

V -uniform Ergodicity for State-dependent Single Class Queueing Networks

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Abstract

We consider single class queueing networks with state-dependent arrival and service rates. Under the uniform (in state) stability condition, it is shown that the queue length process is V -uniformly ergodic; that is, it has a transition probability kernel which converges to its limit geometrically quickly in the V -norm sense. Among several asymptotic properties of V -uniformly ergodic processes, we present a Strassen-type functional law of the iterated logarithm result.

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AMS 2000 subject classifications. Primary 60J65; secondary 60H35, 60K25, 93E15

1 Introduction

We consider single class queueing networks in which arrival and service rates depend on the state (i.e., queue length) of the network. The network consists of K service stations, each of which has an associated infinite capacity buffer. Arrivals of jobs can be from outside the system and/or from internal routing. Upon completion of service at a station, the customer is routed to one of the other service stations (or exits the system) according to a probabilistic routing matrix. We study state-dependent arrival and service processes that would arise if the arrival and work processes were allowed to depend on the current queue levels. Such signatures of queueing systems are common in applications to communications and computer networks. For example, congested queue levels would cause balking (i.e., customers refusing to join a queue) or reneging (i.e., customers abandoning the queue after waiting a long time). It is natural to model such state-dependent behavior directly by birth-death processes. In particular, we will consider the state-dependent form of Poisson-type input and service processes (see (2.1)-(2.4) below). Such models with many examples can be found in the literature (see, for example, [25], [41], [40], [24], [23]). The queueing network model studied in this work is similar to those in [25] and [41]. These papers are concerned with the transient analysis of networks via fluid and/or diffusion limit approximations. We, on the other hand, are interested in steady-state behavior and the rate of convergence to invariant distribution of a state-dependent network model.

Steady-state analysis of state-dependent queueing networks was initiated by the seminal work of Jackson [18]. (See also [37].) The stability properties of (generalized) Jackson type networks with constant arrival and service rates have been studied (see, for instance, [38], [10], [9], [30]). In particular, in the case where the interarrival times possess geometrically decaying tails, it is shown in [30] that the state process is geometrically ergodic. Also, in [33], the standard Foster-Lyapunov approach was used to establish state-dependent drift conditions that are sufficient for the geometric ergodicity of Markov chains. The discrete-time Markovian model (without the state-dependent feature) is considered in [27], and Theorem 4 therein shows that *uniform* convergence of the scaled queue length process to a *stable* fluid limit implies geometric ergodicity of the queue length process. (See also Chapters 8 and 10 in [28].) Results of [14] suggest that this uniform convergence assumption cannot be weakened by providing an example in which the interarrival/service times have exponential distributions and the corresponding fluid limit is stable, and yet the queue length process has heavy tails in steady-state and is not exponentially ergodic (cf. Theorem 2.3 in [14]). The main contribution of this paper lies in establishing the V -uniform ergodicity (see Definition 2.11) and related longtime asymptotic properties for the continuous time state-dependent queueing network model.

The primary result of this paper is concerned with a rate of convergence to the steady-state distribution. Under the uniform (in state) stability condition, it is shown that the queue length process is V -uniformly ergodic; that is, it has a transition probability kernel which converges to its limit geometrically quickly in the V -norm sense. Knowledge of this rate of convergence is useful in several practical situations. For example, it provides insights for (i) the performance of a stochastic simulation for underlying queueing systems; (ii) how accurately one is approximating the system state, at a given time instant t , by simulation schemes (cf. [1]); (iii) how long one should run a simulation experiment to yield satisfactory approximations. As consequences of V -uniform ergodicity, one can obtain several meaningful asymptotic properties of the process such as a strong form of the large deviations principle and mixing results, a functional central limit theorem and a Strassen-type functional law of the iterated logarithm result (cf. [22], [16], [4], and Chapter 16 in [34]).

Our proofs rely on a critical use of the Lyapunov function methods developed in [34] and [11] (see also Chapters 8 and 10 in [28]). For the stability analysis of the underlying queueing network model, we use a “hitting time to the origin” function of related deterministic dynamical systems (see (3.13)). As a result, the growth estimate on the first moment of a queue length process is obtained in terms of its initial conditions. Once positive (Harris) recurrence is assured, our main efforts lie in proving Theorem 4.2, for which related fluid limit analysis is conducted and a perturbed Lyapunov function technique is used (cf. [29], [28]) to show uniform convergence of the stable fluid limit model to zero.

The remainder of the paper is organized as follows. We begin, in Section 2, by describing the queueing network model with state-dependency, and provide a description of the dynamics of the queue length process in terms of a Skorohod map. A basic uniform (in state) stability assumption on this queueing system is then introduced (see condition (S) below). In Section 2.1, we collect some standard concepts and terminologies on stochastic stability for Markov processes. Section 3 presents moment stability results that lead to the positive Harris recurrence of the queue length process Q , and Section 4 is devoted to the study of the V -uniform ergodicity of this state process. In Theorem 4.3, we present a Strassen-type law of the iterated logarithm result, and finally in the

Appendix, we provide a proof of Lemma 3.5.

The following notation is used. For a metric space X , let $\mathcal{B}(X)$ denote the Borel σ -field on X . For a real-valued measurable function f on X and a measure ν on $\mathcal{B}(X)$, let $\nu(f) \equiv \int_X f d\nu$. The Dirac measure at the point x is denoted by δ_x . Let \mathbb{N} denote the set of natural numbers, \mathbb{N}_0 denote $\mathbb{N} \cup \{0\}$, and $\mathbb{N}_0^d = \{x = (x_1, \dots, x_d) : x_i \in \mathbb{N}_0, i = 1, \dots, d\}$. Denote the set of real numbers by \mathbb{R} and non-negative real numbers by \mathbb{R}_+ . Let \mathbb{R}^d be the d -dimensional Euclidean space with the norm of $u \in \mathbb{R}^d$, $|u| = \sum_{k=1}^d |u_k|$. For a given matrix M , denote by M^T its transpose and by M_i the i^{th} row of M . Let $\mathbb{I} = \mathbb{I}_{K \times K}$ denote the identity matrix for some K . When it is clear from the context, we will omit the subscript. For a set $A \subseteq \mathbb{R}^d$, denote its interior and boundary by A° and ∂A , respectively. For sets $A, B \subseteq \mathbb{R}^d$, $\text{dist}(A, B)$ will denote the distance between two sets, i.e., $\inf\{|x - y| : x \in A, y \in B\}$. When $\sup_{0 \leq s \leq t} |f_n(s) - f(s)| \rightarrow 0$ as $n \rightarrow \infty$ for all $t \geq 0$, we say that $f_n \rightarrow f$ uniformly on compact sets. The class of continuous functions $f : X \rightarrow Y$ is denoted by $C(X, Y)$ and real continuous bounded functions on X by $C_b(X)$. Finally, let $D(X, Y)$ denote the class of right continuous functions having left limits defined from X to Y , equipped with the usual Skorohod topology. A stochastic process will be denoted interchangeably by $\{Z(t)\}$ and $\{Z_t\}$. Inequalities for vectors are interpreted componentwise. We will denote generic constants by c_1, c_2, \dots , and their values may change from one proof to another.

2 The queueing network model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Unless specified otherwise, all the random variables considered in this work are assumed to be defined on this probability space. We consider a network with K service stations, denoting the i^{th} station by P_i , $i \in \mathbb{K} \equiv \{1, \dots, K\}$. We assume that each station has an infinite capacity buffer. Arrivals of jobs can be from outside the system and/or from internal routing. Routing is assumed to be Bernoulli, so that, upon completion of service at station P_i , a customer is routed to another service station P_j with probability p_{ij} (or exits the system with probability $1 - \sum_{\ell=1}^K p_{i\ell}$) according to a probabilistic routing matrix $\mathbb{P} = (p_{ij})_{i,j \in \mathbb{K}}$. (See [25] for state-dependent routing.) At every station the jobs are assumed to be processed by the First-In-First-Out discipline. We allow arrival and service rates to be time-varying random processes having state-dependent features.

A precise description of the model is as follows. We assume that $(\Omega, \mathcal{F}, \mathbb{P})$ is a probability space on which are given unit-rate, right continuous with left-limits (RCLL) Poisson processes, N_i, N_{ij} , $i \in \mathbb{K}, j \in \mathbb{K} \cup \{0\}$. For $i \in \mathbb{K}$, N_i will be used to define the stream of jobs entering the i^{th} buffer and for $i, j \in \mathbb{K}$, N_{ij} will be used to represent the flow of jobs to buffer j from buffer i . For $i \in \mathbb{K}$ and $j = 0$, N_{ij} will be associated with jobs that leave the system after service at station P_i . Throughout this paper, S will be used to denote the K -dimensional non-negative integer lattice \mathbb{N}_0^K . Let $\lambda_i, \mu_i, i \in \mathbb{K}$ be measurable functions from $S \rightarrow \mathbb{R}_+$ and write $\lambda = (\lambda_1, \dots, \lambda_K)^T$; μ is defined similarly. These functions will be used to define state-dependent arrival and service rates. External arrivals are assumed to occur for any $i \in \mathbb{K}_e$, where \mathbb{K}_e is a subset of \mathbb{K} . Thus $\lambda_i(x) = 0$ for $x \in S$ and $i \in \mathbb{K} \setminus \mathbb{K}_e$. The state of the system $Q = (Q_1, \dots, Q_K)^T$ is given by the following

equations:

$$Q(t) = Q(0) + A(t) + B(t) - D(t), \quad (2.1)$$

$$A_i(t) = N_i \left(\int_0^t \lambda_i(Q(s)) ds \right), \quad (2.2)$$

$$B_i(t) = \sum_{j=1}^K N_{ji} \left(p_{ji} \int_0^t 1_{\{Q_j(s) > 0\}} [\mu_j(Q(s))] ds \right), \quad (2.3)$$

$$D_i(t) = \sum_{j=0}^K N_{ij} \left(p_{ij} \int_0^t 1_{\{Q_i(s) > 0\}} [\mu_i(Q(s))] ds \right), \quad (2.4)$$

where $t \geq 0, i \in \mathbb{K}, A = (A_1, \dots, A_K)^T$ and similarly for B and D . In the above display $p_{i0} \equiv 1 - \sum_{j=1}^K p_{ij}$. The random quantities in (2.1)-(2.4) have the following interpretation: $Q(0) \in S$ is an initial queue vector; $A = \{A(t), t \geq 0\}$, $B = \{B(t), t \geq 0\}$, $D = \{D(t), t \geq 0\}$ are counting processes, where the i^{th} coordinates $A_i(t), B_i(t), D_i(t)$ represent the cumulative number of exogenous and endogenous arrivals to station P_i during time interval $[0, t]$ and the cumulative number of departures from P_i during $[0, t]$, respectively. Throughout this paper, we make the following standing assumption on the network.

MAIN ASSUMPTION ON PRIMITIVES (A).

(A1) The spectral radius of \mathbb{P} is strictly less than 1 and $p_{ii} = 0$ for all $i \in \mathbb{K}$.

(A2) The random quantities $Q(0), (N_i, i \in \mathbb{K}), (N_{ij}, i \in \mathbb{K}, j \in \mathbb{K} \cup \{0\})$ are mutually independent.

(A3) For each $i \in \mathbb{K}$, $\lambda_i, \mu_i \in C_b(S)$.

Remark 2.1. *The spectral radius assumption in (A1) says that the network is open; that is, any customer entering the network eventually leaves it. In particular, it implies the existence of a regular Skorohod map associated with the network data (see Proposition 2.5 below and also [17], [12]), which is a key ingredient for many estimates in this paper. The process Q defined in (2.1) is a (strong) Markov jump process on the K -dimensional non-negative integer lattice (cf. Theorem 4.1, Chapter 6 in [13]). The sample paths of Q are S -valued functions, which are RCLL and piecewise constant. Our queueing model defined in (2.1)-(2.4) can be viewed as a state-dependent adaptation of the time-dependent network model proposed by Massey and Whitt [26].*

Next, we define the “centered” processes as follows: For $i \in \mathbb{K}, j \in \mathbb{K} \cup \{0\}$,

$$M_{0i}(t) \equiv N_i \left(\int_0^t \lambda_i(Q(s)) ds \right) - \int_0^t \lambda_i(Q(s)) ds, \quad (2.5)$$

$$\begin{aligned} M_{ij}(t) &\equiv N_{ij} \left(p_{ij} \int_0^t 1_{\{Q_i(s) > 0\}} [\mu_i(Q(s))] ds \right) \\ &\quad - p_{ij} \int_0^t 1_{\{Q_i(s) > 0\}} [\mu_i(Q(s))] ds. \end{aligned} \quad (2.6)$$

Remark 2.2. *Although one can assume that all the random quantities in (2.1)-(2.4) are given on a single probability space, there is some subtlety in defining an appropriate filtration. In general,*

a filtration, for which the processes in (2.5) and (2.6) are adapted and martingales, would depend on the corresponding state process. Lemma 2.3 below provides the mathematical framework on this matter (cf. Section 12 in [25]). Its proof is based on the multiparameter time change arguments in [13] (Section 2, Chapter 6), the optional sampling theorem, and the integration theorem in [7] (Theorem T8, page 27).

Lemma 2.3. ([25], Lemma 3.9) *Let $(\Omega, \mathcal{F}, \mathbb{P})$ denote the common complete probability space on which the random quantities involved in (2.1)-(2.4) are defined. Suppose that the assumptions (A1)-(A3) are satisfied. Then there exists a filtration $(\mathcal{F}_t, t \geq 0)$ on $(\Omega, \mathcal{F}, \mathbb{P})$, satisfying the usual conditions (cf. [19], page 10), such that (M_{0i}, M_{ij}) given by (2.5) and (2.6) are vector-valued, locally square integrable (cf. [35], page 38) $\{\mathcal{F}_t\}$ -martingales, and Q is $\{\mathcal{F}_t\}$ -adapted.*

Letting $M_i \equiv \sum_{j=0}^K M_{ji} - \sum_{j=0}^K M_{ij}$ for $i \in \mathbb{K}$ and $R \equiv [\mathbb{I} - \mathbb{P}^T]$, we can rewrite the evolution (2.1) as

$$Q_i(t) = Q_i(0) + \int_0^t \left[\lambda_i(Q(s)) + \sum_{j=1}^K p_{ji} \mu_j(Q(s)) - \mu_i(Q(s)) \right] ds + M_i(t) + [RY(t)]_i,$$

where

$$Y_i(t) = \sum_{j=0}^K N_{ij} \left(p_{ij} \int_0^t 1_{\{Q_i(s)=0\}} [\mu_i(Q(s))] ds \right), \quad i \in \mathbb{K}.$$

Note that Y_i is an RCLL, non-decreasing, $\{\mathcal{F}_t\}$ -adapted process, and Y_i increases only when $Q_i(t) = 0$, i.e., $\int_0^\infty 1_{\{Q_i(t) \neq 0\}} dY_i(t) = 0$ a.s. Set $a(\cdot) \equiv \lambda(\cdot) - R\mu(\cdot)$; then the state evolution can be expressed succinctly by the following vector equation:

$$Q(t) = Q(0) + \int_0^t a(Q(s)) ds + M(t) + RY(t), \quad t \geq 0. \quad (2.7)$$

The above dynamics can equivalently be described in terms of a Skorohod map as described below.

Definition 2.4. *Let $\psi \in D([0, \infty), \mathbb{R}^K)$ be given with $\psi(0) \in S$. Then $(\phi, \eta) \in D([0, \infty), \mathbb{R}^K) \times D([0, \infty), \mathbb{R}^K)$ solves the Skorohod problem for ψ with respect to S and R if and only if the following hold:*

- (i) $\phi(t) = \psi(t) + R\eta(t) \in S$, for all $t \geq 0$;
- (ii) η satisfies, for $i \in \mathbb{K}$, (a) $\eta_i(0) = 0$, (b) η_i is non-decreasing, and (c) η_i can increase only when ϕ is on the i^{th} face of S , that is, $\int_0^\infty 1_{\{\phi_i(s) \neq 0\}} d\eta_i(s) = 0$.

Let $D_S([0, \infty), \mathbb{R}^K) \equiv \{\psi \in D([0, \infty), \mathbb{R}^K) : \psi(0) \in S\}$. On the domain $D \subset D_S([0, \infty), \mathbb{R}^K)$ on which there is a unique solution to the Skorohod problem, we define the Skorohod map Γ as

$$\Gamma(\psi) \equiv \phi,$$

if $(\phi, R^{-1}[\phi - \psi])$ is the unique solution of the Skorohod problem posed by ψ . The following result (see [17], [12]) gives the regularity of the Skorohod map, which is a consequence of Assumption (A1).

Proposition 2.5. *The Skorohod map is well-defined on all of $D_S([0, \infty), \mathbb{R}^K)$, i.e. $D = D_S([0, \infty), \mathbb{R}^K)$, and the Skorohod map is Lipschitz continuous in the following sense: There exists a constant $L \in (1, \infty)$ such that for all $\psi_1, \psi_2 \in D_S([0, \infty), \mathbb{R}^K)$,*

$$\sup_{0 \leq t < \infty} |\Gamma(\psi_1)(t) - \Gamma(\psi_2)(t)| < L \sup_{0 \leq t < \infty} |\psi_1(t) - \psi_2(t)|.$$

The dynamics in (2.7) can now be equivalently described in terms of the Skorohod map as follows:

$$Q(t) = \Gamma \left(Q(0) + \int_0^t a(Q(s)) ds + M(\cdot) \right) (t), \quad \text{for } t \geq 0. \quad (2.8)$$

When $Q(0) \equiv x$, we will sometimes write the corresponding state process as Q_x .

Remark 2.6. *Many estimates in the current paper make use of the Lipschitz property of the Skorohod map as in Proposition 2.5. However, there is a rich collection of multiclass queueing networks for which the state dynamics are not given in terms of such a well-behaved Skorohod map. Also, in the setting of the state-dependent routing scheme considered in [25], Skorohod map need not be Lipschitz. We note that the proofs in the current paper cannot be easily extended to cover such settings.*

We now introduce our main stability condition on the queueing system that will be assumed throughout this paper:

(S) There exists a $\theta > 0$, such that $\sup_{x \in S, i \in \mathbb{K}} [R^{-1}a(x)]_i < -\theta$.

Remark 2.7. *The assumption (S) will be particularly used to stipulate the permissible drift vector field as in (3.12) below, which in turn enables us to use the function $T(\cdot)$, the hitting time to the origin (see (3.13)), for the stability analysis of the underlying queueing network model. We remark that such a condition on the drift vector field (a so called cone condition) was made in [2] to address stability properties for a class of diffusion processes with general state-dependent coefficients, constrained to take values in a convex polyhedral cone in a positive orthant. Study of such diffusions is motivated by queueing networks with state-dependent arrival and service rates as our model.*

Hereafter, explicit reference to (S) in statements of our results will be omitted.

2.1 Modes of stability

The term ‘stability’ in stochastic processes literature does not have a single definition, but rather could have different meanings depending on the context of its use. In this section, we collect standard Markov processes terminologies and definitions on a series of increasingly stronger concepts of ‘stability’ for a continuous time Markov process Φ (cf. [32], [34], [11], [6]).

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ be a filtered measurable space on which is given an X -valued stochastic process $\Phi = \{\Phi_t : t \in \mathbb{R}_+\}$ and a collection of probability measures $\{\mathbb{P}_x, x \in X\}$ such that $(\Phi, \{\mathbb{P}_x\})$ form a time-homogeneous Markov family. More precisely, under each of the measures \mathbb{P}_x , (Φ_t) is a Markov process with initial distribution δ_x and transition probability kernel $P^t(y, A) \equiv \mathbb{P}_y(\Phi_t \in A | \Phi_0 = y)$.

$A), y \in X, A \in \mathcal{B}(X)$. Frequently, when the family $\{\mathbb{P}_x\}$ is clear, we will suppress it from the notation and refer to Φ as the Markov process. Here, X is a locally compact, complete and separable metric space. We assume that Φ is a strong Markov process with RCLL paths and $\{P^t\}$ maps Borel functions to Borel functions; i.e. for all bounded Borel maps $f : X \rightarrow \mathbb{R}$, the map $x \mapsto \int_X f(y)P^t(x, dy)$ is a Borel measurable map. Let $\mathbb{E}_x[\cdot]$ denote the expectation with respect to the measure \mathbb{P}_x . For a real bounded measurable function f and a σ -finite measure ν on X , define $P^t f(x) \equiv \int P^t(x, dy)f(y)$, $\nu P^t(A) \equiv \int \nu(dx)P^t(x, A)$. For a measurable set $A \in \mathcal{B}(X)$ let $\tau_A \equiv \inf\{t \geq 0 : \Phi_t \in A\}$ and $\eta_A \equiv \int_0^\infty 1_{\{\Phi_t \in A\}} dt$, the first hitting time of A and a sojourn time in A , respectively. By convention, all measures ν on $(X, \mathcal{B}(X))$ in this section will be non-trivial (i.e., $\nu(X) \neq 0$).

Definition 2.8. (φ -irreducibility [32]) *Let φ be a σ -finite measure on $(X, \mathcal{B}(X))$. The Markov process Φ is called φ -irreducible if whenever $A \in \mathcal{B}(X)$ is such that $\varphi(A) > 0$, we have $\mathbb{E}_x[\eta_A] > 0$ for all $x \in X$. The measure φ is called an irreducibility measure for the process Φ .*

We now introduce the next stronger notion of stability, namely Harris recurrence.

Definition 2.9. (Harris recurrence [32]) *The process Φ is called Harris recurrent if for some σ -finite measure φ on $(X, \mathcal{B}(X))$, $\mathbb{P}_x(\{\eta_A = \infty\}) = 1$ whenever $\varphi(A) > 0$.*

Taking φ as the counting measure, one sees that the notion of φ -irreducibility and Harris recurrence coincide with the usual concepts of irreducibility and recurrence for countable state space Markov processes. Sometimes in order to emphasize the special choice of φ , we will say Φ is φ -Harris recurrent. It can be shown (see Theorem 2.4 in [31], [20]) that Φ is φ -Harris recurrent if and only if there exists a σ -finite measure ν on $(X, \mathcal{B}(X))$ such that $\mathbb{P}_x(\{\tau_A < \infty\}) = 1$ whenever $\nu(A) > 0$. Clearly φ -Harris recurrence implies φ -irreducibility. Furthermore, it can be shown that if Φ is Harris recurrent then it has a unique (up to a scalar multiplier) invariant measure π (cf. [15] and p. 491 in [32]). We recall that a σ -finite measure π on $(X, \mathcal{B}(X))$ is called an *invariant measure* for Φ if and only if for all $A \in \mathcal{B}(X)$, $x \in X$ and $t \geq 0$, $\pi(A) = \pi P^t(x, A)$.

The next stronger notion of stability is the finiteness of the invariant measure π . If π is finite, then it can be normalized to a probability measure. With an abuse of notation, we denote the normalized probability measure once more by π . This unique invariant probability measure π plays a central role in the study of asymptotic properties of Φ .

Definition 2.10. (Positive Harris recurrence [32]) *Suppose that Φ is Harris recurrent with a finite invariant measure π . Then Φ is called positive Harris recurrent.*

Once positive Harris recurrence is assured, then the next important and practical question of interest is the convergence and speed of convergence to the steady-state. For example, does the transition function P^t converge to invariant measure π as $t \rightarrow \infty$; if so, how fast is the convergence? Exponential rate of convergence to the steady-state is perhaps one of the most sought-after ergodic properties of a Markov process. For any signed measure ν on $\mathcal{B}(X)$ and $f \geq 1$, define its *f-norm* as $\|\nu\|_f \equiv \sup_{|g| \leq f} |\nu(g)| = \sup_{|g| \leq f} |\int \nu(dy)g(y)|$. Now we introduce the notion of *f-uniform ergodicity* from [11].

Definition 2.11. (f-uniform ergodicity [11]) *Let a Markov process Φ be positive Harris recurrent with invariant probability measure π . For a measurable function $f : X \rightarrow [1, \infty)$, Φ is called *f-uniformly ergodic* if there exist constants $D \in (0, \infty)$, $\rho \in (0, 1)$ such that for all $t \in \mathbb{R}_+$ and $x \in X$, $\|P^t(x, \cdot) - \pi\|_f \leq f(x)D\rho^t$.*

3 Some stability results

The following moment stability properties are key ingredients in the proofs.

Proposition 3.1. *Let Q_x be defined by (2.8) with $Q_x(0) \equiv x \in S$. Then there exists a $\delta \in (0, \infty)$ such that,*

$$\lim_{|x| \rightarrow \infty} \frac{1}{|x|} \mathbb{E}(|Q_x(\delta|x|)|) = 0. \quad (3.9)$$

Before we prove Proposition 3.1, we present its implication in Proposition 3.3, which gives practical criteria of positive Harris recurrence in terms of *petite* sets.

For a probability measure a on $(0, \infty)$ and a locally compact, separable metric state space $(X, \mathcal{B}(X))$, define the Markov transition function $K_a : X \times \mathcal{B}(X) \rightarrow [0, 1]$ as $K_a(x, A) \equiv \int_0^\infty P^t(x, A) a(dt)$. Note that if a is the distribution of increment for the (undelayed) renewal process $\{t_k\}$, then K_a is the transition function of the Markov chain “sampled” at time-points drawn successfully according to distribution a , that is, $\{\Phi_{t_k} : k \in \mathbb{N}\}$ from a Markov process Φ .

Definition 3.2. *A non-empty set $A \in \mathcal{B}(X)$ is said to be ν_a -petite if ν_a is a non-trivial measure on $(X, \mathcal{B}(X))$ (i.e., $\nu_a(X) > 0$) and a is a probability distribution on $(0, \infty)$ satisfying $K_a(x, \cdot) \geq \nu_a(\cdot)$ for all $x \in A$; ν_a is then called a petite measure. The set A will be called simply petite when the specific measure ν_a is unimportant.*

Petiteness of A can be interpreted as the property that each set in $\mathcal{B}(X)$ is “equally accessible” from any $x \in A$ with respect to the measure ν_a . The following result is established in [9], Theorem 3.1.

Proposition 3.3. *Let Q_x be defined by (2.8) with $Q_x(0) \equiv x \in S$. If (3.9) holds for $B = \{x \in S : |x| \leq \kappa\}$ for some $\kappa > 0$, then $\sup_{x \in B} \mathbb{E}_x[\tau_B(\delta)] < \infty$, where $\tau_B(\delta) \equiv \inf\{t \geq \delta : Q(t) \in B\}$. In particular, if B is a closed petite set, Q is positive Harris recurrent.*

Remark 3.4. *In order to satisfy the petite set requirement in Proposition 3.3, the following two conditions, in addition to the i.i.d. assumption, on the distribution of the interarrival times $(\xi(i) : i \geq 1)$ are made in [9] (also see [30]). The first is that $\xi(1)$ is unbounded, that is, $P(\xi(1) \geq t) > 0$ for all $t > 0$. The second is the spread-out condition, that is, there exists some positive function $q(\cdot)$ with $\int_0^\infty q(t)dt > 0$, such that $P(\xi(1) + \dots + \xi(\ell) \in [c, d]) \geq \int_c^d q(t)dt$ for all $c < d$. Note that even the i.i.d. assumption may be violated in our setting due to state-dependent arrival rate. Nevertheless, the lemma below justifies that $B = \{x \in S : |x| \leq \kappa\}$ is indeed a petite set in our model as well. This can be shown in a way similar to the proof of Lemma 3.7 in [30], and the proof is relegated to the Appendix.*

Lemma 3.5. *Let $B = \{x \in S : |x| \leq \kappa\}$ for some $\kappa > 0$. Then B is a closed petite set in S .*

Proof of Proposition 3.1. Fix $x \in S$. We write (2.8) as,

$$Q_x(t) \equiv \Gamma(x + r(\cdot) + M(\cdot))(t),$$

where

$$r(t) = \int_0^t a(Q(s))ds \equiv \int_0^t b(s)ds, \quad t \geq 0.$$

Define

$$Z_x(t) \equiv \Gamma(x + r(\cdot))(t), \quad t \geq 0. \quad (3.10)$$

Using the Lipschitz property of Γ (Proposition 2.5), we have

$$|Q_x(t) - Z_x(t)| \leq L \sup_{0 \leq s \leq t} |M(s)|, \quad \text{for all } t \geq 0. \quad (3.11)$$

Next, denoting by $\mathcal{C} \equiv \{v \in \mathbb{R}^K : R^{-1}v \leq 0\}$ we see from condition (S) that there exists a $\beta \in (0, \infty)$ satisfying

$$\inf_{s \geq 0} \text{dist}(b(s), \partial\mathcal{C}) \geq \beta. \quad (3.12)$$

Thus for $s \geq 0$, $b(s) \in \mathcal{C}_\beta \equiv \{v \in \mathcal{C} : \text{dist}(v, \partial\mathcal{C}) \geq \beta\}$. For $z_0 \in S$ denote by $\mathcal{K}(z_0)$ the collection of all trajectories $\psi : [0, \infty) \rightarrow S$ of the form

$$\psi(t) \equiv \Gamma\left(z_0 + \int_0^t \varpi(s) ds\right)(t), \quad t \geq 0,$$

where $\varpi : [0, \infty) \rightarrow \mathbb{R}^K$ is a measurable map satisfying

$$\text{for all } t \in [0, \infty), \quad \int_0^t |\varpi(s)| ds < \infty, \quad \varpi(t) \in \mathcal{C}_\beta.$$

Define the ‘‘hitting time to the origin’’ function as follows,

$$T(z_0) \equiv \sup_{\psi \in \mathcal{K}(z_0)} \inf\{t \in [0, \infty) : \psi(t) = 0\}. \quad (3.13)$$

Lemma 3.1 of [2] shows that

$$T(z_0) \leq \frac{4L^2}{\beta} |z_0|, \quad \text{and for all } \psi \in \mathcal{K}(z_0), \quad \psi(t) = 0 \text{ for all } t \geq T(z_0). \quad (3.14)$$

Combining this observation with (3.12), we now have that $Z_x(t) = 0$, for all $t \geq \delta_0|x|$, where $\delta_0 \equiv \frac{4L^2}{\beta}$. Using this in (3.11) we now see that

$$|Q_x(t|x)| \leq L \sup_{0 \leq s \leq t|x|} |M(s)|, \quad (3.15)$$

for all $t \geq \delta_0$ and for all initial conditions x . Next we obtain an estimate on the first moment of the right side of (3.15). Since M_i is an $\{\mathcal{F}_t\}$ square integrable martingale, one gets using Doob’s inequality that

$$\begin{aligned} \mathbb{E} \sup_{0 \leq s \leq t} |M_i(s)| &\leq (\mathbb{E} \sup_{0 \leq s \leq t} |M_i(s)|^2)^{1/2} \leq (4\mathbb{E}|M_i(t)|^2)^{1/2} \\ &\leq c_1 \left(\mathbb{E} \left[\int_0^t [\lambda_i(Q(s)) + \sum_{j=1}^K \mu_j(Q(s))] ds \right] \right)^{1/2} \leq c_2(1+t)^{1/2}, \end{aligned} \quad (3.16)$$

for some $c_1, c_2 \in (0, \infty)$ and the last inequality follows from boundedness of $\lambda(\cdot)$ and $\mu(\cdot)$. Applying this estimate in (3.15), we now have that for all $t \geq \delta_0$ and $x \in S$, $\mathbb{E}|Q_x(t|x)| \leq c_3(1+t|x|)^{1/2}$ for some $c_3 \in (0, \infty)$. The result now follows on choosing $\delta = \delta_0$. \blacksquare

By Propositions 3.1 and 3.3 we establish the positive Harris recurrence of Q .

Theorem 3.6. *Let Q_x be defined by (2.8) with $Q_x(0) \equiv x \in S$. Then Q is positive Harris recurrent.*

Henceforth, we denote the unique invariant probability measure of Q by π .

4 V -uniform ergodicity of Q

We will now study V -uniform ergodic properties of Q . Fluid models and fluid limit are useful for stability analysis for underlying queueing systems (cf. [9], [8], [10], [5], [6] among others). Roughly speaking we will show (a) the *fluid* scaled queue length process converges *uniformly* (in a sufficiently strong sense as in [27] assumption (U1)) to a fluid limit model and (b) this fluid limit model is *stable*, i.e., the origin of the coordinates is the absorbing point and the fluid limit model eventually reaches the origin. (See Definition 4.1 and Theorem 5.1 in [9].) Consider a sequence of queueing networks indexed by $n = 1, 2, \dots$, each of which is specified by (2.1)-(2.4) and satisfies Main Assumption (A) in Section 2. A superscript n will be used to indicate the corresponding quantity related to the n^{th} network. Introduce the fluid scaled processes $q^n = \{q^n(t) : t \geq 0\}$, $n = 1, 2, \dots$, by $q^n(t) = Q^n(t)/n$. Many works that study fluid approximations for queueing systems consider scaled processes where time is scaled up by a factor of n and space is scaled down by the same factor. For the scaling considered here, which is adapted from [24], the time parameter is left unchanged and the factor of n is absorbed in the arrival and service rates. We make the following standing assumption on the primitives λ^n , μ^n , and $q^n(0)$.

MAIN ASSUMPTION ON THE PRIMITIVES FOR FLUID MODELS (B).

- (B1) For $\xi \in S$, $\frac{1}{n}\lambda^n(n\xi) \rightarrow \lambda(\xi)$, $\frac{1}{n}\mu^n(n\xi) \rightarrow \mu(\xi)$ uniformly on compact sets as $n \rightarrow \infty$, where λ, μ are given vector-valued locally Lipschitz functions.
- (B2) For $\xi \in S$, $\max(|\lambda^n(n\xi)|_\infty, |\mu^n(n\xi)|_\infty) \leq nL_1(1 + |\xi|)$, $n = 1, 2, \dots$, where $|a|_\infty \equiv \max_k |a_k|$ and L_1 is a given positive constant.
- (B3) As $n \rightarrow \infty$, $q^n(0) \xrightarrow{P} q(0)$, where $q(0)$ is a given deterministic vector and the sequence $\{\mathbb{E}|q^n(0)|\}$ is bounded uniformly in n .

The following theorem describes the asymptotic behavior of $\{q^n\}$ (cf. Theorem 4.6 of [24]).

Theorem 4.1. *The fluid scaled queue length process $\{q^n\}$ converges uniformly on compact sets over $[0, \infty)$ in probability, as $n \rightarrow \infty$, to a deterministic, absolutely continuous function q . This q is the unique solution to the following differential equation:*

$$q(t) = q(0) + \int_0^t a(q(s))ds + Ry(t) \geq 0, \quad t \geq 0, \quad (4.17)$$

$$y \text{ is non-decreasing in each coordinate, } y(0) = 0, \quad (4.18)$$

$$\int_0^\infty e^T [q(t) > 0] dy(t) = 0, \text{ where } e_i = 1 \text{ for each } i \in \mathbb{K}. \quad (4.19)$$

Hereafter, q will be referred to as the *fluid limit* model, which can be also expressed as

$$\frac{d}{dt}q(t) = a(q(t)) + Ry(t).$$

We now present the main result of this work.

Theorem 4.2. *The Markov process Q is V -uniformly ergodic; i.e., there exist constants $D \in (0, \infty)$, $\rho \in (0, 1)$ such that for all $t \in \mathbb{R}_+$ and $x \in S$, $\|P^t(x, \cdot) - \pi\|_V \leq V(x)D\rho^t$.*

Proof. We start with constructing a Lyapunov function following the perturbation technique in [29] (see also Sections 4.9 and 8.4 in [28]). Define $V_0 : \mathbb{R}^K \rightarrow \mathbb{R}$ as $V_0(x) \equiv e^T R^{-1}x$. Then whenever $q \in S^\circ$,

$$\frac{d}{dt}V_0(q(t)) = e^T R^{-1}a(q(t)) \leq -K\theta,$$

where $\theta \in (0, \infty)$ is as in (S). To deal with the boundaries of S , consider the following perturbation of V_0 through a change of variables: For fixed $\gamma \geq 1$, we denote $\tilde{x}_i \equiv x_i + \gamma(e^{-x_i/\gamma} - 1)$ for $i \in \mathbb{K}$ and $x \in \mathbb{R}^K$, and let \tilde{x} denote the corresponding vector $\tilde{x} \equiv (\tilde{x}_1, \dots, \tilde{x}_K)^T \in S$. Set $V_1(x) \equiv V_0(\tilde{x})$ for $x \in S$. We then have by the chain rule

$$\frac{d}{dt}V_1(q(t)) = \sum_{i,j \in \mathbb{K}} R_{ij}^{-1}([a(q(t))]_j + [Ry(t)]_j)(1 - e^{-q_j(t)/\gamma}).$$

From (4.19) we get $y_j(t)(1 - e^{-q_j(t)/\gamma}) \equiv 0$ and hence

$$\frac{d}{dt}V_1(q(t)) = \sum_{i,j \in \mathbb{K}} R_{ij}^{-1}[a(q(t))]_j(1 - e^{-q_j(t)/\gamma}) \leq \sum_{i,j \in \mathbb{K}} R_{ij}^{-1}[a(q(t))]_j.$$

It follows from (S) that

$$\frac{d}{dt}V_1(q(t)) \leq -K\theta, \tag{4.20}$$

which establishes condition (ii) of Theorem 4 in [27]. (Condition (i) of the cited theorem is fulfilled by Assumption (B) and Theorem 4.1 above.) Given the uniform convergence of $\{q^n\}$ and the inequality (4.20) for the limit, V_1 also acts as a Lyapunov function for the network. Applying Theorem 4 in [27] (see also Proposition A.5.7 in [28] and Theorem 16.3.1 in [34]), it follows that the process is V -uniformly ergodic with $V \equiv V_\epsilon = e^{\epsilon V_1}$, for sufficiently small $\epsilon > 0$. ■

Theorem 4.2 asserts that expected value of unbounded (and possibly exponentially growing) functionals of the state process converges to the expectation under the invariant measure at an exponential rate. In addition to the very strong total variation norm convergence that V -uniformly ergodic processes satisfy by definition, a strong form of the large deviations principle and mixing result may be obtained for these stochastic processes (cf. [3], [21], [22], Section 16.1.2 of [34]). Furthermore, one can also establish a functional central limit result for $S_t \equiv \int_{[0,t)} F(Q(s))ds$, as $t \rightarrow \infty$, for a broad family of measurable functions $F : S \rightarrow \mathbb{R}$ (cf. [16]). The next result is a Strassen-type functional law of the iterated logarithm, which is lesser known in the literature. It follows directly from Theorem 2.7 in [4] and Theorem 6 in [39].

Theorem 4.3. *Let $F : S \rightarrow \mathbb{R}$ be a measurable function such that $F^2(x) \leq V(x)$ for all $x \in S$ and suppose $Q(0)$ is distributed according to π . Then the sequence of random functions*

$$\left\{ \frac{1}{\sqrt{2n \log \log n}} \int_0^{nt} F(Q_s) ds : 0 \leq t \leq 1 \right\}, \quad n = 2, 3, \dots,$$

is relatively compact in $C[0, 1]$ and the set of limit points is the set of all absolutely continuous functions ζ on $[0, 1]$ satisfying $\zeta(0) = 0$, $\int_0^1 \zeta'(t)^2 dt \leq \sigma_F^2$ for some constant $\sigma_F \in [0, \infty)$.

Remark 4.4. *The above conclusion holds for every initial condition x , since we have the V -norm convergence of the transition kernel $P^t(x, \cdot)$ to π (cf. Theorem 2.7 (b) in [4]).*

Appendix

Proof of Lemma 3.5. We begin by observing that any non-empty, measurable subset of a petite set is also petite. Let $[\kappa]$ denote the integer part of $\kappa > 0$. To prove $B = \{x \in S : |x| \leq \kappa\}$ is petite, it suffices to show that for arbitrary $\kappa \in (0, \infty)$ the set $G_\kappa \equiv \{0, 1, \dots, [\kappa] + 1\}^K$ is petite. This can be shown in a way similar to the proof of Lemma 3.7 in [30], where the authors made the unboundedness and spread-out assumption (cf. Remark 3.4) in addition to the i.i.d. condition on the interarrival times distribution. In our setting, the i.i.d. assumption is invalid due to the state-dependent arrival rate. Nevertheless, by recalling Assumption (A3) we can think of the non-homogeneous process $\{A(t), t \geq 0\}$ in (2.2) as being a random sample from a homogeneous Poisson process (see, e.g., Section 2.4 in [36]). Specifically, let $\bar{\lambda} \in (0, \infty)$ be such that

$$K \sup_{x \in S, i \in K} \lambda_i(x) \leq \bar{\lambda}$$

and consider a Poisson process with rate $\bar{\lambda}$. Now if we suppose that an event of the Poisson process that occurs at time t is counted with probability $\lambda_i(Q(t))/\bar{\lambda}$, then the process of counted events is a non-homogeneous Poisson process $A_i(t) \equiv N_i \left(\int_0^t \lambda_i(Q(s)) ds \right)$. The following argument is adapted from the proof of Lemma 3.7 in [30]. For arbitrary $\kappa \in (0, \infty)$, we estimate, for large $n \in \mathbb{N}$, the probability $P^n(x, \{0\})$ for $x \in G_\kappa$ as follows. Let v_r denote the r^{th} interarrival time to the network. For large n we have,

$$\begin{aligned} P^n(x, \{0\}) &\geq \mathbb{P}_x \left\{ Q_n = 0, v_1 + \dots + v_{k_0} \leq \frac{n}{2}, v_{k_0+1} \geq 2n \right\} \\ &\geq \mathbb{P} \left\{ E_{L, [\kappa]+1}, v_1 + \dots + v_{k_0} \leq \frac{n}{2}, v_{k_0+1} \geq 2n \right\}, \end{aligned}$$

where $E_{L, [\kappa]+1}$ denotes the event that no customer is routed to the same queue twice and that each of the first $K(K([\kappa] + 1) + k_0)$ services takes no more than L units of time and L is chosen large enough so that $\epsilon_0 \equiv \mathbb{P}(E_{L, [\kappa]+1}) > 0$. We have from Assumption (A2) that

$$\begin{aligned} P^n(x, \{0\}) &\geq \epsilon_0 \mathbb{P} \left\{ v_1 + \dots + v_{k_0} \leq \frac{n}{2}, v_{k_0+1} \geq 2n \right\} \\ &= \epsilon_0 \sum_{M_2=1}^{\infty} \sum_{M_1=k_0}^{\infty} \mathbb{P} \left\{ \sum_{i=1}^{\eta_{k_0}} X_i \leq \frac{n}{2}, \sum_{i=1}^{\eta_{k_0+1}} Y_i \geq 2n \mid \eta_{k_0} = M_1, \eta_{k_0+1} = M_2 \right\} \\ &\quad \times \mathbb{P} \{ \eta_{k_0} = M_1, \eta_{k_0+1} = M_2 \}, \end{aligned}$$

where η_{k_0}, η_{k_0+1} are non-negative, integer-valued random variables counting total number of arrivals based on a Poisson process with rate $\bar{\lambda}$ and $\{X_i\}_{i \geq 1}, \{Y_i\}_{i \geq 1}$ denote independent exponential random variables with parameter $\bar{\lambda}$. Then one gets

$$P^n(x, \{0\}) \geq \epsilon_0 \sum_{M_2=1}^{\infty} \sum_{M_1=k_0}^{\infty} \int_0^{\infty} \int_0^{\infty} 1(s \leq \frac{n}{2}) 1(r \geq 2n) p_{M_1}(ds) q_{M_2}(dr) \\ \times \mathbb{P} \{ \eta_{k_0} = M_1, \eta_{k_0+1} = M_2 \},$$

where $p_{M_1}(\cdot), q_{M_2}(\cdot)$ denote probability density functions of exponential random variables with parameters $M_1 \bar{\lambda}$ and $M_2 \bar{\lambda}$, respectively. Define

$$\bar{\nu}_{[\kappa]}(\{0\}) \equiv \epsilon_0 \sum_{M_2=1}^{\infty} \sum_{M_1=k_0}^{\infty} \int_0^{\infty} \int_0^{\infty} 1(s \leq \frac{n}{2}) 1(r \geq 2n) p_{M_1}(ds) q_{M_2}(dr) \\ \times \mathbb{P} \{ \eta_{k_0} = M_1, \eta_{k_0+1} = M_2 \}.$$

By construction we have, for any set $G \in \mathcal{B}(S)$ and $x \in G_{\kappa}$, that

$$P^n(x, G) \geq \delta_0(G) \bar{\nu}_{[\kappa]}(\{0\}),$$

where δ_0 is the unit mass concentrated on $0 \in S$. For $\kappa \in (0, \infty)$ and $G \in \mathcal{B}(S)$, define $\nu_{[\kappa]}(x, G) \equiv 1\{x \in G_{\kappa}\} \delta_0(G) \bar{\nu}_{[\kappa]}(\{0\})$, then we see that the measure $\nu_{[\kappa]}(x, \cdot)$ is non-trivial for $x \in G_{\kappa}$. For $l \in \mathbb{N}$, let n_l denote an integer time at which P^{n_l} admits a non-trivial measure ν_l on $G_{l-1} \equiv \{0, 1, \dots, l\}^K$. Finally, we define the distribution a on $(0, \infty)$ as $a \equiv \sum_{l=1}^{\infty} 2^{-l} \delta_{n_l}$ and the measure ν_a as $\nu_a \equiv \sum_{l=1}^{\infty} 2^{-l} \nu_l$. Then ν_a is a non-trivial measure on $(S, \mathcal{B}(S))$ and a is a probability distribution satisfying $K_a(x, \cdot) \geq \nu_a(\cdot)$ for all $x \in G_{\kappa}$. This completes the proof. ■

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