values of $\alpha_1$ and $\alpha_2$ may be found such that $\frac{\partial}{\partial \rho} \{P[L \leq \rho \leq U]\}_{\rho=\rho_T} = 0$. Of course, a result analogous to Proposition 3 would have to be established to verify the unequal-tailed interval is locally unbiased.

Further research is needed to find useful analytical approximations for expected lengths of LM confidence intervals since such expressions can help circumvent the need for computer intensive numerical calculations.

The mixed linear models in this paper were presented in the context of animal breeding applications. Many other fields of studies utilize this modeling approach. In forestry, mixed linear models are used to assess the impact of environment effects on the growth of trees. Evaluating intraclass correlation, a function of variance components, in data sets where observations are not independent is of concern in psychological testing procedures. Industrial applications employ mixed linear models to investigate the impact of variations in machinery and labor practices on the production process.

The data sets used in this paper exemplify the fact that in many cases there does not
exist a uniformly minimum expected length LM confidence interval. The computed expected
lengths depend heavily on the true value of the parameter. This prompts one to consider
additional criteria such as minimax or minimizing the average expected length, where the av-
eraging is over the possible parameter values having preassigned weights. Similarly, bayesian
analysis could be employed by assuming an appropriate prior distribution for $\rho$.

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Figure 1: Probability of Coverage for Loin eye Data
Figure 2: Expected Length for Loin eye Data
Figure 3: Expected Length for Simple Data Set
APPENDIX

Proof of Proposition 1

Let

\[
G(\rho) = \frac{\sum_{j \in Set_2} \frac{Q_j}{1 + \rho(\Delta_j - 1)}}{\sum_{i \in Set_1} \frac{r_i}{1 + \rho(\Delta_i - 1)}} \left( \frac{\sum_{i \in Set_1} r_i}{\sum_{i \in Set_1} \frac{Q_i}{1 + \rho(\Delta_i - 1)}} \right)^2 .
\]  \hspace{1cm} (A.1)

Then

\[
\frac{\partial}{\partial \rho} G(\rho) = \frac{\sum_{i \in Set_1} r_i}{\sum_{j \in Set_2} r_j} \frac{1}{\left( \frac{\sum_{i \in Set_1} r_i}{\sum_{i \in Set_1} \frac{Q_i}{1 + \rho(\Delta_i - 1)}} \right)^2} \left[ \sum_{i \in Set_1} \sum_{j \in Set_2} \frac{Q_i Q_j}{[1 + \rho(\Delta_i - 1)]^2 [1 + \rho(\Delta_j - 1)]^2} (\Delta_i - \Delta_j) \right].
\]  \hspace{1cm} (A.2)

Recall that \(0 \leq \Delta_1 < \ldots < \Delta_d\). Suppose \(\Delta_i < \Delta_j\) for all \(i \in Set_1, j \in Set_2\). Then \(\frac{\partial}{\partial \rho} G(\rho) < 0\) with probability one. It follows that \(G(\rho)\) is a monontone decreasing function of \(\rho\).

Suppose \(G(\rho)\) is a monontone decreasing function of \(\rho\). Then \(\frac{\partial}{\partial \rho} G(\rho) < 0\) with probability one. Let \(\Omega_- = \{(i, j) \in Set_1 \times Set_2 | \Delta_i - \Delta_j < 0\}\) and \(\Omega_+ = \{(i, j) \in Set_1 \times Set_2 | \Delta_i - \Delta_j > 0\}\). Then

\[
P \left[ \frac{\partial}{\partial \rho} G(\rho) < 0 \right] = P \left[ \sum_{i \in Set_1} \sum_{j \in Set_2} \frac{Q_i Q_j}{[1 + \rho(\Delta_i - 1)]^2 [1 + \rho(\Delta_j - 1)]^2} (\Delta_i - \Delta_j) < 0 \right]
\]

\[
= P \left[ \sum_{(i, j) \in \Omega_-} \frac{Q_i Q_j}{[1 + \rho(\Delta_i - 1)]^2 [1 + \rho(\Delta_j - 1)]^2} (\Delta_i - \Delta_j) \right]
\]

\[
+ \sum_{(i, j) \in \Omega_+} \frac{Q_i Q_j}{[1 + \rho(\Delta_i - 1)]^2 [1 + \rho(\Delta_j - 1)]^2} (\Delta_i - \Delta_j) < 0 \right] .
\]  \hspace{1cm} (A.3)

Note that \(P \left[ \frac{\partial}{\partial \rho} G(\rho) < 0 \right] < 1\) since the coefficients of the nonnegative random variables \(Q_i Q_j\) are negative when \((i, j) \in \Omega_-\) and positive when \((i, j) \in \Omega_+\). It follows that \(\Delta_i < \Delta_j\) for all \(i \in Set_1, j \in Set_2\).

Proof of Proposition 2

Let \(X_j\) be independent random variables having distributions \(\chi^2(r_j)\) for \(j = 1, \ldots, d\), respectively.

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Then

\[
   P \left[ F_{\alpha_1} \leq \frac{\sum_{j=k+1}^{d} \frac{Q_j}{1 + \rho(\Delta_j - 1)/r_j}}{\sum_{j=k+1}^{d} r_j} \leq F_{1-\alpha_2} \right] =
\]

\[
   P \left[ \frac{1}{\sum_{j=k+1}^{d} r_j} \sum_{j=k+1}^{d} \frac{1 + \rho_T(\Delta_j - 1)}{1 + \rho(\Delta_j - 1)} X_j - \frac{F_{1-\alpha_2}}{\sum_{j=1}^{k} r_j} \sum_{j=1}^{k} \frac{1 + \rho_T(\Delta_j - 1)}{1 + \rho(\Delta_j - 1)} X_j < 0 \right]
\]

\[
   - P \left[ \frac{1}{\sum_{j=k+1}^{d} r_j} \sum_{j=k+1}^{d} \frac{1 + \rho_T(\Delta_j - 1)}{1 + \rho(\Delta_j - 1)} X_j - \frac{F_{\alpha_1}}{\sum_{j=1}^{k} r_j} \sum_{j=1}^{k} \frac{1 + \rho_T(\Delta_j - 1)}{1 + \rho(\Delta_j - 1)} X_j < 0 \right]
\]  (A.4)

since \( Q_j \sim \sigma_j^2(1 + \frac{\rho_T}{1 - \rho_T}\Delta_j)\chi^2(r_j) \) are independent for \( j = 1, \ldots, d \) and \( \rho_T \) denotes the true value of the parameter \( \rho \). Note that \( \frac{1}{\sum_{j=k+1}^{d} r_j} \sum_{j=k+1}^{d} \frac{1 + \rho_T(\Delta_j - 1)}{1 + \rho(\Delta_j - 1)} X_j - \frac{F_{1-\alpha_2}}{\sum_{j=1}^{k} r_j} \sum_{j=1}^{k} \frac{1 + \rho_T(\Delta_j - 1)}{1 + \rho(\Delta_j - 1)} X_j \) is a linear combination of independent \( \chi^2 \) random variables and thus has characteristic function, call it \( \phi_2(t) \), given by

\[
   \phi_2(t) = \prod_{j=k+1}^{d} \frac{1}{1 - 2i\alpha_j t} \prod_{j=1}^{k} \frac{1}{1 + 2i\alpha_j b_j t} \]  (A.5)

where

\[
   a_j = \frac{1}{\sum_{j=k+1}^{d} r_j} \frac{(1 + \rho_T(\Delta_j - 1))}{(1 + \rho(\Delta_j - 1))} \]  (A.6)

and

\[
   b_j = \frac{1}{\sum_{j=1}^{k} r_j} \frac{(1 + \rho_T(\Delta_j - 1))}{(1 + \rho(\Delta_j - 1))}. \]  (A.7)

Similarly, \( \frac{1}{\sum_{j=k+1}^{d} r_j} \sum_{j=k+1}^{d} \frac{1 + \rho_T(\Delta_j - 1)}{1 + \rho(\Delta_j - 1)} X_j - \frac{F_{\alpha_1}}{\sum_{j=1}^{k} r_j} \sum_{j=1}^{k} \frac{1 + \rho_T(\Delta_j - 1)}{1 + \rho(\Delta_j - 1)} X_j \) has characteristic function \( \phi_1(t) \) given by

\[
   \phi_1(t) = \prod_{j=k+1}^{d} \frac{1}{1 - 2i\alpha_j t} \prod_{j=1}^{k} \frac{1}{1 + 2i\alpha_j b_j t}. \]  (A.8)

From Davies (1973), if \( X \) is a random variable having characteristic function \( \phi(t) \), then

\[
   P[X < x] = \frac{1}{2} - \int_{-\infty}^{\infty} \text{Im} \left( \frac{\phi(t)e^{-itx}}{2\pi t} \right) dt \]  (A.9)

so that

\[
   P[X < 0] = \frac{1}{2} - \int_{-\infty}^{\infty} \text{Im} \left( \frac{\phi(t)}{2\pi t} \right) dt. \]  (A.10)
It follows that

$$P \left[ F_{\alpha_1} \leq \frac{d}{\sum_{j=k+1}^{d} \frac{Q_j}{1+\rho(\Delta_j-1)} / \sum_{j=1}^{d} r_j} \leq F_{1-\alpha_2} \right] = \int_{-\infty}^{\infty} \text{Im} \left( \frac{\phi_1(t) - \phi_2(t)}{2\pi t} \right) dt. \quad (A.11)$$

We now evaluate

$$\frac{\partial}{\partial \rho} \left[ P \left[ F_{\alpha_1} \leq \frac{\sum_{j=k+1}^{d} \frac{Q_j}{1+\rho(\Delta_j-1)} / \sum_{j=1}^{k} r_j}{\sum_{j=1}^{k} \frac{Q_j}{1+\rho(\Delta_j-1)} / \sum_{j=1}^{k} r_j} \leq F_{1-\alpha_2} \right] \right] \bigg|_{\rho=\rho_T}. \quad (A.12)$$

Applying the Dominated Convergence Theorem,

$$\frac{\partial}{\partial \rho} P \left[ F_{\alpha_1} \leq \frac{\sum_{j=k+1}^{d} \frac{Q_j}{1+\rho(\Delta_j-1)} / \sum_{j=1}^{k} r_j}{\sum_{j=1}^{k} \frac{Q_j}{1+\rho(\Delta_j-1)} / \sum_{j=1}^{k} r_j} \leq F_{1-\alpha_2} \right] = \text{Im} \left( \int_{-\infty}^{\infty} \frac{1}{2\pi t} \frac{\partial}{\partial \rho} \left[ \phi_1(t) - \phi_2(t) \right] dt \right). \quad (A.12)$$

Let $S_1 = \sum_{j=1}^{k} r_j$ and $S_2 = \sum_{j=k+1}^{d} r_j$. One can verify that

$$\frac{\partial}{\partial \rho} \phi_1(t) = \text{it} \left[ \sum_{j=1}^{k} \frac{r_j(\Delta_j-1)(1+\rho T(\Delta_j-1))}{S_1(1+\rho(\Delta_j-1))^2} \frac{F_{\alpha_1}}{(1+2iF_{\alpha_1}b_j t)} \phi_1(t) \right. \left. - \sum_{j=k+1}^{d} \frac{r_j(\Delta_j-1)(1+\rho T(\Delta_j-1))}{S_2(1+\rho(\Delta_j-1))^2} \frac{1}{(1-2ia_j t)} \phi_1(t) \right] \quad (A.13)$$

and

$$\frac{\partial}{\partial \rho} \phi_2(t) = \text{it} \left[ \sum_{j=1}^{k} \frac{r_j(\Delta_j-1)(1+\rho T(\Delta_j-1))}{S_1(1+\rho(\Delta_j-1))^2} \frac{F_{1-\alpha_2}}{(1+2iF_{1-\alpha_2}b_j t)} \phi_2(t) \right. \left. - \sum_{j=k+1}^{d} \frac{r_j(\Delta_j-1)(1+\rho T(\Delta_j-1))}{S_2(1+\rho(\Delta_j-1))^2} \frac{1}{(1-2ia_j t)} \phi_2(t) \right] \quad (A.14)$$

so that

$$\frac{\partial}{\partial \rho} \left[ \phi_1(t) - \phi_2(t) \right] \bigg|_{\rho=\rho_T} = \text{it} \left[ \sum_{j=k+1}^{d} \frac{r_j(\Delta_j-1)}{S_2(1+\rho T(\Delta_j-1))(1-2ia_j t)} \left( \phi_2(t) - \phi_1(t) \right) \right. \left. + \sum_{j=1}^{k} \frac{r_j(\Delta_j-1)}{S_1(1+\rho T(\Delta_j-1))} \left( \frac{F_{\alpha_1}}{(1+2iF_{\alpha_1}t)} \phi_1(t) - \frac{F_{1-\alpha_2}}{(1+2iF_{1-\alpha_2}t)} \phi_2(t) \right) \right] \quad (A.15)$$

It follows that (interchanging summation and integration)

$$\frac{\partial}{\partial \rho} \left[ P \left[ F_{\alpha_1} \leq \frac{\sum_{j=k+1}^{d} \frac{Q_j}{1+\rho(\Delta_j-1)} / \sum_{j=1}^{k} r_j}{\sum_{j=1}^{k} \frac{Q_j}{1+\rho(\Delta_j-1)} / \sum_{j=1}^{k} r_j} \leq F_{1-\alpha_2} \right] \right] \bigg|_{\rho=\rho_T} =$$

$$\frac{1}{2\pi} \left[ \sum_{j=k+1}^{d} \frac{r_j(\Delta_j-1)}{S_2(1+\rho T(\Delta_j-1))} \left( \int_{-\infty}^{\infty} \phi_2(t) dt - \int_{-\infty}^{\infty} \phi_1(t) dt \right) + \sum_{j=1}^{k} \frac{r_j(\Delta_j-1)}{S_1(1+\rho T(\Delta_j-1))} \left( F_{\alpha_1} \int_{-\infty}^{\infty} \phi_1(t) dt - F_{1-\alpha_2} \int_{-\infty}^{\infty} \phi_2(t) dt \right) \right] \quad (A.16)$$
since it turns out that the integrals are real valued.

When \( \rho = \rho_T \),

\[
\int_{-\infty}^{\infty} \frac{\phi_2(t)}{(1 - 2it})^{m_2}} dt = \int_{-\infty}^{\infty} \frac{1}{(1 - 2it \frac{S_2}{S_1} + \frac{S_2}{S_1}_2)} \frac{S_1}{S_1}^{m_2} dt, \quad \text{(A.17)}
\]

\[
\int_{-\infty}^{\infty} \frac{\phi_1(t)}{(1 - 2it \frac{S_1}{S_1})} dt = \int_{-\infty}^{\infty} \frac{1}{(1 - 2it \frac{S_2}{S_1} + \frac{S_1}{S_1}_2)} \frac{S_1}{S_1}^{m_1} dt, \quad \text{(A.18)}
\]

\[
\int_{-\infty}^{\infty} \frac{\phi_1(t)}{(1 + 2it \frac{S_1}{S_1})} dt = \int_{-\infty}^{\infty} \frac{1}{(1 - 2it \frac{S_2}{S_1} + \frac{S_1}{S_1}_2 + 1)} \frac{S_1}{S_1}^{m_1 + 1} dt, \quad \text{(A.19)}
\]

\[
\int_{-\infty}^{\infty} \frac{\phi_2(t)}{(1 + 2it \frac{S_1}{S_1})} dt = \int_{-\infty}^{\infty} \frac{1}{(1 - 2it \frac{S_2}{S_1} + \frac{S_1}{S_1}_2 + 1)} \frac{S_1}{S_1}^{m_2 + 1} dt. \quad \text{(A.20)}
\]

The four integrals given above can be evaluated by finding an expression for the integral of the form

\[
\int_{-\infty}^{\infty} \frac{1}{(1 - 2it \frac{S_1}{S_1})^m (1 + 2it \frac{S_1}{S_1})^n} dt \quad \text{(A.21)}
\]

where \( a \) is a constant. Using the residue theorem (Goodstein, 1965, p.125), we obtain

\[
\int_{-\infty}^{\infty} \frac{1}{(1 - 2it \frac{S_1}{S_1})^m (1 + 2it \frac{S_1}{S_1})^n} dt = \pi \frac{\Gamma(m + n - 1)}{\Gamma(m) \Gamma(n)} \frac{S_1^n S_2^m a^{m-1}}{(S_1 + aS_2)^{m+n-1}}. \quad \text{(A.22)}
\]

It follows that

\[
\int_{-\infty}^{\infty} \frac{1}{(1 - 2it \frac{S_1}{S_1})^{m_2} + 1} \frac{S_1}{S_1}^{m_2} dt = 2\pi \frac{\Gamma(\frac{1}{2}(S_1 + S_2))}{\Gamma(\frac{1}{2}S_2) \Gamma(\frac{1}{2}S_1)} \frac{S_1^{\frac{1}{2}} S_2^{\frac{1}{2}}}{F_{1-a_2} \frac{S_2}{S_1}}, \quad \text{(A.23)}
\]

\[
\int_{-\infty}^{\infty} \frac{1}{(1 - 2it \frac{S_2}{S_1}) + 1} \frac{S_1}{S_1}^{\frac{1}{2}} dt = 2\pi \frac{\Gamma(\frac{1}{2}(S_1 + S_2))}{\Gamma(\frac{1}{2}S_2) \Gamma(\frac{1}{2}S_1)} \frac{S_1^{\frac{1}{2}} S_2^{\frac{1}{2}}}{F_{1-a_2} \frac{S_2}{S_1}}, \quad \text{(A.24)}
\]

\[
\int_{-\infty}^{\infty} \frac{1}{(1 - 2it \frac{S_1}{S_1})^{m_1 + 1}} \frac{S_1}{S_1}^{m_1 + 1} dt = 2\pi \frac{\Gamma(\frac{1}{2}(S_1 + S_2))}{\Gamma(\frac{1}{2}S_2) \Gamma(\frac{1}{2}S_1)} \frac{S_1^{\frac{1}{2}} S_2^{\frac{1}{2}}}{F_{1-a_2} \frac{S_2}{S_1}}, \quad \text{and (A.25)}
\]

\[
\int_{-\infty}^{\infty} \frac{1}{(1 - 2it \frac{S_2}{S_1})^{m_2 + 1}} \frac{S_1}{S_1}^{m_2 + 1} dt = 2\pi \frac{\Gamma(\frac{1}{2}(S_1 + S_2))}{\Gamma(\frac{1}{2}S_2) \Gamma(\frac{1}{2}S_1)} \frac{S_1^{\frac{1}{2}} S_2^{\frac{1}{2}}}{F_{1-a_2} \frac{S_2}{S_1}^{m_2}}, \quad \text{(A.26)}
\]

This results in

\[
\frac{\partial}{\partial \rho} \{ P[L \leq \rho \leq U] \} \bigg|_{\rho = \rho_T} = \frac{\Gamma(\frac{1}{2}(S_1 + S_2))}{\Gamma(\frac{1}{2}S_2) \Gamma(\frac{1}{2}S_1)} \frac{S_1^{\frac{1}{2}} S_2^{\frac{1}{2}}}{S_2^{\frac{1}{2}} (S_1 + F_{1-a_2} S_2)} \left( \frac{F_{1-a_2} \frac{S_2}{S_1}}{(S_1 + F_{1-a_2} S_2) \frac{S_1 + S_2}{S_1}} - \frac{F_{1-a_2} \frac{S_2}{S_1}}{(S_1 + F_{1-a_2} S_2) \frac{S_1 + S_2}{S_1}} \right)
\]

\[
\left[ \frac{1}{S_2} \sum_{j=k+1}^{d} \frac{r_j (\Delta_j - 1)}{(1 + \rho_T (\Delta_j - 1))} \right] - \frac{1}{S_1} \sum_{j=1}^{k} \frac{r_j (\Delta_j - 1)}{(1 + \rho_T (\Delta_j - 1))} \quad \text{(A.27)}
\]
which completes the proof.

Proof of Proposition 3

Let $\alpha_1 = \alpha_2$ and $S_1 = S_2$. Using procedures similar to those given in the proof of Proposition 2, one can show that

\[
\frac{\partial^2}{\partial \rho^2} \{ P[L \leq \rho \leq U] \} \bigg|_{\rho=\rho_T} = \frac{1}{2} \frac{F^\frac{S}{2} (1 - F)}{F + 1} \frac{1}{S + 1} \frac{1}{\Gamma(\frac{S}{2} + 1) \Gamma(\frac{1}{2})} \left[ -\frac{2}{S + 2} \left\{ \sum_{i=k+1}^{d} r_i \left( \frac{\Delta_i - 1}{1 + \rho_T (\Delta_i - 1)} \right)^2 - \frac{1}{S} \left( \sum_{i=k+1}^{d} r_i \left( \frac{\Delta_i - 1}{1 + \rho_T (\Delta_i - 1)} \right) \right)^2 \right\} - \frac{2}{S + 2} \left\{ \sum_{i=1}^{k} r_i \left( \frac{\Delta_i - 1}{1 + \rho_T (\Delta_i - 1)} \right)^2 - \frac{1}{S} \left( \sum_{i=1}^{k} r_i \left( \frac{\Delta_i - 1}{1 + \rho_T (\Delta_i - 1)} \right) \right)^2 \right\} - \frac{1}{S} \left\{ \sum_{i=k+1}^{d} r_i \left( \frac{\Delta_i - 1}{1 + \rho_T (\Delta_i - 1)} \right) - \sum_{i=1}^{k} r_i \left( \frac{\Delta_i - 1}{1 + \rho_T (\Delta_i - 1)} \right) \right\} \right] ^2 \]

(A.28)

where $S = S_1 = S_2$ and $F = F_{\alpha_1} = 1/F_{1-\alpha_2}$. It follows that $\frac{\partial^2}{\partial \rho^2} \{ P[L \leq \rho \leq U] \} \bigg|_{\rho=\rho_T} < 0$. 

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