

On moment stability properties for a class of state-dependent stochastic networks

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Abstract

We consider a class of stochastic networks with state-dependent arrival and service rates. The state-dependency is described via multi-dimensional birth/death processes, where the birth/death rates are dependent upon the current population size in the system. Under the uniform (in state) stability condition, we establish several moment stability properties of the system:

- (i) the unique invariant measure of the state process admits a finite moment generating function in a neighborhood of zero;
- (ii) the expected value of unbounded functionals of the state process converges to the expectation under the invariant measure at an exponential rate;
- (iii) uniform (in time and initial condition) estimates on exponential moments of the process;
- (iv) growth estimates of polynomial moments of the process as a function of the initial conditions.

Our approach provides elementary proofs of these stability properties without resorting to the convergence of the scaled process to a stable fluid limit model.

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AMS 2000 subject classifications. Primary 60J25; Secondary 93D20

1 Introduction

This paper is devoted to the study of long time moment stability properties for a class of stochastic networks with state-dependent arrival and service rates. The stochastic network under consideration consists of K service stations, each of which has an associated infinite capacity buffer. Arrivals

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of jobs can be from outside the system and/or from internal routing. Upon completion of service at a station, the customer is routed to one of the another service stations (or exits the system) according to a probabilistic routing matrix. We allow the dependence of arrival and service rates on the current queue level in the system. State-dependent features can be easily found in applications to complex manufacturing systems, service engineering, communications and computer networks. For example, longer queues would lead to customers refusing to join a queue or leaving before service, or to faster processing (cf. Mandelbaum & Pats (1998); Whitt (1990); Kushner (2001)). The stochastic network model studied in this work is similar to those in Mandelbaum & Pats (1998); Yamada (1995). These papers are concerned with the transient analysis of networks via fluid and/or diffusion limit approximations. We, on the other hand, are interested in long time stability properties of such queueing systems with a state-dependent form of the Poisson arrival and exponential service times (see (2.1) for a precise model description).

In a recent work of Lee (2010), it has been established that, under the uniform (in state) stability condition, the queue length process is V -uniformly ergodic; that is, it has a transition probability kernel which converges to its limit geometrically quickly in the V -norm sense (see Section 2.2 for the definition of the V -norm). The proofs of results in Lee (2010) rely on a critical use of a perturbed Lyapunov function technique (cf. Chapter 8 of Meyn (2008)) to show uniform convergence of the stable fluid limit model to zero. On the contrary, for the stability analysis in the current paper, we employ different Lyapunov function methods (as described in the next paragraph). We also note that the asymptotic results in Lee (2010) were obtained under further assumptions on the primitives for the fluid models (see Main Assumption (B1)–(B3) in Lee (2010)). Those assumptions on the fluid models are unnecessary in the current work, since our approach provides elementary proofs of stability properties without resorting to the convergence of the scaled process to a stable fluid limit model (cf. Dai (1995); Bramson (2008)).

Our first choice of Lyapunov functions is the “hitting time to the origin” function $T(\cdot)$ of related deterministic dynamical systems obtained from a fluid limit analysis of the underlying stochastic networks (see (3.8)). Using the function $T(\cdot)$, main efforts lie in proving Theorem 3.4, which obtains suitable bounds on exponential moments of hitting times of compact sets. Once these estimates are available, the results of Down et al. (1995) (cf. Theorem 4.4) yield another Lyapunov function f for which the drift inequality in Theorem 3.6 holds and as a consequence the process is f -uniformly ergodic. Furthermore, in Lemma 3.7, explicit exponential upper and lower bounds of f are obtained, so that the Lyapunov function f is shown to be unbounded and satisfy an exponential growth condition. Most of technical ideas in the proofs are adaptations of those in Budhiraja & Lee (2007). One of the major differences is that the estimates available for constrained diffusion processes considered in Budhiraja & Lee (2007) are now replaced by the counterparts for a (strong) Markov jump process on the K -dimensional, non-negative integer lattice. Some of the new estimates are proved along the way to accommodate our model assumptions and the recent results established in Lee (2010) are critically used in the analysis.

We next show several long time stability properties of the state process, which were not established in Lee (2010). We prove that the unique invariant measure of the queue length process admits a finite moment generating function in a neighborhood of zero (Theorem 3.8). As a result, we can also prove that expected value of unbounded (and possibly exponentially growing) functionals of the state process converges to the expectation under the invariant measure at an exponential

rate (Theorem 3.9). This result then leads us to obtain uniform (in time and initial condition in a compact set) estimates on exponential moments of the process (Theorem 3.10) and growth estimates of polynomial moments of the process as a function of the initial conditions (Theorem 3.11).

The remainder of the paper is organized as follows. We begin, in Section 2.1, by describing the stochastic network model with state-dependency, and provide a description of the dynamics of the queue length process in terms of a Skorokhod map. A basic uniform (in state) stability assumption on this queueing system is then introduced (see condition (S) below). In Section 2.2, we collect some standard concepts on stochastic stability for Markov processes. Section 3 is devoted to the study of long time asymptotic properties of the state process. Finally, in the Appendix, we provide a proof of Lemma 3.7.

The following notation is used in this paper. For a metric space X , let $\mathcal{B}(X)$ denote the Borel σ -field on X . For a real-valued measurable function f on X and a measure ν on $\mathcal{B}(X)$, let $\nu(f) \equiv \int_X f d\nu$. The Dirac measure at the point x is denoted by δ_x . Let \mathbb{N} denote the set of natural numbers. Denote the set of real numbers by \mathbb{R} and non-negative real numbers by \mathbb{R}_+ . Let \mathbb{R}^d be the d -dimensional Euclidean space with the norm of $u \in \mathbb{R}^d$, $|u| = \sum_{k=1}^d |u_k|$. For a given matrix M denote by M' its transpose and by M_i the i^{th} row of M . Let $\mathbb{I} = \mathbb{I}_{K \times K}$ denote the identity matrix for some K . When it is clear from the context, we will omit the subscript. For a set $A \subset \mathbb{R}^d$, denote its boundary by ∂A . For sets $A, B \subset \mathbb{R}^d$, $dist(A, B)$ will denote the distance between two sets, i.e., $\inf\{|x - y| : x \in A, y \in B\}$. The symbol $1_A(\cdot)$ denotes the indicator function of a set A . The class of continuous functions $f : X \rightarrow Y$ is denoted by $C(X, Y)$ and real continuous bounded functions on X by $C_b(X)$. Finally, let $D(X, Y)$ denote the class of right continuous functions having left limits defined from X to Y , equipped with the usual Skorokhod topology. A stochastic process will be denoted interchangeably by $\{Z(t)\}$ and $\{Z_t\}$. Inequalities for vectors are interpreted componentwise. We will denote generic constants by c_1, c_2, \dots , and their values may change from one proof to another.

2 The stochastic network model and Markov process preliminaries

2.1 Model description

We consider a network with K service stations equipped with infinite capacity buffers, denoting the i^{th} station by P_i , $i \in \mathbb{K} \equiv \{1, \dots, K\}$. Upon completion of service at station P_i a customer is routed to other service station P_j with probability p_{ij} (or exits the system with probability $1 - \sum_{\ell=1}^K p_{i\ell}$) according to a probabilistic routing matrix $\mathbb{P} = (p_{ij})_{i,j \in \mathbb{K}}$. The jobs are assumed to be processed by the First-In-First-Out discipline at each station. We require

- (A) the spectral radius of \mathbb{P} is strictly less than 1 and $p_{ii} = 0$ for all $i \in \mathbb{K}$,

so that the network is open; that is, any customer entering the network eventually leaves it. We allow arrival and service rates to be time varying random processes, which exhibit state (i.e., current queue levels in a system) dependent features.

A precise description of the model is as follows. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ be a filtered probability space. Assume that unit-rate independent Poisson processes, N_i, N_{ij} with $i \in \mathbb{K}, j \in \mathbb{K} \cup \{0\}$ are $\{\mathcal{F}_t\}$ adapted and a family of random variables

$$\left\{ N_i(t) - N_i(s), N_{ij}(t) - N_{ij}(s) : i \in \mathbb{K}, j \in \mathbb{K} \cup \{0\}, 0 \leq s < t < \infty \right\}$$

is independent of $\{\mathcal{F}_s : 0 \leq s < t < \infty\}$. For $i \in \mathbb{K}$, N_i will be used to define the stream of jobs entering the i^{th} buffer and for $i, j \in \mathbb{K}$, N_{ij} will be used to represent the flow of jobs to buffer j from buffer i . For $i \in \mathbb{K}$ and $j = 0$, N_{ij} will be associated with jobs that leave the system after service at station P_i .

Let $\lambda_i, \mu_i, i \in \mathbb{K}$ be measurable functions from $\mathbb{R}_+^K \rightarrow \mathbb{R}_+$. We write $\lambda = (\lambda_1, \dots, \lambda_K)'$; μ is defined similarly. These functions will be used to define state-dependent arrival and service rates. External arrivals are assumed to occur for any $i \in \mathbb{K}_e$, where \mathbb{K}_e is a subset of \mathbb{K} . Thus $\lambda_i(x) = 0$ for $x \in \mathbb{R}_+^K$ and $i \in \mathbb{K} \setminus \mathbb{K}_e$. Set

$$S \equiv \mathbb{R}_+^K, R \equiv [\mathbb{I} - \mathbb{P}'], \text{ and } a(\cdot) \equiv \lambda(\cdot) - R\mu(\cdot).$$

Throughout this paper, we assume that $\lambda, \mu \in C_b(S)$.

Let $Q = (Q_1, \dots, Q_K)'$ denote the queue length vector process. Then the state of the system is given by the following equation: For $t \geq 0$,

$$\begin{aligned} Q_i(t) &= Q_i(0) + N_i \left(\int_0^t \lambda_i(Q(s)) ds \right) \\ &\quad + \sum_{j=1}^K N_{ji} \left(p_{ji} \int_0^t \mathbf{1}_{\{Q_j(s) > 0\}} [\mu_j(Q(s))] ds \right) \\ &\quad - \sum_{j=0}^K N_{ij} \left(p_{ij} \int_0^t \mathbf{1}_{\{Q_i(s) > 0\}} [\mu_i(Q(s))] ds \right), \quad i \in \mathbb{K}. \end{aligned} \quad (2.1)$$

In the above display $p_{i0} \equiv 1 - \sum_{j=1}^K p_{ij}$. We require that M_{ij} defined below are $\{\mathcal{F}_t\}$ martingales (cf. Lemma 3.9 Mandelbaum & Pats (1998)):

$$M_{0i}(t) \equiv N_i \left(\int_0^t \lambda_i(Q(s)) ds \right) - \int_0^t \lambda_i(Q(s)) ds, \quad (2.2)$$

$$\begin{aligned} M_{ij}(t) &\equiv N_{ij} \left(p_{ij} \int_0^t \mathbf{1}_{\{Q_i(s) > 0\}} [\mu_i(Q(s))] ds \right) \\ &\quad - p_{ij} \int_0^t \mathbf{1}_{\{Q_i(s) > 0\}} [\mu_i(Q(s))] ds. \end{aligned} \quad (2.3)$$

In general, a filtration, for which the processes in (2.2) and (2.3) are adapted and martingales, would depend on the corresponding state process. According to Lemma 3.9 in Mandelbaum & Pats (1998), there exists a filtration $(\mathcal{F}_t, t \geq 0)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the usual conditions (cf. Karatzas & Shreve (1991), page 10), such that (M_{0i}, M_{ij}) given by (2.2) and (2.3) are vector-valued, locally square integrable (cf. Protter (2004), page 38) $\{\mathcal{F}_t\}$ -martingales, and Q is $\{\mathcal{F}_t\}$ -adapted. From $\lambda, \mu \in C_b(S)$, it follows that the martingales defined above are square integrable.

Let $M_i \equiv \sum_{j=0}^K M_{ji} - \sum_{j=0}^K M_{ij}$. Then we can rewrite the evolution (2.1) as

$$Q_i(t) = Q_i(0) + \int_0^t \left[\lambda_i(Q(s)) + \sum_{j=1}^K p_{ji} \mu_j(Q(s)) - \mu_i(Q(s)) \right] ds + M_i(t) + [RY(t)]_i,$$

where

$$Y_i(t) = \int_0^t 1_{\{Q_i(s)=0\}} [\mu_i(Q(s))] ds, \quad i \in \mathbb{K}.$$

Note that Y_i is an RCLL non-decreasing $\{\mathcal{F}_t\}$ adapted process and Y_i increases only when $Q_i(t) = 0$, i.e.,

$$\int_0^\infty 1_{\{Q_i(t) \neq 0\}} dY_i(t) = 0 \quad \mathbb{P}\text{-a.s.}$$

The state evolution can be expressed succinctly by the following vector equation:

$$Q(t) = Q(0) + \int_0^t a(Q(s)) ds + M(t) + RY(t). \quad (2.4)$$

The above dynamics can equivalently be described in terms of a Skorokhod map as described below.

Definition 2.1. (Skorokhod Problem) Let $\psi \in D([0, \infty), \mathbb{R}^K)$ be given with $\psi(0) \in S$. Then we say that $(\phi, \eta) \in D([0, \infty), \mathbb{R}^K) \times D([0, \infty), \mathbb{R}^K)$ solves the Skorokhod problem for ψ with respect to S and R if and only if the following hold:

- (i) $\phi(t) = \psi(t) + R\eta(t) \in S$, for all $t \geq 0$;
- (ii) η satisfies, for $i \in \mathbb{K}$,
 - (a) $\eta_i(0) = 0$,
 - (b) η_i is non-decreasing, and
 - (c) η_i can increase only when ϕ is on the i^{th} face of S , that is, $\int_0^\infty 1_{\{\phi_i(s) \neq 0\}} d\eta_i(s) = 0$.

Let $D_S([0, \infty), \mathbb{R}^K) \equiv \{\psi \in D([0, \infty), \mathbb{R}^K) : \psi(0) \in S\}$. We define the Skorokhod map $\Gamma : D_S([0, \infty), \mathbb{R}^K) \rightarrow D([0, \infty), S)$ as

$$\Gamma(\psi) \equiv \phi,$$

if $(\phi, R^{-1}[\phi - \psi])$ is the unique solution of the Skorokhod problem posed by ψ . It is known that the Skorokhod map is well defined on all of $D_S([0, \infty), \mathbb{R}^K)$ (see Harrison & Reiman (1981); Dupuis & Ishii (1991)), that is, there is a unique solution to the Skorokhod problem.

The following Proposition 2.2 gives the regularity of the Skorokhod map, which is a consequence of Assumption (A). We refer the reader to the proof of (10) in page 305 of Harrison and Reiman Harrison & Reiman (1981) and arguments in pages 164–165 of Dupuis & Ramanan (1999) for its proof. See also Dupuis & Ishii (1991); Dupuis & Ramanan (1999) for more sufficient conditions under which this regularity property holds.

Proposition 2.2. *The Skorokhod map is Lipschitz continuous in the following sense: There exists a constant $L \in (1, \infty)$ such that for all $\psi_1, \psi_2 \in D_S([0, \infty), \mathbb{R}^K)$, and $t \geq 0$,*

$$\sup_{0 \leq s \leq t} |\Gamma(\psi_1)(s) - \Gamma(\psi_2)(s)| < L \sup_{0 \leq s \leq t} |\psi_1(s) - \psi_2(s)|.$$

The dynamics in (2.4) can now be equivalently described in terms of the Skorokhod map as follows:

$$Q(t) = \Gamma \left(Q(0) + \int_0^t a(Q(s)) ds + M(\cdot) \right) (t), \quad \text{for } t \geq 0. \quad (2.5)$$

When $Q(0) \equiv x$, we will sometimes write the corresponding state process as Q_x .

We now introduce our main stability condition on the queueing system that will be assumed throughout this paper:

(S) *There exists a $\theta > 0$, such that $\sup_{x \in S, i \in \mathbb{K}} [R^{-1}a(x)]_i < -\theta$.*

Remark 2.3. *For the case of constant intensities $\lambda(\cdot) \equiv \lambda$ and $\mu(\cdot) \equiv \mu$ ($\lambda, \mu \in S$), the assumption (S) amounts to the well-known ergodicity condition on the flow equation in the classical Jackson networks setting. See, for example, Theorem 2.1 in Sigman (1990).*

The assumption (S) will be particularly used to stipulate the permissible drift vector field (see (3.7) below), which in turn enables us to use the Lyapunov function $T(\cdot)$, the hitting time to the origin (see (3.8)), for the stability analysis of the underlying stochastic network model. More precisely, we make use of the basic properties of the Lyapunov function $T(\cdot)$ to prove the uniform upper bound on exponential hitting times and return times of the process to the compact sets (see Theorems 3.3, 3.4), which constitute the crucial steps towards the desired ergodicity of the process in Theorem 3.6 and Theorem 3.9.

Remark 2.4. *The condition (S) is generally more restrictive than the following fluid limit model requirements: (i) The scaled queue length process (e.g., $\{Q(nt)/n : n \geq 1\}$) converges uniformly to a fluid limit model, which is obtained formally using a functional law of large numbers. (ii) This fluid limit model is stable, i.e., the origin of the coordinates is the absorbing point and fluid limit model eventually reaches the origin. (See Definition 4.1 and Theorem 5.1 in Dai (1995). See also Section 4 in Mandelbaum & Pats (1998) for a related fluid approximation model.) Fluid models and fluid limits are mostly used for stability analysis for underlying queueing systems (cf. Dai (1995); Chen (1995); Bramson (2008) among others).*

Hereafter, explicit reference to (S) in statements of our results will be omitted.

2.2 Stability concepts for Markov processes

For the reader's convenience, we collect the standard Markov processes terminologies and notions on a 'stability' for a continuous time Markov process Φ (cf. Down et al. (1995); Meyn & Tweedie

(2009)). The term ‘stability’ in stochastic processes literature could have different meanings depending on the context of its use and we present them as a series of increasingly stronger order.

Let X be a locally compact, complete and separable metric space. Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ be a filtered measurable space on which is given an X -valued stochastic process $\Phi = \{\Phi_t : t \in \mathbb{R}_+\}$. Assume that under each probability measure $\{\mathbb{P}_x\}_{x \in X}$, Φ is a Markov process with initial distribution δ_x and transition probability kernel $P^t(y, A) \equiv \mathbb{P}_y(\Phi_t \in A)$, $y \in X, A \in \mathcal{B}(X)$. For a real bounded measurable function f and a σ -finite measure ν on X , define

$$P^t f(x) \equiv \int P^t(x, dy) f(y), \quad \nu P^t(A) \equiv \int \nu(dx) P^t(x, A).$$

By convention, all measures ν on $(X, \mathcal{B}(X))$ in this section will be non-trivial (i.e., $\nu(X) \neq 0$).

We assume that Φ is a strong Markov process with RCLL paths and $\{P^t\}$ maps Borel functions to Borel functions. That is, for all bounded Borel maps $f : X \rightarrow \mathbb{R}$, the map $x \mapsto P^t f(x)$ is a Borel measurable map. Let $\mathbb{E}_x[\cdot]$ denote the expectation with respect to the measure \mathbb{P}_x . For a measurable set $A \in \mathcal{B}(X)$ let $\eta_A \equiv \int_0^\infty 1_{\{\Phi_t \in A\}} dt$, a sojourn time in A .

The process Φ is called *Harris recurrent* if for some σ -finite measure φ on $(X, \mathcal{B}(X))$, we have that $\mathbb{P}_x(\{\eta_A = \infty\}) = 1$ for all $x \in X$ whenever $\varphi(A) > 0$. It can be shown that if Φ is Harris recurrent then it has a unique (up to a scalar multiplier) invariant measure π . A σ -finite measure π on $(X, \mathcal{B}(X))$ is called an *invariant measure* for Φ if and only if for all $A \in \mathcal{B}(X)$, $x \in X$ and $t \geq 0$, $\pi(A) = \pi P^t(x, A)$. If π is finite, then it can be normalized to a probability measure. With an abuse of notation, we denote the normalized probability measure once more by π . The Harris recurrent process Φ is called *positive Harris recurrent* if it has a finite invariant measure π .

Next we introduce a stronger notion of stability. For a signed measure ν on $\mathcal{B}(X)$ define its *total variation norm* $\|\nu\|$ as $\|\nu\| \equiv \sup_{f:|f| \leq 1} |\nu(f)|$. A Markov process Φ is called *ergodic* if it has a unique invariant probability measure π and

$$\lim_{t \rightarrow \infty} \|P^t(x, \cdot) - \pi\| = 0$$

for all $x \in X$. Ergodicity ensures the convergence of the expectation $\mathbb{E}_x[f(\Phi_t)]$ to the steady state value $\pi(f)$, for all bounded measurable functions f , as $t \rightarrow \infty$. To investigate such convergence for an *unbounded* function f , we need the concept of the so-called f -norm. For any signed measure ν on $\mathcal{B}(X)$ and $f \geq 1$, define its f -norm as

$$\|\nu\|_f \equiv \sup_{\{g|g \leq f\}} |\nu(g)|.$$

Among several asymptotic properties of a Markov process, an exponential rate of convergence to the steady-state is perhaps one of the most sought-after ergodic property. Let a Markov process Φ be positive Harris recurrent with invariant probability measure π .

- ◇ For a measurable function $f : X \rightarrow [1, \infty)$, Φ is called *f -uniformly ergodic* if there exist constants $r \in (0, \infty)$, $\rho \in (0, 1)$ such that for all $t \in \mathbb{R}_+$ and $x \in X$, $\|P^t(x, \cdot) - \pi\|_f \leq f(x)r\rho^t$.

3 Stability analysis

We begin by recalling the positive Harris recurrence of Q , which was established in Theorem 1 of Lee (2010).

Theorem 3.1. *Let Q_x be defined by (2.5) with $Q_x(0) \equiv x \in S$. Then Q is positive Harris recurrent.*

Henceforth, we denote the unique invariant probability measure of Q by π . We will now study the stronger stability properties of Q . Define

$$Z_x(t) \equiv \Gamma(x + u(\cdot))(t), \quad t \geq 0, \quad (3.6)$$

where

$$u(t) = \int_0^t b(s)ds \equiv \int_0^t a(Q(s))ds, \quad t \geq 0.$$

Next, denoting by $\mathcal{C} \equiv \{v \in \mathbb{R}^K : R^{-1}v \leq 0\}$ we see from condition (S) that there exists a $\beta \in (0, \infty)$ satisfying

$$\begin{aligned} \inf_{s \geq 0} \text{dist}(b(s), \partial\mathcal{C}) &= \inf_{s \geq 0} \inf \{|b(s) - v| : R^{-1}v = 0, v \in \mathbb{R}^K\} \\ &\geq \inf_{s \geq 0} \frac{1}{\|R^{-1}\|} \inf \{|R^{-1}b(s) - R^{-1}v| : R^{-1}v = 0, v \in \mathbb{R}^K\} \\ &= \inf_{s \geq 0} \frac{1}{\|R^{-1}\|} |R^{-1}b(s)| \\ &\geq \frac{1}{\|R^{-1}\|} \theta K \equiv \beta > 0, \end{aligned} \quad (3.7)$$

where $\theta \in (0, \infty)$ is a constant as in condition (S), and the first inequality is owing to the fact $|Ax| \leq \|A\| \cdot |x|$ for a square matrix A and a vector x .

Thus for $s \geq 0$,

$$b(s) \in \mathcal{C}_\beta \equiv \{v \in \mathcal{C} : \text{dist}(v, \partial\mathcal{C}) \geq \beta\}.$$

For $z_0 \in S$ denote by $\mathcal{K}(z_0)$ the collection of all trajectories $\psi : [0, \infty) \rightarrow S$ of the form

$$\psi(t) \equiv \Gamma\left(z_0 + \int_0^t \varpi(s)ds\right)(t), \quad t \geq 0,$$

where $\varpi : [0, \infty) \rightarrow \mathbb{R}^K$ is a measurable map satisfying

$$\text{for all } t \in [0, \infty), \quad \int_0^t |\varpi(s)|ds < \infty, \quad \varpi(t) \in \mathcal{C}_\beta.$$

Define the ‘‘hitting time to the origin’’ function as follows,

$$T(z_0) \equiv \sup_{\psi \in \mathcal{K}(z_0)} \inf\{t \in [0, \infty) : \psi(t) = 0\}. \quad (3.8)$$

Then Lemma 3.1 of Atar et al. (2001) shows that the trajectory in (3.6) hits the origin in a finite amount of time and stays at the origin afterwards:

$$T(z_0) \leq \frac{4L^2}{\beta}|z_0|, \text{ and for all } \psi \in \mathcal{K}(z_0), \psi(t) = 0 \text{ for all } t \geq T(z_0),$$

where $\beta, L \in (0, \infty)$ are as in (3.7) and Proposition 2.2, respectively. The following properties of the Lyapunov function $T(\cdot)$ from Atar et al. (2001) (cf. Lemma 3.1) are critical in our analysis.

Lemma 3.2. *The function $T : S \rightarrow [0, \infty)$ in (3.8) satisfies the following properties.*

- (1) For some $c_1 \in (0, \infty)$, $|T(x) - T(y)| \leq c_1|x - y|$ for all $x, y \in S$.
- (2) For some $c_2, c_3 \in (0, \infty)$, $c_2|x| \leq T(x) \leq c_3|x|$ for all $x \in S$.
- (3) $T(Z_x(t)) \leq (T(x) - t)^+$, \mathbb{P}_x -a.s. for all $x \in S, t \geq 0$, where $Z_x(t)$ is defined in (3.6).

Fix $\bar{M}_1 \in (0, \infty)$ and define $B_1 \equiv \{x \in S : |x| \leq \bar{M}_1\}$. From Lemma 3.2 (2), one can choose $\bar{M}_2 \in (0, \infty)$ large enough so that $B_1 \subset B_2 \equiv \{x \in S : |T(x)| \leq \bar{M}_2\}$. Define a stopping time

$$\sigma_1 \equiv \inf\{t \geq 0 : Q(t) \in B_2\}.$$

Most of the proofs below are adaptations of those in Budhiraja & Lee (2007); the estimates available for constrained diffusion processes considered in Budhiraja & Lee (2007) are now replaced by those for (strong) Markov jump process on the K -dimensional, non-negative integer lattice, and also some new estimates are proved along the way as needed.

Theorem 3.3. *There exist $\bar{\beta} \in (0, \infty)$ and $\alpha_1, \alpha_2 \in (0, \infty)$ such that for all $x \in S$,*

$$\mathbb{E}_x[e^{\bar{\beta}\sigma_1}] < \alpha_1 e^{\alpha_2|x|}.$$

In particular, for any compact set $E \subset S$,

$$\sup_{x \in E} \mathbb{E}_x[e^{\bar{\beta}\sigma_1}] < \infty.$$

Proof. Let $(\Omega, \mathcal{F}, \mathbb{P}, \{\mathcal{F}_t\}_{t \geq 0})$ be a filtered probability space on which $\{M_t\}$ is an $\{\mathcal{F}_t\}$ martingale. Let $T(\cdot)$ be as in (3.8). Fix $\Delta, t \in (0, \infty)$. Using Lemma 3.2 (1), (3) along with Lipschitz property of Skorokhod map (Proposition 2.2) we have

$$T(Q_x(t + \Delta)) \leq [T(Q_x(t)) - \Delta]^+ + c_4 \sum_{i=1}^K \sum_{j=0}^K \sup_{0 \leq s \leq \Delta} |M_{ij}(t + s) - M_{ij}(t)|,$$

where $c_4 = 2Lc_1$ and c_1 is as in Lemma 3.2. For $n \in \mathbb{N}$, define

$$E_n \equiv \left\{ \omega \in \Omega : \inf_{s \in [0, n\Delta]} T(Q(s)) > \bar{M}_2 \right\}.$$

Then we have, along the lines of Theorem 4.1 of Atar et al. (2001), that

$$\mathbb{P}_x(E_n) \leq \mathbb{P}_x \left(\bar{M}_2 < T(Q(n\Delta)) \leq T(x) - n\Delta + c_4 \sum_{m=1}^n \nu_m \right),$$

where for $1 \leq m \leq n$,

$$\nu_m \equiv \sum_{i=1}^K \sum_{j=0}^K \sup_{0 \leq s \leq \Delta} |M_{ij}((m-1)\Delta + s) - M_{ij}((m-1)\Delta)|.$$

Thus,

$$\begin{aligned} \mathbb{P}_x(E_n) &\leq \mathbb{P}_x \left(c_4 \sum_{m=1}^n \nu_m \geq n\Delta + \bar{M}_2 - T(x) \right) \\ &\leq \frac{\mathbb{E}[\exp(\alpha c_4 \sum_{m=1}^n \nu_m)]}{\exp(\alpha(n\Delta + \bar{M}_2 - T(x)))}, \end{aligned} \quad (3.9)$$

where $\alpha > 0$ is arbitrary. We make use of the following estimate (proof of which follows along the lines of Lemma 4.4 in Atar et al. (2001); see the proof of (7.3) in Budhiraja et al. (2010) for detailed arguments): There are positive constants γ, Δ, η such that

$$\sup_{m \in \mathbb{N}} e^{-\gamma\Delta} \mathbb{E} \left(e^{2\gamma c_4 \nu_m} \mid \mathcal{F}_{(m-1)\Delta} \right) \leq e^{-\eta\Delta}.$$

Using this estimate with $\alpha = 2\gamma$ in (3.9) we get

$$\begin{aligned} \mathbb{P}_x(E_n) &\leq \frac{\exp(-\eta\Delta n)}{\exp(\alpha(n\Delta - \frac{\Delta}{2} + \bar{M}_2 - T(x)))} \\ &\leq c_5 \exp(\alpha T(x)) \exp(-\eta' n\Delta), \end{aligned}$$

where $\eta' = \eta + 2\gamma > 0$ and for some $c_5 \in (0, \infty)$. Let $t \in (0, \infty)$ be arbitrary and pick $n_0 \in \mathbb{N}$ such that $t \in [n_0\Delta, (n_0 + 1)\Delta]$. Then

$$\begin{aligned} \mathbb{P}_x(\sigma_1 > t) &\leq \mathbb{P}_x(E_{n_0}) \\ &\leq c_5 e^{\alpha T(x)} e^{-\eta' t}. \end{aligned}$$

Finally, we have

$$\begin{aligned} \mathbb{E}_x[e^{\bar{\beta}\sigma_1}] &= 1 + \int_0^\infty \bar{\beta} e^{\bar{\beta}t} \mathbb{P}_x[\sigma_1 > t] dt \\ &\leq 1 + c_5 \bar{\beta} e^{\alpha T(x)} \int_0^\infty e^{(\bar{\beta} - \eta')t} dt \\ &= 1 + \frac{c_5 \bar{\beta}}{\eta' - \bar{\beta}} e^{\alpha T(x)}, \end{aligned}$$

for $\bar{\beta} \in (0, \eta')$. Hence the result follows from the above estimates on recalling Lemma 3.2 (2) and this completes the proof. \blacksquare

Our objective now is to establish suitable bounds on exponential moments of hitting times of compact sets. Let $B_3 \equiv \{x \in S : |T(x)| \leq \bar{M}_3\}$, where $\bar{M}_3 \geq \bar{M}_2$ is chosen large enough so that $\text{dist}(B_2, \partial B_3) > 1$. Note that $B_1 \subset B_2 \subset B_3$. Let

$$\sigma_0 \equiv \inf\{t \geq 0 : Q(t) \in \partial B_3\}, \quad \tau_1 \equiv \inf\{t \geq \sigma_0 : Q(t) \in \partial B_2\},$$

and recall $\sigma_1 \equiv \inf\{t \geq 0 : Q(t) \in B_2\}$.

Theorem 3.4. *There exist $\beta_1 \in (0, \infty)$ and $A \in (0, \infty)$ such that $\sup_{x \in B_2} \mathbb{E}_x[e^{\beta_1 \tau_1}] < A$. Furthermore, for fixed $\delta \in (0, \infty)$,*

$$\sup_{x \in B_2} \mathbb{E}_x \left[e^{\beta_1 \tau_{B_2}(\delta)} \right] < \infty. \quad (3.10)$$

Proof. Recalling the positive Harris recurrence of Q (Theorem 3.1) and in view of Lemma 3.2 (2) we have the following result: For each fixed $k \in \mathbb{N}$ there exists an $\epsilon_0 \equiv \epsilon_0(k) \in (0, 1)$ such that $\sup_{x \in B_3} \mathbb{P}_x(\sigma_0 > k) < \epsilon_0$ and $\sup_{x \in \partial B_3} \mathbb{P}_x(\sigma_1 < k) < \epsilon_0$.

Let

$$F_{x,k-1} \equiv \mathbb{E}_x \left[1_{\{\sigma_0 > k\}} | \mathcal{F}_{k-1} \right] 1_{\{\sigma_0 > k-1\}}.$$

Note that $\mathbb{P}_x[\sigma_0 > k] = \mathbb{E}_x(F_{x,k-1})$ and

$$\begin{aligned} F_{x,k-1} &\leq \sup_{x \in B_3} \mathbb{P}_x[\sigma_0 > 1] 1_{\{\sigma_0 > k-1\}} \\ &< \epsilon_0(1) 1_{\{\sigma_0 > k-1\}}. \end{aligned}$$

On iterating the above conditioning argument k times we have that there exists $\vartheta_0 \in (0, \infty)$ such that for all $k \in \mathbb{N}$ and $x \in B_2$, $\mathbb{P}_x[\sigma_0 > k] < e^{-\vartheta_0 k}$. Hence, there exists $\beta_0 \in (0, \infty)$ such that $\sup_{x \in B_2} \mathbb{E}_x[e^{\beta_0 \sigma_0}] < \infty$ and by strong Markov property of Q and Theorem 3.3, the first result follows.

To show the second result, we define stopping time sequences $\{v_i\}, \{\varrho_i\}, i \in \mathbb{N}$ as follows: Let $v_0 \equiv 0$ and

$$\varrho_n \equiv \inf\{t \geq v_{n-1} : Q(t) \in \partial B_3\}, \quad v_n \equiv \inf\{t \geq \varrho_n : Q(t) \in B_2\},$$

$n = 1, 2, \dots$. Let $\vartheta \equiv \epsilon_0(\delta)$. Also for $n \in \mathbb{N}$, set $m_n \equiv v_n - v_{n-1}$ and let $m_0 \equiv 0$. Then we have

$$\mathbb{P}_x[v_n - v_{n-1} > \delta | \mathcal{F}_{v_{n-1}}] \geq 1 - \vartheta$$

for all $n \geq 1$ and from second Borel-Cantelli Lemma $v_n \rightarrow \infty$ a.s. as $n \rightarrow \infty$. Therefore, for $x \in B_2$

$$\mathbb{E}_x \left[e^{\beta_1 \tau_{B_2}(\delta)} \right] = e^{\beta_1 \delta} \mathbb{E}_x \sum_{n=1}^{\infty} 1_{\delta \in (v_{n-1}, v_n]} e^{\beta_1 (\tau_{B_2}(\delta) - \delta)}. \quad (3.11)$$

Since

$$1_{\delta \in (v_{n-1}, v_n]} e^{\beta_1 (\tau_{B_2}(\delta) - \delta)} \leq 1_{\delta \in (v_{n-1}, v_n]} e^{\beta_1 (v_n - v_{n-1})},$$

we have for $n \geq 1$,

$$\begin{aligned} \mathbb{E}_x \left[1_{\delta \in (v_{n-1}, v_n]} e^{\beta_1(\tau_{B_2}(\delta) - \delta)} \right] &\leq \mathbb{E}_x \left[e^{\beta_1(v_n - v_{n-1})} \prod_{i=1}^{n-1} 1_{\{m_i < \delta\}} \right] \\ &\leq \mathbb{E}_x \left[\prod_{i=1}^{n-1} 1_{\{m_i < \delta\}} \mathbb{E}_x \left[e^{\beta_1(v_n - v_{n-1})} | \mathcal{F}_{v_{n-1}} \right] \right]. \end{aligned} \quad (3.12)$$

As a consequence of the strong Markov property of Q and $\sup_{x \in B_2} \mathbb{E}_x[e^{\beta_1 \tau_1}] < A$, we obtain

$$\mathbb{E}_x \left[1_{\delta \in (v_{n-1}, v_n]} e^{\beta_1(\tau_{B_2}(\delta) - \delta)} \right] \leq A \mathbb{E}_x \prod_{i=1}^{n-1} 1_{\{m_i < \delta\}}.$$

Observing that for $n \geq 2$, $\mathbb{E}_x[1_{m_{n-1} < \delta} | \mathcal{F}_{v_{n-2}}] < \vartheta$, and by a successive conditioning argument we now have that the right side of (3.12) is bounded by $A\vartheta^{n-1}$. The result now follows on substituting this bound in (3.11), that is,

$$\sup_{x \in B_2} \mathbb{E}_x \left[e^{\beta_1 \tau_{B_2}(\delta)} \right] \leq A e^{\beta_1 \delta} \sum_{n=1}^{\infty} \vartheta^{n-1} < \infty. \quad \blacksquare$$

Remark 3.5. We note that similar return time result as in (3.10) was obtained for a class of multi-dimensional reflected fractional Brownian motion (RFBM) process on the positive orthant with Hurst parameter $H \in (1/2, 1)$ (see Lee (2011)). The RFBM process appears as a limiting workload process for fluid queueing network models fed by a large number of heavy-tailed ON/OFF sources in heavy traffic.

We proceed on recalling an extended generator of a Markov process. Denote by $\mathcal{D}(\tilde{\mathcal{A}})$ the set of all measurable functions $V : S \rightarrow \mathbb{R}$ for which for all $x \in S$, $\mathbb{E}_x|V(Q_t)| < \infty$ and there exists a measurable function $W : S \rightarrow \mathbb{R}$ satisfying

$$\mathbb{E}_x[V(Q_t)] = V(x) + \mathbb{E}_x \left[\int_0^t W(Q_s) ds \right], \quad \int_0^t \mathbb{E}_x[|W(Q_s)|] ds < \infty,$$

for $t > 0$. For $V \in \mathcal{D}(\tilde{\mathcal{A}})$ and a corresponding W , we write $(V, W) \in \tilde{\mathcal{A}}$ and $W = \tilde{\mathcal{A}}V$. We refer to the map $\tilde{\mathcal{A}}$ as the extended generator of Q and $\mathcal{D}(\tilde{\mathcal{A}})$ the domain of the extended generator.

Henceforward, let $P^t(x, A)$, $x \in S$, $A \in \mathcal{B}(S)$, $t \geq 0$, denote the transition probability of Q , defined as $P^t(x, A) \equiv \mathbb{P}_x(Q_t \in A)$. Define the Markov transition function $\mathfrak{R}_\kappa : S \times \mathcal{B}(S) \rightarrow [0, 1]$ as

$$\mathfrak{R}_\kappa(x, A) \equiv \int_0^\infty P^t(x, A) \kappa \exp(-\kappa t) dt$$

and call it by *the resolvent kernel*; if $\kappa = 1$, we write \mathfrak{R}_κ as merely \mathfrak{R} .

Under stability condition (S), the Markov process Q satisfies the following important ‘‘drift criteria.’’

Theorem 3.6. *Let*

$$f_0(x) \equiv \frac{1}{\beta_1} [\mathbb{E}_x e^{\beta_1 \tau_{B_2}(\delta)} - 1] + 1,$$

where β_1 and B_2 are as in Theorem 3.4. Then for all $\kappa > 0$,

$$(f_\kappa, W_\kappa) \in \tilde{\mathcal{A}}, \text{ where } f_\kappa \equiv \mathfrak{R}_\kappa f_0 \text{ and } W_\kappa \equiv f_\kappa - f_0.$$

Furthermore, there exist $b, c \in (0, \infty)$ such that

$$\tilde{\mathcal{A}}f_\kappa(x) \leq -cf_\kappa(x) + b1_{B_2}(x), \text{ for all } x \in S. \quad (3.13)$$

Proof. From Theorem 3.4, we have

$$\mathbb{E}_x[e^{\beta_1 \tau_{B_2}(\delta)}] < \infty \text{ for all } x \in S \text{ and } \sup_{x \in B_2} f_0(x) < \infty.$$

Then the result follows from these two facts and applying Theorem 6.2 (b) in Down et al. (1995) by taking $f \equiv 1$ therein along with Theorem 5.1 (a) in the same paper. We note that the cited results in Down et al. (1995) are formulated in terms of petite sets. However, according to Lemma 2 in Lee (2010), B_2 can be shown indeed a petite set in S . ■

The following martingale estimate will be used several times. Recalling that M_i is an $\{\mathcal{F}_t\}$ square integrable martingale, one gets using Doob's inequality that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |M_i(s)| &\leq \left(\mathbb{E} \sup_{0 \leq s \leq t} |M_i(s)|^2 \right)^{1/2} \\ &\leq (4\mathbb{E}|M_i(t)|^2)^{1/2} \\ &\leq c_1 \left(\mathbb{E} \left[\int_0^t \left[\lambda_i(Q(s)) + \sum_{j=1}^K \mu_j(Q(s)) \right] ds \right] \right)^{1/2} \\ &\leq c_2(1+t)^{1/2} \end{aligned} \quad (3.14)$$

for some $c_1, c_2 \in (0, \infty)$ and the last inequality follows from boundedness of $\lambda(\cdot)$ and $\mu(\cdot)$.

The next lemma obtains explicit exponential upper and lower bounds of Lyapunov functions f_0 and f_κ . These estimates play a crucial role in our stability analysis.

Lemma 3.7. *Let f_0 be as in Theorem 3.6, then there exist $a_1, a_2, A_1, A_2 \in (0, \infty)$, such that*

$$a_1 e^{a_2|x|} \leq f_0(x) \leq A_1 e^{A_2|x|} \text{ for each } x \in S. \quad (3.15)$$

Furthermore, there is a $A_3 \in (0, \infty)$ such that for every $\kappa \in (A_3, \infty)$ there are $\tilde{a}_1, \tilde{a}_2, \tilde{A}_1, \tilde{A}_2 \in (0, \infty)$, such that

$$\tilde{a}_1 e^{\tilde{a}_2|x|} \leq f_\kappa(x) \leq \tilde{A}_1 e^{\tilde{A}_2|x|} \text{ for each } x \in S, \quad (3.16)$$

where f_κ is as in Theorem 3.6.

The above lemma can be proved by combining the Lipschitz property of the Skorokhod map (Proposition 2.2) and martingale estimate in (3.14). Its proof is deferred to the Appendix.

Henceforth, we will fix a $\kappa > A_3$ and denote the corresponding f_κ by f . As a consequence of Theorem 3.6 and Lemma 3.7, we obtain finiteness of the moment generating function of invariant measure in a neighborhood of zero.

Theorem 3.8. *Let π be the unique invariant distribution for Q and $\tilde{a}_2 \in (0, \infty)$ be as in Lemma 3.7. Then for all $c \in \mathbb{R}^K$ with $|c| \leq \tilde{a}_2$, we have*

$$\int_S e^{c \cdot x} \pi(dx) < \infty.$$

Proof. We see that Theorem 5.1 (c) of Down et al. (1995) along with equation (3.13) implies the existence of $\bar{r} \in (0, 1)$ and $\tilde{b} \in (0, \infty)$ such that

$$\mathfrak{R}f(x) \leq f(x) - (1 - \bar{r})f(x) + \tilde{b}1_{B_2}(x)$$

for all $x \in S$. Combining the above drift condition with Theorem 3.1 of Meyn & Tweedie (1991) and Theorem 14.3.7 of Meyn & Tweedie (2009), we get

$$\int_S f(x)\pi(dx) \leq \frac{b\pi(B_2)}{1 - \bar{r}} < \infty.$$

Let $\tilde{a}_1 \in (0, \infty)$ be as in Lemma 3.7. Then

$$\begin{aligned} \tilde{a}_1 \int_S e^{c \cdot x} \pi(dx) &\leq \tilde{a}_1 \int_S e^{\tilde{a}_2 |x|} \pi(dx) \\ &\leq \int_S f(x)\pi(dx) < \infty, \end{aligned}$$

which completes the proof. ■

Due to Theorem 5.2 (c) of Down et al. (1995) and Theorem 3.6 we conclude that Q is f -uniformly ergodic.

Theorem 3.9. *The Markov process Q is f -uniformly ergodic; i.e., there exist constants $r \in (0, \infty)$, $\rho \in (0, 1)$ such that for all $t \in \mathbb{R}_+$ and $x \in S$,*

$$\|P^t(x, \cdot) - \pi\|_f \leq f(x)r\rho^t.$$

As immediate consequences of Theorem 3.9, we obtain the following results on uniform (in time and initial condition in a compact set) estimates on exponential moments and growth estimates on polynomial moments of the process Q as a function of the initial conditions. For a function $U : S \rightarrow [1, \infty)$, let L_∞^U be the vector space of functions $h : S \rightarrow \mathbb{R}$ such that $\|h\|_U \equiv \sup_{x \in S} \frac{|h(x)|}{U(x)} < \infty$.

Theorem 3.10. For every $g \in L_\infty^f$, there exists a $\tilde{r} \in (0, \infty)$ such that for all $x \in S$ and $t \geq 0$,

$$\mathbb{E}_x g(Q_t) \leq \tilde{r}[1 + f(x)\rho^t],$$

where $\rho \in (0, \infty)$ is as in Theorem 3.9. In particular, for a suitable $\tilde{r} \in (0, \infty)$,

$$\mathbb{E}_x e^{\tilde{a}_2|Q_t|} \leq \tilde{r}[1 + f(x)\rho^t],$$

where $\tilde{a}_2 \in (0, \infty)$ is as in Lemma 3.7, and for every compact set $E \subset S$ we have

$$\sup_{t \geq 0} \sup_{x \in E} \mathbb{E}_x e^{\tilde{a}_2|Q_t|} < \infty.$$

Proof. For $g \in L_\infty^f$, let $\tilde{g} \equiv \frac{g}{\|g\|_f}$. Then $\tilde{g} \leq f$ and by Theorem 3.9, we have that for all $t \in \mathbb{R}_+$ and $x \in S$,

$$\left| \mathbb{E}_x \left\{ \tilde{g}(Q_t) - \int_S \tilde{g}(y)\pi(dy) \right\} \right| \leq f(x)r\rho^t,$$

where $r \in (0, \infty)$ is as in Theorem 3.9. Hence we get

$$\mathbb{E}_x \tilde{g}(Q_t) \leq \int_S \tilde{g}(y)\pi(dy) + f(x)r\rho^t.$$

Since $\int_S \tilde{g}(y)\pi(dy) \leq \pi(f) < \infty$, there is a $\tilde{r} \in (0, \infty)$ such that

$$\mathbb{E}_x g(Q_t) \leq \tilde{r}[1 + f(x)\rho^t].$$

Choosing $g(x)$ to be $\tilde{a}_1 e^{\tilde{a}_2|x|}$ as in Lemma 3.7 so that $|g| \leq f$, yields that

$$\mathbb{E}_x e^{\tilde{a}_2|Q_t|} \leq \tilde{r}[1 + f(x)\rho^t],$$

and hence again from Lemma 3.7 the last assertion of the theorem follows. ■

We now proceed on proving growth estimates of polynomial moments of the state process as a function of the initial conditions.

Theorem 3.11. There exists a $\bar{\delta}_0 > 0$ such that for all $p \geq 1$,

$$\lim_{|x| \rightarrow \infty} \sup_{\bar{\delta} \geq \bar{\delta}_0} \frac{1}{|x|^p} \mathbb{E}_x (|Q(\bar{\delta}|x)|^p) = 0.$$

Proof. Fix $p \geq 1$ and choose $\varsigma \equiv \varsigma(p) \in (0, \infty)$ small enough so that $\varsigma x^p \leq e^{\tilde{a}_2 x}$ for $x \in (0, \infty)$, where \tilde{a}_2 is as in (3.16). Then from Theorem 3.10, for $\bar{\delta}_0 \in (0, \bar{\delta}]$

$$\begin{aligned} \frac{1}{|x|^p} \mathbb{E}_x (|Q(\bar{\delta}|x)|^p) &\leq \frac{1}{\varsigma|x|^p} \mathbb{E}_x e^{\tilde{a}_2|Q(\bar{\delta}|x)|} \\ &\leq \frac{\tilde{r}}{\varsigma|x|^p} [1 + f(x)\rho^{\bar{\delta}_0|x|}]. \end{aligned}$$

Applying Lemma 3.7, there exist $c_1, c_2 \in (0, \infty)$ such that

$$\mathbb{E}_x (|Q(\bar{\delta}|x)|^p) \leq \frac{\tilde{r}}{\varsigma} [1 + c_1 e^{c_2|x|} \rho^{\bar{\delta}_0|x|}].$$

Take $\bar{\delta}_0$ large enough so that $\rho^{\bar{\delta}_0} \leq e^{-c_2}$ and then the result follows. ■

Appendix

Proof of Lemma 3.7. Using the Lipschitz property of the Skorokhod map (Proposition 2.2) and martingale upper bound estimate in (3.14), we have the following estimate: There exists a $\bar{C} \in (0, \infty)$ such that for all $x \in S$ and $0 \leq t_1 \leq t_2 < \infty$,

$$\sup_{t_1 \leq s \leq t \leq t_2} |Q_t - Q_s| \leq \bar{C} \left[\sup_{t_1 \leq s \leq t \leq t_2} |M(t) - M(s)| + (t_2 - t_1) \right], \quad \mathbb{P}_x\text{-a.s.} \quad (\text{A1})$$

We begin by showing the first inequality of (3.15). Note that $f_0(x) \geq 1$, so in order to prove the inequality it suffices to show that there exist $M \in (1, \infty)$, $a_1, a_2 \in (0, \infty)$ such that for all $|x| \geq M$, $f_0(x) \geq a_1 e^{a_2|x|}$. By Jensen's inequality, one gets

$$\mathbb{E}_x[e^{\beta_1 \tau_{B_2}(\delta)}] \geq e^{\beta_1 \mathbb{E}_x \tau_{B_2}(\delta)}.$$

For the lower bound on $\mathbb{E}_x \tau_{B_2}(\delta)$, let \bar{M} be large enough so that for all $|x| \geq \bar{M}$, $\text{dist}(x, B_2) \geq \frac{1}{2}|x|$. Then for $\Theta \in (0, 1)$ and $|x| \geq \bar{M}$, we see that

$$\mathbb{E}_x \tau_{B_2}(\delta) \geq \mathbb{E}_x \tau_{B_2}(\delta) \mathbb{1} \left\{ \sup_{0 \leq s \leq \Theta|x|} |Q_s - x| \leq \frac{1}{2}|x| \right\}$$

and hence

$$\mathbb{E}_x \tau_{B_2}(\delta) \geq \Theta|x| \mathbb{P}_x \left[\sup_{0 \leq s \leq \Theta|x|} |Q_s - x| \leq \frac{1}{2}|x| \right]. \quad (\text{A2})$$

Applying Markov inequality and from (A1), we get

$$\begin{aligned} \mathbb{P}_x \left[\sup_{0 \leq s \leq \Theta|x|} |Q_s - x| \geq \frac{1}{2}|x| \right] &\leq \frac{4\bar{C} \left[\mathbb{E}_x \sup_{0 \leq s \leq \Theta|x|} |M_s| + \Theta|x| \right]}{|x|} \\ &\leq \tilde{C} \Theta^{1/2}, \end{aligned} \quad (\text{A3})$$

where \bar{C} as in (A1), $\tilde{C} \in (0, \infty)$ is some constant and the second inequality in (A3) follows from martingale upper bound estimate in (3.14). Choosing $\Theta < 1/(2\tilde{C})^2$, we now have from (A2) that for all $|x| \geq \bar{M}$, $\mathbb{E}_x \tau_{B_2}(\delta) \geq \frac{\Theta}{2}|x|$. Then one gets the lower bound in (3.15).

Next we show the lower bound in (3.16). Recall that $f_\kappa(x) \equiv \mathfrak{A}_\kappa f_0 = \int_0^\infty \mathbb{E}_x f_0(Q_t) \kappa e^{-\kappa t} dt$, and one gets

$$\begin{aligned} f_\kappa(x) &\geq \mathbb{E}_x \int_0^\infty a_1 e^{a_2|Q_t|} \kappa e^{-\kappa t} dt \\ &\geq a_1 e^{a_2|x|} \mathbb{E}_x \int_0^\infty e^{-a_2|Q_t-x|} \kappa e^{-\kappa t} dt. \end{aligned}$$

By noting that $|Q_t - x| \leq \sup_{0 \leq s \leq t} |Q_s - x|$ and applying (A1), we now have from a similar argument leading to (A3) that

$$f_\kappa(x) \geq a_1 e^{a_2|x|} \int_1^\infty \kappa e^{-\bar{b}t} e^{-\kappa t} dt,$$

for some constant $\bar{b} \in (0, \infty)$. This establishes the lower bound in (3.16).

We continue with showing the second inequality of (3.15). Recall the stopping time σ_1 introduced above Theorem 3.3. Note that from a conditioning argument and the strong Markov property, one gets

$$\mathbb{E}_x e^{\beta_1 \tau_{B_2}(\delta)} \leq \mathbb{E}_x e^{\beta_1 \sigma_1} + \mathbb{E}_x [e^{\beta_1 \tau_{B_2}(\delta)} \mathbf{1}_{\sigma_1 \leq \delta}].$$

Thus we have

$$\mathbb{E}_x e^{\beta_1 \tau_{B_2}(\delta)} \leq \mathbb{E}_x e^{\beta_1 \sigma_1} + \sup_{x \in B_2} \mathbb{E}_x [e^{\beta_1 \tau_{B_2}(\delta)}]$$

and then the desired upper bound in (3.15) follows on using Theorems 3.3 and 3.4.

Finally, notice that

$$\begin{aligned} f_\kappa(x) &\leq \mathbb{E}_x \int_0^\infty A_1 e^{A_2 |Q_t|} \kappa e^{-\kappa t} dt \\ &\leq A_1 e^{A_2 |x|} \mathbb{E}_x \int_0^\infty \kappa e^{A_2 |Q_t - x|} e^{-\kappa t} dt. \end{aligned}$$

We have from (A1) that

$$f_\kappa(x) \leq A_1 e^{A_2 |x|} \int_0^\infty \kappa e^{A_3 t} e^{-\kappa t} dt,$$

for some constant $A_3 \in (0, \infty)$. The upper bound in (3.16) now follows on choosing $\kappa \in (A_3, \infty)$. ■

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