

# A geometric drift inequality for a reflected fractional Brownian motion process on the positive orthant

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## Abstract

We study a  $d$ -dimensional reflected fractional Brownian motion process (RFBM) on the positive orthant  $S = \mathbb{R}_+^d$ , with drift  $r^0 \in \mathbb{R}^d$  and Hurst parameter  $H \in (1/2, 1)$ . Under a natural stability condition on the drift vector  $r^0$  and reflection directions, we establish a geometric drift towards a compact set for the 1-skeleton chain  $\check{Z}$  of the RFBM process  $Z$ ; that is, there exist  $\beta, b \in (0, \infty)$  and a compact set  $C \subset S$  such that

$$\Delta V(x) := \mathbb{E}_x[V(\check{Z}(1))] - V(x) \leq -\beta V(x) + b\mathbb{1}_C(x), \quad x \in S,$$

for an exponentially growing Lyapunov function  $V : S \rightarrow [1, \infty)$ . For a wide class of Markov processes, such drift inequality is known as a necessary and sufficient condition for an exponential ergodicity. Indeed, similar drift inequalities have been established for reflected processes driven by standard Brownian motions, and our result can be viewed as their FBM counterpart. We also derive that the return times to the set  $C$  itself are geometrically bounded. Motivation for this study is that RFBM appears as a limiting workload process for fluid queueing network models fed by a large number of heavy-tailed ON/OFF sources in heavy traffic.

*Keywords:* Reflected fractional Brownian motion, heavy traffic theory, a geometric drift inequality, return times.

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# 1 Introduction

We consider a multi-dimensional reflected fractional Brownian motion process  $\{Z(t) : t \geq 0\}$  (abbreviated as RFBM hereafter) on the positive orthant  $S := \mathbb{R}_+^d$ , with drift  $r^0 \in \mathbb{R}^d$  and Hurst parameter  $H \in (1/2, 1)$ . For example, this model serves as an approximation for stochastic networks with a large number of heavy-tailed ON/OFF sources ([4]), or with long range dependent arrival and service time processes ([13, 16]), in the heavy traffic regime. This model was further studied in subsequent papers [5, 6], where the heavy traffic limit theorems for a multi-class multi-server fluid network and various asymptotic behaviors of the (weighted) maximum processes were studied, respectively. The logarithmic asymptotics of tail probabilities of such reflected processes were obtained (cf. [15] and the references therein). In a recent paper [14], a uniform moment estimate on expected return times of the RFBM process to a compact set is established. While there is an ample body of literature for heavy traffic analysis based on short-range dependent (Markovian-like) models and light tails, the study on the long time stability properties of RFBM process is a largely unexplored area. This is mainly due to the lack of semi-martingale and/or Markov properties of a fractional Brownian motion (FBM) and hence many techniques from the classical theory of stochastic calculus are inapplicable to its analysis.

In this work, we establish a few basic stability properties of the RFBM process, mainly, by proving the *geometric drift* towards a compact set  $C \subset S$  for the discretely sampled RFBM process  $\{\check{Z}(n) : n \geq 0\}$ . More precisely, we show that  $\{\check{Z}(n) := Z(n), n = 0, 1, \dots\}$  satisfies the following drift inequality: There exist  $\beta, b \in (0, \infty)$  and a compact set  $C \subset S$  such that

$$\Delta V(x) := \mathbb{E}_x[V(\check{Z}(1))] - V(x) \leq -\beta V(x) + b \mathbb{I}_C(x), \quad x \in S, \quad (1.1)$$

with a suitable Lyapunov function  $V : S \rightarrow [1, \infty)$  satisfying an exponential growth condition. Here (and throughout the paper),  $\mathbb{E}_x$  denotes the expectation conditional on the process  $\check{Z}$  starting from  $x \in S$  and  $\mathbb{I}_C(\cdot)$  denotes an indicator function on a set  $C$ . This result (commonly referred to as a geometric drift (V4) in the literature, cf. Chapter 15 of [19]), combined with a *petite* set requirement for  $C$ , is reminiscent of a necessary and sufficient condition for an exponential ergodicity of a  $\psi$ -irreducible, aperiodic Markov process (see, e.g., [7, 18, 19]). Similar results as

in (1.1) were established for reflected processes driven by standard Brownian motions; indeed, the  $V$ -uniform ergodicity result of [3] (see also [10] for an ergodicity result) in view of Theorem 5.3 in [7] implies (1.1) for semi-martingale reflecting Brownian motions. In this regard, results in this paper can be viewed as a significant step towards further time-asymptotic analysis of RFBM with the aim of establishing similar ergodic properties for reflected processes driven by non-Markovian processes.

The organization of the paper is as follows. In Section 2, we carefully describe our model in Definition 2.1 and make a standard assumption on reflection matrix (see **(HR)** in Section 2), which is used in heavy traffic analysis for invoking a functional central limit theorem in [4, 5]. In addition, similar to [12], we assume a natural stability condition (see **(S)** in Section 2) on the RFBM process. Our proof of the main result is based on exploiting basic properties, such as Lipschitz continuity and linear growth, of a hitting time to the origin function  $T(\cdot)$  (see Lemma 3.1) on a family of certain deterministic dynamical systems obtained from the underlying RFBM process. This result, combined with an exponential moment estimate for the maximal increment of FBM with  $H \in (1/2, 1)$ , yields an estimate on the expected exponential hitting times to the origin in terms of the initial state (Lemma 3.3). In order to connect this result with the desired drift inequality (1.1), we make a suitable choice of  $\delta \in (0, \infty)$  and show that  $V(\check{Z}(n)) := e^{\delta T(\check{Z}(n))}$  is bounded in mean on some compact set  $C \subset S$ , and also it has a uniform negative drift off of the set  $C$  (see Theorem 3.4).

As consequences of geometric drift inequality, we obtain the comparison result (Theorem 3.7) and then establish an exponential regularity result for the sampled process (see Theorem 3.8). Such results are used to conclude that the return times to the set  $C$  itself are geometrically bounded. Lastly, we show in Theorem 3.10 that the drift inequality (1.1) can be extended with a larger compact set  $D \subset S$  (in place of  $C$ ), and hence the subsequent exponential regularity results hold on the set  $D$  as well.

We use the following notation. The set of positive integers is denoted by  $\mathbb{N}$  and let  $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ . The set of real numbers is denoted by  $\mathbb{R}$  and non-negative real numbers by  $\mathbb{R}_+$ . For  $a \in \mathbb{R}$ , let  $a^+ = \max\{a, 0\}$ . Let  $\mathbb{R}^d$  be the  $d$ -dimensional Euclidean space and for  $x \in \mathbb{R}^d$  the  $L_1$ -norm of  $x$ , i.e.,  $\sum_{i=1}^d |x_i|$ , will be denoted by  $|x|$ . For a  $(d \times d)$ -matrix  $A$ , we will use the

norm  $\|A\| = \max_{1 \leq j \leq d} \sum_{i=1}^d |a_{ij}|$ . For a given matrix  $M$ , denote by  $M^T$  its transpose and by  $M_i$  the  $i$ -th row of  $M$ . Let  $\mathcal{I} = \mathcal{I}_{d \times d}$  denote the identity matrix for some  $d$ . When it is clear from the context, we will omit the subscript. For a set  $A \subset \mathbb{R}^d$ , denote its interior and boundary by  $A^\circ$  and  $\partial A$ , respectively. For sets  $A, B \subset \mathbb{R}^d$ ,  $\text{dist}(A, B)$  will denote the distance between two sets, i.e.,  $\inf\{|x - y| : x \in A, y \in B\}$ . The symbol  $\mathbb{1}_A(\cdot)$  denotes the indicator function of a set  $A$ . Let  $C(X, Y)$  denote the space of continuous functions from  $X$  to  $Y$ , endowed with the topology of uniform convergence on compact intervals. Inequalities for vectors are interpreted componentwise.

## 2 Model and assumptions

We begin with the definitions of multi-dimensional FBM and RFBM. Let  $d \in \mathbb{N}$ . A stochastic process  $B_H = \{B_H(t) = (B_H^{(1)}(t), \dots, B_H^{(d)}(t))^T, t \geq 0\}$ , defined on some filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , is called a  $d$ -dimensional FBM of (Hurst) parameter  $H \in (0, 1)$ , starting from  $B_H(0) \in \mathbb{R}^d$ , and associated matrix  $\Lambda$ , if it satisfies the following conditions: The process  $B_H$  is a continuous Gaussian process with initial condition  $B_H(0)$   $\mathbb{P}$ -a.s. and its covariance function is given by

$$\text{Cov}(B_H(t), B_H(s)) = \mathbb{E}((B_H(t) - B_H(0))(B_H(s) - B_H(0))^T) = \Lambda_H(s, t)\Lambda,$$

for any  $s, t \geq 0$ , where  $\Lambda$  is a  $d \times d$  positive definite matrix and

$$\Lambda_H(s, t) := \frac{1}{2} (t^{2H} + s^{2H} - |t - s|^{2H}).$$

Without loss of generality, we will assume throughout that the diagonal entries of  $\Lambda$  are all ones. Also, it is assumed that  $B_H$  is adapted to the filtration  $(\mathcal{F}_t)_{t \geq 0}$ . We will say that  $B_H$  is a  $d$ -dimensional FBM with associated data  $(B_H(0), H, \Lambda)$ .

Fix some column vectors  $r^0, r^1, \dots, r^d \in \mathbb{R}^d$  and define the matrix  $R := [r^1, \dots, r^d]_{d \times d}$ . The quintuple  $(B_H(0), H, \Lambda, r^0, R)$  will be called as the *data* for an RFBM. The following definition is similar to that of [4].

**Definition 2.1. (RFBM)** For  $x \in S$ , a reflected fractional Brownian motion (RFBM) associated with the data  $(x, H, \Lambda, r^0, R)$  is a continuous  $d$ -dimensional process  $Z_x$ , defined on some probability

space  $(\Omega, \mathcal{F}, \mathbb{P})$ , such that

- (i)  $Z_x(t) = x + B_H(t) + r^0 t + RY(t) \in S$  for all  $t \geq 0$ ,  $\mathbb{P}$ -a.s.,
- (ii)  $B_H$  is a  $d$ -dimensional FBM with data  $(0, H, \Lambda)$ ,
- (iii)  $Y$  is a  $d$ -dimensional process such that  $Y_i(0) = 0$  for  $i = 1, \dots, d$ ,  $\mathbb{P}$ -a.s. For each  $i = 1, \dots, d$ ,  $Y_i$  is continuous, non-decreasing and  $Y_i$  can increase only when  $Z(\cdot)$  is on the face  $F^i := \{x \in S : x_i = 0\}$ , i.e.,  $\int_0^t \mathbb{1}_{\{Z_i(s) \neq 0\}} dY_i(s) = 0$  for all  $t \geq 0$ .

The set of directions of reflections on the boundary is defined as follows. For  $y \in \partial S$ ,

$$r(y) := \left\{ \sum_{i=1}^d q_i r^i : \sum_{i=1}^d q_i = 1, q_i \geq 0, \text{ and } q_i > 0 \text{ only if } y_i = 0 \right\}.$$

An RFBM, introduced in the above definition, behaves like an FBM in the interior of the orthant  $S$  and it is confined to the orthant by an instantaneous “reflection” (or, more accurately, “regulation”) at the boundary  $\partial S$ . For each  $i$ , the  $i$ -th column of the *reflection matrix*  $R$  gives the direction of the reflection on the  $i$ -th face  $F^i$ . Specifically, if the boundary  $F^i$  is hit, it is  $Y_i$  that increases, the direction of displacement is given by  $r^i$ , the  $i$ -th column of  $R$ , and the magnitude of the displacement is the minimal amount required to keep  $Z_i$  nonnegative. At an intersection of faces, the allowed directions of reflection are given by the convex combinations of the reflection directions associated with the faces meeting there (cf. [22]).

A square matrix  $R$  is called *completely-S* if for every  $k \times k$  principal submatrix  $G$  of  $R$ , there is a  $k$ -dimensional vector  $v_G$  such that  $v_G \geq 0$  and  $Gv_G > 0$ . The completely- $S$  condition on the reflection matrix  $R$  ensures that for every  $y \in \partial S$ , there exists a convex combination of vectors in  $r(y)$  which points into  $S^\circ$  from  $y$ . Also, the completely- $S$  property is sufficient to ensure the existence of a pair  $(Z, Y)$  satisfying (i) and (iii) in Definition 2.1 (cf. Theorem 2 in [2]). However, this property does not ensure the adaptiveness of process  $Y$  to a filtration to which  $B_H$  is adapted. This problem is overcome under a stronger assumption on  $R$ , that we quote below (cf. Proposition 4.2 in [21] and Section 2 of [4]). We will impose throughout that the reflection matrix  $R$  verifies the following assumption.

**(HR) Assumption on matrix  $R$ :**

- (i) The reflection matrix  $R$  can be expressed as  $\mathcal{I} + \Theta$ , with  $\Theta$  a  $d \times d$  matrix such that  $|\Theta|$ ,

that is the matrix obtained from  $\Theta$  by replacing all the entries in  $\Theta$  by their absolute values, has spectral radius  $r(|\Theta|)$  strictly less than 1.

(ii) Moreover, the matrix  $\Theta = (\theta_{ij})$  satisfies  $\theta_{ij} \leq 0$  and  $\theta_{ii} = 0$  for  $1 \leq i, j \leq d$ .

We now describe some of the important implications of an assumption (HR). Under (HR), it can be shown (cf. [21, 4]) that if  $B_H$  is adapted to some filtration  $\{\mathcal{F}_t : t \geq 0\}$ , then  $(Z, Y)$  is adapted to filtration  $\{\mathcal{G}_t : t \geq 0\}$ , with  $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{N}$ , where  $\mathcal{N}$  denotes the collection of  $\mathbb{P}$ -null sets in  $\mathcal{F}$ . Henceforth, with an abuse of notation, we will assume that  $(Z, Y)$  is adapted to filtration  $\{\mathcal{F}_t : t \geq 0\}$ . We also note that part (i) of (HR) implies the matrix  $R$  is completely- $\mathcal{S}$  and invertible. Furthermore, part (i) of (HR) is a sufficient condition for strong pathwise uniqueness of a solution of the *Skorokhod Problem* described below. The part (ii) of (HR), together with (i), verifies the so-called Harrison-Reiman condition in [11]. As a consequence, (HR) will imply that a solution to the Skorokhod Problem exists, and moreover the Skorokhod map is Lipschitz continuous in the sense of Proposition 2.3 below.

**Definition 2.2. (Skorokhod Problem)** Let  $\psi \in C([0, \infty), \mathbb{R}^d)$  be given with  $\psi(0) \in S$ . Then, we say that  $(\phi, \eta) \in C([0, \infty), \mathbb{R}^d) \times C([0, \infty), \mathbb{R}^d)$  solves the Skorokhod problem for  $\psi$  with respect to  $S$  and  $R$  if and only if the following hold:

- (i)  $\phi(t) = \psi(t) + R\eta(t) \in S$ , for all  $t \geq 0$ ;
- (ii)  $\eta$  satisfies, for  $1 \leq i \leq d$ , (a)  $\eta_i(0) = 0$ , (b)  $\eta_i$  is non-decreasing, and (c)  $\eta_i$  can increase only when  $\phi$  is on the  $i$ -th face of  $S$ , that is,  $\int_0^\infty \mathbb{I}_{\{\phi_i(s) \neq 0\}} d\eta_i(s) = 0$ .

Let  $C_S([0, \infty), \mathbb{R}^d) := \{\psi \in C([0, \infty), \mathbb{R}^d) : \psi(0) \in S\}$ . On the domain  $E \subset C_S([0, \infty), \mathbb{R}^d)$  on which there is a unique solution to the Skorokhod problem, we define the Skorokhod map  $\Gamma$  as

$$\Gamma(\psi) := \phi,$$

if  $(\phi, R^{-1}[\phi - \psi])$  is the unique solution of the Skorokhod problem posed by  $\psi$ . Equivalent form of RFBM in Definition 2.1 in terms of the Skorokhod map can now be written as follows:

$$Z = \Gamma(x + B_H + r^0 \iota), \quad Z - (x + B_H + r^0 \iota) = RY,$$

where  $\iota : [0, \infty) \rightarrow [0, \infty)$  is the identity map, i.e.,  $\iota(t) = t$  for  $t \in [0, \infty)$ .

As a consequence of Assumption (HR), we have the following proposition on the regularity of the Skorokhod map [11]. (Although the Lipschitz continuity is not stated explicitly in [11], it follows easily from the method used to prove existence of solutions and continuity of the Skorokhod map (map  $\phi(\cdot)$  in [11])). See also [8, 9] for more general sufficient conditions under which this regularity property holds.

**Proposition 2.3.** *The Skorokhod map is well-defined on all of  $C_S([0, \infty), \mathbb{R}^d)$ , i.e.  $E = C_S([0, \infty), \mathbb{R}^d)$ , and the Skorokhod map is Lipschitz continuous in the following sense: There exists a constant  $L \in (0, \infty)$  such that for all  $\psi_1, \psi_2 \in C_S([0, \infty), \mathbb{R}^d)$  and  $t \geq 0$ ,*

$$\sup_{0 \leq s \leq t} |\Gamma(\psi_1)(s) - \Gamma(\psi_2)(s)| < L \sup_{0 \leq s \leq t} |\psi_1(s) - \psi_2(s)|.$$

Next, we introduce the main stability condition on the drift vector  $r^0$  and the matrix  $R$  that will be assumed throughout this paper.

**(S) Stability assumption on  $r^0$  and  $R$ :**

There exists a constant  $\theta > 0$ , such that  $\sup_{1 \leq i \leq d} [R^{-1}r^0]_i < -\theta$ .

**Remark 2.4.** *For a model driven by standard Brownian motion (i.e., when  $H = 1/2$ ), the assumption (S) is known as a necessary and sufficient condition for the existence of a unique stationary distribution for the reflected process  $Z$ . We refer the reader to [12] and references therein for the related asymptotic stability results of reflected Brownian motions (RBMs) on a positive orthant.*

### 3 Main results

Let  $Z_x$  be defined by Definition 2.1 with an initial condition  $Z_x(0) = x \in S$  and the Hurst parameter  $H \in (1/2, 1)$ . Fix  $x \in S$  and we write the RFBM  $Z_x$  in Definition 2.1 as,

$$Z_x(t) = \Gamma(x + r^0 \iota + B_H(\cdot))(t), \quad t \geq 0,$$

where  $\iota : [0, \infty) \rightarrow [0, \infty)$  is the identity map. Define the corresponding deterministic trajectory of  $\{Z_x(t) : t \geq 0\}$  by

$$z_x(t) := \Gamma(x + r^0 \iota)(t), \quad t \geq 0.$$

Next, denoting by  $\mathcal{C} := \{v \in \mathbb{R}^d : R^{-1}v \leq 0\}$ , it can be seen from condition (S) that there exists a  $\rho \in (0, \infty)$  satisfying

$$\begin{aligned} \text{dist}(r^0, \partial\mathcal{C}) &\geq \frac{1}{\|R^{-1}\|} \inf \left\{ |R^{-1}r^0 - R^{-1}v| : R^{-1}v = 0, v \in \mathbb{R}^d \right\} \\ &\geq \frac{1}{\|R^{-1}\|} \theta d =: \rho > 0, \end{aligned} \tag{3.1}$$

where  $\theta \in (0, \infty)$  is as in condition (S). Thus,

$$r^0 \in \mathcal{C}_\rho := \{v \in \mathcal{C} : \text{dist}(v, \partial\mathcal{C}) \geq \rho\}.$$

The cone  $\mathcal{C}_\rho$  was considered in [1] to stipulate the permissible drift vector field and to characterize the positive recurrence of a class constrained diffusion processes in some polyhedral domain. In this work, we will treat  $\rho \in (0, \infty)$  as one of the parameters, which can be chosen appropriately. (See the main result presented in Theorem 3.4 below.)

For  $x_0 \in S$ , denote by  $\mathcal{K}_\rho(x_0)$  the collection of all trajectories  $\phi_\rho : [0, \infty) \rightarrow S$  of the form

$$\phi_\rho(t) = \Gamma(x_0 + \varpi \iota)(t), \quad t \geq 0, \tag{3.2}$$

where  $\varpi$  ranges over all of  $\mathcal{C}_\rho$ . For a fixed  $x_0 \in S$ , we now define the ‘‘hitting time to the origin’’ function  $T : S \rightarrow [0, \infty)$  as follows:

$$T(x_0) := \sup_{\phi_\rho \in \mathcal{K}_\rho(x_0)} \inf \{t \in [0, \infty) : \phi_\rho(t) = 0\}. \tag{3.3}$$

We refer the reader to Section 3 of [1] for the various stability properties of the deterministic trajectories defined in (3.2). The following result from [1] (see Lemma 3.1 and its proof in the cited paper) will be crucially used in our analysis.

**Lemma 3.1.** *The function  $T : S \rightarrow [0, \infty)$  defined in (3.3) satisfies the following properties.*

(i) For some  $c_1 \in (0, \infty)$ ,  $|T(x) - T(y)| \leq c_1|x - y|$  for all  $x, y \in S$ .

(ii) For some  $c_2, c_3 \in (0, \infty)$ ,  $c_2|x| \leq T(x) \leq c_3|x|$  for all  $x \in S$ . Thus, in particular, for all  $M \in (0, \infty)$  the set  $\{x \in S : T(x) \leq M\}$  is compact.

(iii) Fix  $x \in S$  and let  $z \in \mathcal{K}_\rho(x)$ . Then for all  $t > 0$ ,  $T(z_x(t)) \leq (T(x) - t)^+$ .

Here, the positive constants  $c_i$  ( $i = 1, 2, 3$ ) depend only on  $L \in (0, \infty)$  and  $\rho \in (0, \infty)$ , appeared in Proposition 2.3 and (3.1), respectively. In particular,  $c_1 = 4L^3\rho^{-1}$ .

We next present an exponential moment estimate for the maximal increment of FBM  $B_H$  with  $H \in (1/2, 1)$ , that may be of independent interest.

**Lemma 3.2.** *Let  $h \in (0, \infty)$  be a constant. For  $n \in \mathbb{N}$ , let  $\nu_n$  be defined as follows:*

$$\nu_n \equiv \sup \left\{ |B_H(s) - B_H((n-1)h)| : (n-1)h \leq s \leq nh \right\},$$

where  $B_H(\cdot)$  is a  $d$ -dimensional FBM with data  $(0, H, \Lambda)$  and Hurst parameter  $H \in (1/2, 1)$ . Then, for any  $\gamma \in (0, \infty)$  and  $n \in \mathbb{N}$ ,

$$\mathbb{E}[e^{\gamma\nu_n}] \leq 4de^{\frac{1}{2}\gamma^2 d^2 h^{2H}},$$

where  $d \geq 1$  is the dimension of the FBM  $B_H$ .

*Proof.* We begin by recalling that for positive random variable  $\nu_n$ ,

$$\mathbb{E}[e^{\gamma\nu_n}] = 1 + \int_0^\infty \gamma e^{\gamma y} \mathbb{P}[\nu_n > y] dy. \quad (3.4)$$

From the stationary increment and self-similar properties of FBM (cf. Theorem 3.3 in [17]), it follows that

$$\begin{aligned} \nu_n &\stackrel{D}{=} \sup \{ |B_H(s) - B_H(0)| : 0 \leq s \leq h \} \\ &\stackrel{a.s.}{=} \sup \{ |B_H(s)| : 0 \leq s \leq h \} \\ &\stackrel{D}{=} h^H \sup \{ |B_H(s)| : 0 \leq s \leq 1 \}, \end{aligned}$$

where the “ $\stackrel{a.s.}{=}$ ” follows from the assumption  $B_H(0) = 0$   $\mathbb{P}$ -a.s. (Recall the part (ii) in Definition 2.1.)

Hence, we obtain

$$\begin{aligned}
\mathbb{P}[\nu_n > y] &= \mathbb{P} \left[ \sup \{ |B_H(s)| : 0 \leq s \leq 1 \} > h^{-H} y \right] \\
&\leq \mathbb{P} \left[ \sum_{i=1}^d \sup \{ |B_H^{(i)}(s)| : 0 \leq s \leq 1 \} > h^{-H} y \right] \\
&\leq d \mathbb{P} \left[ \sup \{ |B_H^{(1)}(s)| : 0 \leq s \leq 1 \} > \frac{h^{-H}}{d} y \right] \\
&\leq 2d \mathbb{P} \left[ \sup \{ B_H^{(1)}(s) : 0 \leq s \leq 1 \} > \frac{h^{-H}}{d} y \right], \tag{3.5}
\end{aligned}$$

where the last inequality follows from the symmetry property of FBM.

For  $H \in (1/2, 1)$ , it is known from Statement 4.2 (part 2) of [20] that

$$\mathbb{P} \left[ \sup \{ B_H^{(1)}(s) : 0 \leq s \leq 1 \} > \lambda \right] \leq \sqrt{\frac{2}{\pi}} \int_{\lambda}^{\infty} e^{-x^2/2} dx \tag{3.6}$$

for any  $\lambda \geq 0$ . More precisely, (3.6) follows from setting  $f = 1$  and  $r = 1$  in Statement 4.2 of [20]. For more direct presentation of this result, we refer the reader to the second to the last displayed inequality on page 1067 in the cited paper. (We note that since the definition of FBM in [20] is slightly different from ours, one should set  $\alpha = 2H - 1$  and also the function  $q_f(s, t)$  in Statement 4.2 should be corrected by multiplying a missing constant factor  $C\alpha(\alpha + 1) = H(2H - 1)$ ; see also the first displayed equality on page 1067.) Applying the estimate in (3.6) to (3.4) and (3.5), we obtain that

$$\begin{aligned}
\mathbb{E}[e^{\gamma \nu_n}] &\leq 1 + 2d \sqrt{\frac{2}{\pi}} \int_0^{\infty} \gamma e^{\gamma y} \left[ \int_{y/(dh^H)}^{\infty} e^{-x^2/2} dx \right] dy \\
&= 1 + 2d \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x^2/2} \left[ \int_0^{dh^H x} \gamma e^{\gamma y} dy \right] dx \\
&= 1 + 2d \sqrt{\frac{2}{\pi}} \int_0^{\infty} e^{-x^2/2} \left[ e^{\gamma dh^H x} - 1 \right] dx.
\end{aligned}$$

Recalling  $\int_0^{\infty} e^{-x^2/2} dx = \frac{\sqrt{2\pi}}{2}$  and by completing the square in the exponent, it can be shown that the last expression is bounded above by

$$2d \sqrt{\frac{2}{\pi}} e^{\frac{1}{2}(\gamma dh^H)^2} \int_0^{\infty} e^{-\frac{1}{2}(x - \gamma dh^H)^2} dx = 4d e^{\frac{1}{2}(\gamma dh^H)^2} (1 - \Phi(-\gamma dh^H)),$$

where  $\Phi(\cdot)$  denotes the cumulative distribution function of the standard normal random variable. Noting that  $\Phi(\cdot) \in [0, 1]$ , we obtain the required bound.  $\square$

Next, we define a “1-skeleton” process  $\check{Z}$  and an associated filtration. Recall the filtered probability space  $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})$ , on which an RFBM process  $\{Z(t) : t \geq 0\}$  is defined. For  $k \geq 0$ , we let

$$\check{Z}(k) := Z(k), \quad \check{\mathcal{F}}_k := \mathcal{F}_k.$$

Combining Lemmas 3.1–3.2 we now have the following result.

**Lemma 3.3.** *For  $\delta \in (0, \infty)$ , there is a constant  $c_4 \in (0, \infty)$  such that*

$$\mathbb{E}_x e^{\delta[T(\check{Z}(1))]} \leq 4d e^{\delta[T(x)-1]} \cdot e^{c_4 \delta^2}, \quad t \geq 0, \quad (3.7)$$

for all  $x \in S_1 := \{x \in S : T(x) > 1\}$ .

*Proof.* Notice that for all  $x \in S$ ,

$$\begin{aligned} T(Z_x(1)) - T(z_x(1)) &\leq c_1 |Z_x(1) - z_x(1)| \\ &\leq c_1 L \cdot \sup_{0 \leq s \leq 1} |B_H(s)|, \end{aligned}$$

where the first inequality follows from the Lipschitz property of  $T$  [Lemma 3.1 (i)], and the second one follows from Proposition 2.3. Then, from Lemma 3.1 (iii), we have that  $\mathbb{P}_x$ -a.s.

$$T(Z_x(1)) \leq (T(x) - 1)^+ + c_1 L \cdot \sup_{0 \leq s \leq 1} |B_H(s)|, \quad \forall x \in S. \quad (3.8)$$

Then, for  $\delta \in (0, \infty)$  and  $x \in S_1$ , we obtain

$$e^{\delta[T(\check{Z}_x(1))]} \leq e^{\delta[T(x)-1]} e^{\delta c_1 L \sup_{0 \leq s \leq 1} |B_H(s)|}, \quad \mathbb{P}_x\text{-a.s.}$$

Taking expectations on both sides along with Lemma 3.2 ( $n = 1, \gamma = \delta c_1 L, h = 1$ ) yields the desired result (3.7) by setting  $c_4 := \frac{1}{2} c_1^2 L^2 d^2$  (which is proportional to  $\rho^{-2}$  in view of Lemma 3.1).  $\square$

Hereafter, we impose an assumption on the parameter  $\rho \in (0, \infty)$  in (3.1) so that the constant

$c_4 \in (0, \infty)$  in Lemma 3.3 can be small enough in the following analysis. Recall that  $c_1 = 4L^3\rho^{-1}$  and  $c_4 = \frac{1}{2}c_1^2L^2d^2$  from Lemma 3.1 and Lemma 3.3, respectively. In what follows, we choose  $\rho \in (0, \infty)$  appropriately so that  $c_4 < (4 \log 4d)^{-1}$ . This amounts to assuming  $\rho \in (\kappa, \infty)$ , where  $\kappa := 4dL^4\sqrt{2 \log 4d}$ . We now present the main result on the geometric drift inequality for the 1-skeleton RFBM process.

**Theorem 3.4. (Geometric drift towards  $C$ )** For  $x \in S$ , let the RFBM process  $\{Z_x(t)\}_{t \geq 0}$  be given as in Definition 2.1 and assume that  $\rho \in (\kappa, \infty)$ . Then, the 1-skeleton process  $\{\check{Z}_x(n) := Z_x(n)\}_{n \in \mathbb{N}_0}$  satisfies the following drift inequality: There are  $\delta, \beta, b \in (0, \infty)$  and a compact set  $C \subset S$  such that

$$\Delta V(x) \leq -\beta V(x) + b\mathbb{1}_C(x), \quad x \in S, \quad (3.9)$$

with  $V(x) := e^{\delta T(x)}$ .

**Proof.** From Lemma 3.3, for  $\delta \in (0, \infty)$  and  $x \in S_1 := \{x \in S : T(x) > 1\}$ ,

$$V(x)^{-1}[\mathbb{E}_x e^{\delta[T(\check{Z}(1))]}] \leq 4de^{-\delta}e^{c_4\delta^2},$$

where  $c_4 \in (0, \infty)$  is a constant (independent of  $\delta$  and  $x$ , and proportional to  $\rho^{-2}$ ) as in (3.7). Now choose  $\delta := \frac{1}{2c_4} \in (0, \infty)$  so that  $c_4\delta^2 - \delta + \log 4d < 0$  and therefore  $e^{c_4\delta^2 - \delta + \log 4d} := (1 - \beta) < 1$ . Then we have

$$\mathbb{E}_x V(\check{Z}(1)) \leq (1 - \beta)V(x), \quad \forall x \in S_1. \quad (3.10)$$

Also for  $x \in C := S \setminus S_1$ , we have from inequality (3.8) that

$$\mathbb{E}_x V(\check{Z}(1)) \leq 4de^{c_4\delta^2} =: b. \quad (3.11)$$

Combining (3.10) and (3.11) we obtain

$$\mathbb{E}_x V(\check{Z}(1)) \leq (1 - \beta)V(x) + b\mathbb{1}_C(x), \quad \forall x \in S \quad (3.12)$$

and the desired result (3.9) follows. □

**Remark 3.5.** In view of Lemma 3.1 (ii), the set  $C = \{x \in S : T(x) \leq 1\}$  is compact and the

function  $V : S \rightarrow [1, \infty)$  is shown to be unbounded and satisfy an exponential growth condition. We remark that similar drift inequalities as in (3.9) were established for a wide class of Markov processes [19, 7, 3]. The geometric drift result in the form of (3.9) is known as the necessary and sufficient condition for the  $V$ -uniform ergodicity of underlying Markov processes. Indeed, the proof that geometric ergodicity of 1-skeleton (which follows from (3.9) for Markov processes) implies  $V$ -uniform ergodicity of the semigroup is a simple consequence of the submultiplicative property of the operator  $V$ -norm (cf. Theorems 5.2–5.3 in [7]). In this respect, the geometric drift result in Theorem 3.4 sheds some light on the time-asymptotic analysis of an RFBM process, anticipating similar long-time properties of reflected processes driven by standard Brownian motions.

**Remark 3.6.** The geometric drift condition (3.9) can be understood as the following simpler contractive form,

$$\mathbb{E}_x V(\check{Z}(1)) \leq (1 - \beta)V(x) + b, \quad \forall x \in S, \quad (3.13)$$

where  $(1 - \beta) < 1$  and  $b$  are as in (3.9). Indeed, if (3.9) holds then (3.13) immediately follows. Conversely, if (3.13) holds then it can be shown

$$\Delta V(x) \leq -\bar{\beta}V(x) + b\mathbb{I}_{\bar{C}}(x), \quad x \in S,$$

where  $\bar{\beta} \equiv \frac{\beta}{2}$  and  $\bar{C} \equiv \{x \in S : V(x) \leq \frac{2b}{\beta}\}$ .

As a consequence of the geometric drift towards a compact set  $C$  in Theorem 3.4, we obtain the following comparison theorem, which will be useful for proving that the return times to the set  $C$  itself are geometrically bounded. For  $m \geq 1$  and a set  $B \subset S$ , define the first entrance time to the set  $B$  by

$$\check{\tau}_B := \inf\{m \geq 1 : \check{Z}(m) \in B\}.$$

**Theorem 3.7.** For any  $r \in (1, (1 - \beta)^{-1})$ , there exists  $\epsilon > 0$  such that for any first entrance time  $\check{\tau}_B$ ,

$$\mathbb{E}_x \left[ \sum_{k=0}^{\check{\tau}_B-1} V(\check{Z}(k))r^k \right] \leq \epsilon^{-1}r^{-1}V(x) + \epsilon^{-1}b\mathbb{E}_x \left[ \sum_{k=0}^{\check{\tau}_B-1} \mathbb{I}_C(\check{Z}(k))r^k \right], \quad (3.14)$$

where the constants  $\beta, b \in (0, \infty)$  and the set  $C \subset S$  are as in Theorem 3.4.

*Proof.* The proof is adapted from Theorem 15.2.5 of [19]. We begin by observing that for any

$r \in (1, (1 - \beta)^{-1})$  one can choose  $\epsilon \in (0, \beta)$  to be the solution to  $r = 1/(1 - \beta + \epsilon)$ . Then, from (3.12) we have the bound

$$\mathbb{E}_x V(\check{Z}(1)) \leq r^{-1}V(x) - \epsilon V(x) + b\mathbb{I}_C(x), \quad \forall x \in S.$$

It follows that

$$\begin{aligned} \mathbb{E} \left[ V(\check{Z}(k+1))r^{k+1} | \check{\mathcal{F}}_k \right] &\leq r^{k+1} \left\{ r^{-1}V(\check{Z}(k)) - \epsilon V(\check{Z}(k)) + b\mathbb{I}_C(\check{Z}(k)) \right\} \\ &= V(\check{Z}(k))r^k - \epsilon V(\check{Z}(k))r^{k+1} + r^{k+1}b\mathbb{I}_C(\check{Z}(k)). \end{aligned} \quad (3.15)$$

For  $n \geq 1$  and  $r \in (1, \infty)$ , define

$$\tau_B^n := \min \left\{ n, \check{\tau}_B, \inf \{ k \geq 0 : V(\check{Z}(k))r^k \geq n \} \right\}.$$

The random time  $\tau_B^n$  is a stopping time with respect to the filtration  $\{\check{\mathcal{F}}_k\}$ , since it is the minimum of stopping times with respect to the same filtration. Note that since  $\{\tau_B^n \geq i\} \in \check{\mathcal{F}}_{i-1}$ , we have

$$\mathbb{E}_x [V(\check{Z}(\tau_B^n))r^{\tau_B^n}] = \mathbb{E}_x [V(\check{Z}(0))] + \mathbb{E}_x \left[ \sum_{k=0}^{\tau_B^n - 1} \left( \mathbb{E}_x [V(\check{Z}(k+1))r^{k+1} | \check{\mathcal{F}}_k] - V(\check{Z}(k))r^k \right) \right]. \quad (3.16)$$

Fix  $N > 0$ . From (3.15) and (3.16) we see that

$$0 \leq \mathbb{E}_x [V(\check{Z}(\tau_B^n))r^{\tau_B^n}] \leq V(x) + \mathbb{E}_x \left[ \sum_{k=0}^{\tau_B^n - 1} \left( r^{k+1}b\mathbb{I}_C(\check{Z}(k)) - \epsilon V(\check{Z}(k))r^{k+1} \wedge N \right) \right]. \quad (3.17)$$

Hence, by adding a finite term to each side of (3.17), we get

$$\begin{aligned} \mathbb{E}_x \left[ \sum_{k=0}^{\tau_B^n - 1} \left( \epsilon V(\check{Z}(k))r^{k+1} \wedge N \right) \right] &\leq V(x) + \mathbb{E}_x \left[ \sum_{k=0}^{\tau_B^n - 1} r^{k+1}b\mathbb{I}_C(\check{Z}(k)) \right] \\ &\leq V(x) + \mathbb{E}_x \left[ \sum_{k=0}^{\check{\tau}_B - 1} r^{k+1}b\mathbb{I}_C(\check{Z}(k)) \right]. \end{aligned}$$

Letting  $n \rightarrow \infty$  and then  $N \rightarrow \infty$  gives the result

$$\mathbb{E}_x \left[ \sum_{k=0}^{\check{\tau}_B-1} \left( \epsilon V(\check{Z}(k)) r^{k+1} \right) \right] \leq V(x) + \mathbb{E}_x \left[ \sum_{k=0}^{\check{\tau}_B-1} r^{k+1} b \mathbb{1}_C(\check{Z}(k)) \right] \quad (3.18)$$

by the monotone convergence theorem. Finally, multiplying both sides of (3.18) by  $\epsilon^{-1} r^{-1} \in (0, \infty)$  yields the desired result (3.14).  $\square$

As an immediate consequence of Theorem 3.7, we obtain the following result.

**Theorem 3.8.** *There exists  $r > 1$  such that*

$$\sup_{x \in C} \mathbb{E}_x \left[ \sum_{k=0}^{\check{\tau}_C-1} V(\check{Z}(k)) r^k \right] < \infty, \quad (3.19)$$

where the set  $C \subset S$  is as in Theorem 3.4. Hence, in particular, we have

$$\sup_{x \in C} \mathbb{E}_x \left[ r^{\check{\tau}_C} \right] < \infty. \quad (3.20)$$

**Remark 3.9.** *A set  $C$  satisfying (3.19) ((3.20), resp.) is known as a  $V$ -Kendall set (Kendall set, resp.) in the Markov processes literature (cf. Chapter 15 of [19]).*

*Proof.* By choosing  $B = C$  in Theorem 3.7, we have

$$V(x) \leq \mathbb{E}_x \left[ \sum_{k=0}^{\check{\tau}_C-1} V(\check{Z}(k)) r^k \right] \leq \epsilon^{-1} r^{-1} V(x) + \epsilon^{-1} b \mathbb{1}_C(x).$$

Therefore, we obtain

$$\begin{aligned} \sup_{x \in C} \mathbb{E}_x \left[ \sum_{k=0}^{\check{\tau}_C-1} V(\check{Z}(k)) r^k \right] &\leq \epsilon^{-1} r^{-1} \sup_{x \in C} V(x) + \epsilon^{-1} b \\ &\leq \epsilon^{-1} r^{-1} e^\delta + \epsilon^{-1} b < \infty. \end{aligned}$$

The result (3.20) is then straightforward by noting  $V(\check{Z}(k)) \geq 1$  for all  $k \geq 0$ .  $\square$

Next we show that Theorems 3.4, 3.7, and Theorem 3.8 hold with the sublevel sets of  $V$ , which can be chosen larger than the set  $C = \{x \in S : T(x) \leq 1\}$ . We refer the reader to Theorem 15.2.6 of [19] for similar results for a class of  $\psi$ -irreducible Markov processes.

**Theorem 3.10.** *Let  $\delta, \beta, b \in (0, \infty)$  and  $C \subset S$  be as in Theorem 3.4. Then, there exist  $\beta', c \in (0, \infty)$  and a compact set  $D \supset C$  in  $S$  such that*

$$\Delta V(x) \leq -\beta'V(x) + c\mathbb{I}_D(x), \quad x \in S, \quad (3.21)$$

and therefore, Theorems 3.7–3.8 hold with  $D$  in place of  $C$ .

*Proof.* Fix  $\ell^{-1} \in (1 - \beta, 1)$  and choose  $M \in (0, \infty)$  sufficiently large so that

$$e^\delta \leq \frac{M + b}{\ell^{-1} - (1 - \beta)}, \quad \text{and define } D := \left\{ x \in S : V(x) \leq \frac{M + b}{\ell^{-1} - (1 - \beta)} \right\}.$$

Then, it follows that  $C \subset D$  and moreover the set  $D$  is compact in view of Lemma 3.1 (ii). Using Theorem 3.4 we have

$$\begin{aligned} \mathbb{E}_x V(\check{Z}(1)) &\leq \ell^{-1}V(x) - (\ell^{-1} - (1 - \beta))V(x) + b\mathbb{I}_C(x) \\ &\leq \ell^{-1}V(x) - M \leq \ell^{-1}V(x), \quad \forall x \in D^c, \end{aligned}$$

from the definition of the set  $D$ . Also, it follows that

$$\mathbb{E}_x V(\check{Z}(1)) \leq (1 - \beta)V(x) + b \leq c, \quad \forall x \in D,$$

for some  $c \in (0, \infty)$ , since  $V$  is bounded on  $D$  by construction. Thus we have shown that

$$\mathbb{E}_x V(\check{Z}(1)) \leq \ell^{-1}V(x) + c\mathbb{I}_D(x), \quad \forall x \in S,$$

and hence (3.21) follows with  $\beta' := 1 - \ell^{-1} \in (0, \infty)$ . Hence using Theorem 3.7 (with replacing the sets  $B$  and  $C$  by the set  $D$ ), there exists  $s \in (1, \infty)$  and  $\varepsilon \in (0, \infty)$  such that

$$\mathbb{E}_x \left[ \sum_{k=0}^{\check{\tau}_D-1} V(\check{Z}(k))s^k \right] \leq \varepsilon^{-1}s^{-1}V(x) + \varepsilon^{-1}c\mathbb{I}_D(x).$$

Since  $V$  is bounded on  $D$ , this leads to

$$\sup_{x \in D} \mathbb{E}_x \left[ \sum_{k=0}^{\check{\tau}_D - 1} V(\check{Z}(k)) s^k \right] < \infty, \text{ and hence } \sup_{x \in D} \mathbb{E}_x \left[ s^{\check{\tau}_D} \right] < \infty.$$

□

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## References

- [1] R. Atar, A. Budhiraja, and P. Dupuis. On positive recurrence of constrained diffusion processes. *Ann. Probab.*, 29(2):979–1000, 2001.
- [2] A. Bernard and A. el Kharroubi. Régulations déterministes et stochastiques dans le premier “orthant” de  $\mathbf{R}^n$ . *Stochastics Stochastics Rep.*, 34(3-4):149–167, 1991.
- [3] A. Budhiraja and C. Lee. Long time asymptotics for constrained diffusions in polyhedral domains. *Stochastic Process. Appl.*, 117(8):1014–1036, 2007.
- [4] R. Delgado. A reflected fBm limit for fluid models with ON/OFF sources under heavy traffic. *Stochastic Process. Appl.*, 117(2):188–201, 2007.
- [5] R. Delgado. State space collapse for asymptotically critical multi-class fluid networks. *Queueing Syst.*, 59(2):157–184, 2008.
- [6] R. Delgado. On the reflected fractional Brownian motion process on the positive orthant: Asymptotics for a maximum with application to queueing networks. *Stochastic Models*, 26(2):272–294, 2010.
- [7] D. Down, S. P. Meyn, and R. L. Tweedie. Exponential and uniform ergodicity of Markov processes. *Ann. Probab.*, 23(4):1671–1691, 1995.
- [8] P. Dupuis and H. Ishii. On Lipschitz continuity of the solution mapping to the Skorohod problem, with applications. *Stochastics*, 35:31–62, 1991.
- [9] P. Dupuis and K. Ramanan. Convex Duality and the Skorokhod Problem. I, II. *Probab. Theory Related Fields*, 115(2):153–195, 197–236, 1999.
- [10] P. Dupuis and R. J. Williams. Lyapunov functions for semimartingale reflecting Brownian motions. *Ann. Probab.*, 22(2):680–702, 1994.
- [11] J. M. Harrison and M. I. Reiman. Reflected Brownian motion on an orthant. *Ann. Probab.*, 9(2):302–308, 1981.

- [12] J. M. Harrison and R. J. Williams. Brownian models of open queueing networks with homogeneous customer populations. *Stochastics*, 22(2):77–115, 1987.
- [13] T. Konstantopoulos and S. Lin. Fractional Brownian approximations of queueing networks. In *Stochastic networks (New York, 1995)*, volume 117 of *Lecture Notes in Statist.*, pages 257–273. Springer, New York, 1996.
- [14] C. Lee. On the return time for a reflected fractional Brownian motion process on the positive orthant. *J. Appl. Prob.*, 48(1):145–153, 2011.
- [15] K. Majewski. Large deviations for multi-dimensional reflected fractional Brownian motion. *Stoch. Stoch. Rep.*, 75(4):233–257, 2003.
- [16] K. Majewski. Fractional Brownian heavy traffic approximations of multiclass feedforward queueing networks. *Queueing Syst.*, 50(2-3):199–230, 2005.
- [17] B. B. Mandelbrot and J. W. Van Ness. Fractional Brownian motions, fractional noises and applications. *SIAM Rev.*, 10:422–437, 1968.
- [18] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes III: Foster-Lyapunov criteria for continuous-time processes. *Adv. in Appl. Probab.*, 25:518–548, 1993.
- [19] S. P. Meyn and R. L. Tweedie. *Markov chains and stochastic stability*. Cambridge University Press, Cambridge, second edition, 2009.
- [20] A. A. Ruzmaikina. Stieltjes integrals of Hölder continuous functions with applications to fractional Brownian motion. *J. Statist. Phys.*, 100(5-6):1049–1069, 2000.
- [21] R. J. Williams. An invariance principle for semimartingale reflecting Brownian motions in an orthant. *Queueing Syst.*, 30(1-2):5–25, 1998.
- [22] R. J. Williams. Reflecting diffusions and queueing networks. In *Proceedings of the International Congress of Mathematicians, Vol. III (Berlin, 1998)*, pages 321–330 (electronic), 1998.