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V-uniform ergodicity for state-dependent single class queueing networks

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Abstract We consider single class queueing networks with state-dependent arrival and service rates. Under the uniform (in state) stability condition, it is shown that the queue length process is V-uniformly ergodic; that is, it has a transition probability kernel which converges to its limit geometrically quickly in V-norm sense. Among several asymptotic properties of V-uniformly ergodic processes, we present a Strassen-type functional law of the iterated logarithm result.

Keywords State-dependent networks · V-uniform ergodicity · functional law of the iterated logarithm

Mathematics Subject Classification (2000) 60J65 · 60H35 · 60K25 · 93E15

1 Introduction

We consider single class queueing networks in which arrival and service rates depend on the state (i.e., queue length) of the network. The network consists of K service stations, each of which has an associated infinite capacity buffer. Arrivals of jobs can be from outside the system and/or from internal routing. Upon completion of service at a station, the customer is routed to one of the other service stations (or exits the system) according to a probabilistic routing matrix. We study state-dependent arrival and service processes that would arise if the arrival and work processes were allowed to depend on the current queue levels. Such signature of queueing systems is common in applications to communications and computer networks. For example, congested queue levels would cause balking (i.e., customers refusing to join a queue) or reneging (i.e., customers abandoning the queue after waiting a long time). It is natural to model such state-dependent behavior directly by birth-death processes. In particular, we will consider the state-dependent form of Poisson-type input and service processes (see (1)-(4) below). Many such models can be found in applications to communications

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and computer networks (see, for example, [25], [41], [40], [24], [23]). The queueing network model studied in this work is similar to those in [25] and [41]. These papers are concerned with transient analysis of networks via fluid and/or diffusion limit approximations. We, on the other hand, are interested in steady-state behavior and rate of convergence to invariant distribution of a state-dependent network model.

Steady-state analysis of state-dependent queueing networks was initiated by the seminal work of Jackson [18]. (See also [37].) The stability properties of (generalized) Jackson type networks with constant arrival and service rates have been studied (see, for instance, [38], [10], [9], [30]). In particular, in the case where the interarrival times possess geometrically decaying tails, it is shown in [30] that the state process is geometrically ergodic. Also, in [33] the standard Foster-Lyapunov approach was used to establish state-dependent drift conditions that are sufficient for geometric ergodicity of Markov chains. The discrete-time Markovian model (without the state-dependent feature) is considered in [27] and Theorem 4 therein shows that *uniform* convergence of the scaled queue length process to a *stable* fluid limit implies geometric ergodicity of the queue length process. (See also Chapters 8 and 10 in [28].) Results of [14] suggest that this uniform convergence assumption cannot be weakened by providing an example in which the interarrival/service times have exponential distributions, and the corresponding fluid limit is stable, and yet the queue length process has heavy tails in steady-state and is not exponentially ergodic (cf. Theorem 2.3 in [14]). The main contribution of this paper lies in establishing the V-uniform ergodicity (see Definition 5) and related longtime asymptotic properties for continuous time state-dependent queueing network model.

The primary result of this paper is concerned with a rate of convergence to the steady-state distribution. Under the uniform (in state) stability condition, it is shown that the queue length process is V-uniformly ergodic; that is, it has a transition probability kernel which converges to its limit geometrically quickly in V-norm sense. Knowledge of this rate of convergence is useful in several practical situations. For example, it provides insights for (i) performance of a stochastic simulation for underlying queueing systems; (ii) how accurately one is approximating the system state, at a given time instant t, by simulation schemes (cf. [1]); (iii) how long one should run the simulation experiment to yield satisfactory approximations. As consequences of V-uniform ergodicity, one can obtain several meaningful asymptotic properties of the process such as strong form of large deviations principle and mixing results, functional central limit theorem and Strassen-type functional law of the iterated logarithm results (cf. [22], [16], [4], and Chapter 16 in [34]).

Our proofs rely on critical use of Lyapunov function methods developed in [34] and [11] (see also Chapters 8 and 10 in [28]). For the stability analysis of the underlying queueing network model, we use "hitting time to the origin" function of related deterministic dynamical systems (see (13)) so that the growth estimate of the first moment for queue length is obtained in terms of its initial condition. Once positive (Harris) recurrence is assured, main efforts are in proving Theorem 3, for which related fluid limit analysis is conducted and a perturbed Lyapunov function technique is used (cf. [29], [28]) to show uniform convergence of the stable fluid limit model to zero.

The remainder of the paper is organized as follows. We begin, in Section 2, by describing the queueing network model with state-dependency, and provide a description of the dynamics of the queue length process in terms of a Skorohod map. A basic uniform (in state) stability assumption on this queueing system is then introduced (see condition (S) below). In Section 2.1 we collect some standard concepts and terminologies on stochastic stability for Markov processes. Section 3 presents moment stability results that lead to the positive Harris recurrence of the queue length process Q and Section 4 is devoted to the study of V-uniform ergodicity of this state process. In Theorem 4 we present a Strassen-type law of the iterated logarithm result and finally in the Appendix we provide a proof of Lemma 2.

The following notation is used. For a metric space X, let $\mathcal{B}(X)$ denote the Borel σ-field on X. For a real valued measurable function f on X and a measure ν on $\mathcal{B}(X)$, let $\nu(f) \doteq \int_X f d\nu$. The Dirac measure at the point x is denoted by δ_x. Let $\mathbb{N}$ denote the set of natural numbers, $\mathbb{N}_0$ denote $\mathbb{N} \cup \{0\}$, and $\mathbb{N}_0^d = \{x = (x_1, \ldots, x_d) : x_i \in \mathbb{N}_0, i = 1, \ldots, d\}$. Denote the set of real numbers by $\mathbb{R}$ and non-negative real numbers by $\mathbb{R}_+$. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space with the norm of $u \in \mathbb{R}^d$, $|u| = \sum_{k=1}^d |u_k|$. For a given matrix M denote by M^T its transpose and by M_i the i^{th} row of M. Let $\mathbb{I} = \mathbb{I}_{K\times K}$ denote the identity matrix for some K. When clear from the context, we will omit the subscript. For a set $A \subseteq \mathbb{R}^d$, denote its interior and boundary by $A^\circ, \partial A$, respectively. For sets $A, B \subseteq \mathbb{R}^d$, $dist(A, B)$ will denote the distance between two sets, i.e., $\inf\{|x - y| : x \in A, y \in B\}$. When $\sup_{0\le s\le t} |f_n(s) - f(s)| \to 0$ as $n \to \infty$, for all $t \ge 0$, we say that $f_n \to f$ uniformly on compact sets. The class of continuous functions $f : X \to Y$ is denoted by $C(X, Y)$ and real continuous bounded functions on X by $C_b(X)$. Finally, let $D(X, Y)$ denote the class of right continuous functions with having left limit defined from X to Y, equipped with the usual Skorohod topology. A stochastic process will be denoted interchangeably by $\{Z(t)\}$ as well as $\{Z_t\}$. Inequalities for vectors are interpreted componentwise. We will denote generic constants by $c_1, c_2, \ldots$, and their values may change from one proof to another.

2 The queueing network model

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Unless specified otherwise, all the random variables considered in this work are assumed to be defined on this probability space. We consider a network with K service stations, denoting the i^{th} station by P_i, $i \in \mathbb{K} \doteq \{1, \ldots, K\}$. We assume that each station has an infinite capacity buffer. Arrivals of jobs can be from outside the system and/or from internal routing. Routing is assumed to be Bernoulli, so that upon completion of service at station P_i a customer is routed to other service station P_j with probability p_{ij} (or exits the system with probability $1 - \sum_{\ell=1}^K p_{i\ell}$) according to a probabilistic routing matrix $\mathbb{P} = (p_{ij})_{i,j\in\mathbb{K}}$. (See [25] for state-dependent routing.) At every station the jobs are assumed to be processed by First-In-First-Out discipline. We allow arrival and service rates to be time varying random processes having state-dependent feature.

A precise description of the model is as follows. We assume that $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space on which are given unit rate, right continuous with left-limits (RCLL) Poisson processes, N_i, N_{ij}, $i \in \mathbb{K}, j \in \mathbb{K} \cup \{0\}$. For $i \in \mathbb{K}$, N_i will be used to define the stream of jobs entering the i^{th} buffer and for $i, j \in \mathbb{K}$, N_{ij} will be used to represent the flow of jobs to buffer j from buffer i. For $i \in \mathbb{K}$ and $j = 0$, N_{ij} will be associated with jobs that leave the system after service at station P_i. Throughout this paper S will be used to denote the K-dimensional non-negative integer lattice $\mathbb{N}_0^K$. Let λ_i, μ_i, $i \in \mathbb{K}$ be measurable functions from $S \to \mathbb{R}_+$ and write $\lambda = (\lambda_1, \ldots, \lambda_K)^T$; μ is defined similarly. These functions will be used to define state-dependent arrival and service rates. External arrivals are assumed to occur any for $i \in \mathbb{K}_e$, where $\mathbb{K}_e$ is a subset of $I\!K$. Thus $\lambda_i(x) = 0$ for $x \in S$ and $i \in I\!K \backslash I\!K_e$. The state of the system $Q = (Q_1, \ldots, Q_K)^T$ is given by the following equations:

$$Q(t) = Q(0) + A(t) + B(t) - D(t), \tag{1}$$

$$A_i(t) = N_i\left(\int_0^t \lambda_i(Q(s))ds\right), \tag{2}$$

$$B_i(t) = \sum_{j=1}^{K} N_{ji}\left(p_{ji}\int_0^t 1_{\{Q_j(s)>0\}}[\mu_j(Q(s))]ds\right), \tag{3}$$

$$D_i(t) = \sum_{j=0}^{K} N_{ij}\left(p_{ij}\int_0^t 1_{\{Q_i(s)>0\}}[\mu_i(Q(s))]ds\right), \tag{4}$$

where $t \geq 0, i \in I\!K, A = (A_1, \ldots, A_K)^T$ and similarly for B and D. In the above display $p_{i0} \doteq 1 - \sum_{j=1}^{K} p_{ij}$. The random quantities in (1)-(4) have the following interpretation: $Q(0) \in S$ is an initial queue vector; $A = \{A(t), t \geq 0\}$, $B = \{B(t), t \geq 0\}$, $D = \{D(t), t \geq 0\}$ are counting processes, where the i^{th} coordinates $A_i(t), B_i(t), D_i(t)$ represent the cumulative number of exogenous and endogenous arrivals to station P_i during time interval $[0, t]$ and the cumulative number of departures from P_i during $[0, t]$, respectively. Throughout this paper, we make the following standing assumptions on the network.

Main Assumptions on Primitives (A).

(A1) The spectral radius of $\mathbb{P}$ is strictly less than 1 and $p_{ii} = 0$ for all $i \in I\!K$.
(A2) The random quantities $Q(0)$, $(N_i, i \in I\!K)$, $(N_{ij}, i \in I\!K, j \in I\!K \cup \{0\})$ are mutually independent.
(A3) For each $i \in I\!K$, $\lambda_i, \mu_i \in C_b(S)$.

Remark 1 The spectral radius assumption in (A1) says that the network is open, that is, any customer entering the network eventually leaves it. It in particular implies ([17], [12]) the existence of a regular Skorohod map associated with the network data (see Proposition 1), which is a key ingredient for many estimates in this paper. The process Q defined in (1) is a (strong) Markov jump process on the K-dimensional non-negative integer lattice (cf. Theorem 4.1, Chapter 6 in [13]). The sample paths of Q are S-valued functions, which are RCLL and piecewise constant. Our queueing model defined in (1)-(4) can be viewed as a state-dependent adaptation of time-dependent network model proposed by Massey and Whitt [26].

Next, we define the "centered" processes as follows: For $i \in I\!K, j \in I\!K \cup \{0\}$,

$$M_{0i}(t) \doteq N_i\left(\int_0^t \lambda_i(Q(s))ds\right) - \int_0^t \lambda_i(Q(s))ds, \tag{5}$$

$$\begin{aligned} M_{ij}(t) \doteq\ & N_{ij}\left(p_{ij}\int_0^t 1_{\{Q_i(s)>0\}}[\mu_i(Q(s))]ds\right) \\ & -p_{ij}\int_0^t 1_{\{Q_i(s)>0\}}[\mu_i(Q(s))]ds. \end{aligned} \tag{6}$$

Remark 2 Although one can assume that all the random quantities in (1)-(4) are given on a single probability space, there is some subtlety in defining an appropriate filtration. In general, a filtration, for which the processes in (5), (6) are adapted and martingales, would depend on the corresponding state process. Lemma 1 below provides the mathematical framework on this matter (cf. Section 12 in [25]). Its proof is based on multiparameter time change arguments in [13] (Section 2, Chapter 6), optional sampling theorem, and the integration theorem in [7] (Theorem T8, page 27).

Lemma 1 ([25], Lemma 3.9) *Let $(\Omega, \mathcal{F}, I\!P)$ denote the common complete probability space on which the random quantities involved in* (1)-(4) *are defined. Suppose that the assumptions (A1)-(A3) are satisfied. Then there exists a filtration $(\mathcal{F}_t, t \geq 0)$ on $(\Omega, \mathcal{F}, I\!P)$, satisfying the usual conditions (cf. [19], page 10), such that (M_{0i}, M_{ij}) given by* (5) *and* (6) *are vector-valued locally square integrable (cf. [35], page 38) $\{\mathcal{F}_t\}$-martingales and Q is $\{\mathcal{F}_t\}$-adapted.*

Letting $M_i \doteq \sum_{j=0}^{K} M_{ji} - \sum_{j=0}^{K} M_{ij}$ for $i \in I\!K$ and $R \doteq [\mathbb{I} - \mathbb{P}^T]$, we can rewrite the evolution (1) as

$$Q_i(t) = Q_i(0) + \int_0^t \left[\lambda_i(Q(s)) + \sum_{j=1}^{K} p_{ji}\mu_j(Q(s)) - \mu_i(Q(s)) \right] ds + M_i(t) + [RY(t)]_i,$$

where

$$Y_i(t) = \sum_{j=0}^{K} N_{ij} \left(p_{ij} \int_0^t 1_{\{Q_i(s)=0\}}[\mu_i(Q(s))]ds \right), \quad i \in I\!K.$$

Note that Y_i is an RCLL non-decreasing $\{\mathcal{F}_t\}$ adapted process and Y_i increases only when $Q_i(t) = 0$, i.e., $\int_0^\infty 1_{\{Q_i(t) \neq 0\}} dY_i(t) = 0$ a.s. Set $a(\cdot) \doteq \lambda(\cdot) - R\mu(\cdot)$ then the state evolution can be described succinctly by the following vector equation:

$$Q(t) = Q(0) + \int_0^t a(Q(s))ds + M(t) + RY(t), \quad t \geq 0. \tag{7}$$

The above dynamics can equivalently be described in terms of a Skorohod map as described below.

Definition 1 Let $\psi \in D([0,\infty), I\!R^K)$ be given with $\psi(0) \in S$. Then $(\phi, \eta) \in D([0,\infty), I\!R^K) \times D([0,\infty), I\!R^K)$ solves the Skorohod problem for ψ with respect to S and R if and only if the following hold:

(i) $\phi(t) = \psi(t) + R\eta(t) \in S$, for all $t \geq 0$;
(ii) η satisfies, for $i \in I\!K$, (a) $\eta_i(0) = 0$, (b) η_i is non-decreasing, and (c) η_i can increase only when ϕ is on the i^{th} face of S, that is, $\int_0^\infty 1_{\{\phi_i(s) \neq 0\}} d\eta_i(s) = 0$.

Let $D_S([0,\infty), I\!R^K) \doteq \{\psi \in D([0,\infty), I\!R^K) : \psi(0) \in S\}$. On the domain $D \subset D_S([0,\infty), I\!R^K)$ on which there is a unique solution to the Skorohod problem we define the Skorohod map Γ as $\Gamma(\psi) \doteq \phi$, if $(\phi, R^{-1}[\phi - \psi])$ is the unique solution of the Skorohod problem posed by ψ. The following result (see [17], [12]) gives the regularity of the Skorohod map, which is a consequence of Assumption (A1).

Proposition 1 *The Skorohod map is well defined on all of $D_S([0,\infty), I\!R^K)$, i.e. $D = D_S([0,\infty), I\!R^K)$, and the Skorohod map is Lipschitz continuous in the following sense: There exists a constant $L \in (1,\infty)$ such that for all $\psi_1, \psi_2 \in D_S([0,\infty), I\!R^K)$,*

$$\sup_{0 \leq t < \infty} |\Gamma(\psi_1)(t) - \Gamma(\psi_2)(t)| < L \sup_{0 \leq t < \infty} |\psi_1(t) - \psi_2(t)|.$$

The dynamics in (7) can now be equivalently described in terms of the Skorohod map as follows:

$$Q(t) = \Gamma\left(Q(0) + \int_0^{\cdot} a(Q(s))ds + M(\cdot)\right)(t), \quad \text{for } t \geq 0. \tag{8}$$

When $Q(0) \equiv x$, we will sometimes write the corresponding state process as Q_x.

Remark 3 Many estimates in the current paper make use of the Lipschitz property of the Skorohod map as in Proposition 1. However, there is a rich collection of *multiclass* queueing networks for which the state dynamics are not given in terms of such a well behaved Skorohod map. Also, in the setting of state-dependent routing scheme considered in [25], Skorohod map needs not be Lipschitz. We note that proofs in the current paper cannot be easily extended to cover such settings.

We now introduce our main stability condition on the queueing system that will be assumed throughout this paper:

(S) There exists a $\theta > 0$, such that $\sup_{x \in S, i \in I\!K} [R^{-1}a(x)]_i < -\theta$.

Remark 4 The assumption (S) will be particularly used to stipulate the permissible drift vector field as in (12) below, which in turn enables us to use the function $T(\cdot)$, the hitting time to the origin (see (13)), for the stability analysis of the underlying queueing network model. We remark that such condition on the drift vector field (so called cone condition) was made in [2] to address stability properties for a class of diffusion processes with general state-dependent coefficients, constrained to take values in a convex polyhedral cone in a positive orthant. Study of such diffusions is motivated by queueing networks with state-dependent arrival and service rates as in our model.

Hereafter, explicit reference to (S) in statement of our results will be omitted.

2.1 Modes of stability

The term 'stability' in stochastic processes literature does not have a single definition, but rather could have different meanings depending on the context of use. In this section, we collect standard Markov processes terminologies and definitions on a series of increasingly stronger concepts of 'stability' for a continuous time Markov process $\mathbf{\Phi}$ (cf. [32], [34], [11], [6]).

Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\})$ be a filtered measurable space on which is given an X-valued stochastic process $\mathbf{\Phi} = \{\Phi_t : t \in I\!R_+\}$ and a collection of probability measures $\{I\!P_x, x \in X\}$ such that $(\mathbf{\Phi}, \{I\!P_x\})$ form a time homogeneous Markov family. More precisely, under each of the measures $I\!P_x$, (Φ_t) is a Markov process with initial distribution δ_x and transition probability kernel $P^t(y, A) \doteq I\!P_y(\Phi_t \in A), y \in X, A \in \mathcal{B}(X)$. Frequently, when the family $\{I\!P_x\}$ is clear, we will suppress it from the notation and refer to $\mathbf{\Phi}$ as the Markov process. Here X is a locally compact, complete and separable metric space. We assume that $\mathbf{\Phi}$ is a strong Markov process with RCLL paths and $\{P^t\}$ maps Borel functions to Borel functions. I.e. for all bounded Borel maps $f : X \to I\!R$, the map $x \mapsto \int_X f(y)P^t(x, dy)$ is a Borel measurable map. Let $I\!E_x[\,\cdot\,]$ denote the expectation with respect to the measure $I\!P_x$. For a real bounded measurable function f and a σ-finite measure ν on X, define $P^t f(x) \doteq \int P^t(x, dy)f(y)$, $\nu P^t(A) \doteq \int \nu(dx)P^t(x, A)$. For a measurable set $A \in \mathcal{B}(X)$ let $\tau_A \doteq \inf\{t \geq 0 : \Phi_t \in A\}$ and $\eta_A \doteq \int_0^\infty 1_{\{\Phi_t \in A\}} dt$, the first hitting time of A and a sojourn time in A, respectively. By convention, all measures ν on $(X, \mathcal{B}(X))$ in this section will be non-trivial (i.e., $\nu(X) \neq 0$).

Definition 2 **(φ-irreducibility [32])** Let φ be a σ-finite measure on $(X, \mathcal{B}(X))$. The Markov process $\mathbf{\Phi}$ is called φ-irreducible if whenever $A \in \mathcal{B}(X)$ is such that $\varphi(A) > 0$, we have $\mathbb{E}_x[\eta_A] > 0, \quad \forall x \in X$. The measure φ is called an irreducibility measure for the process $\mathbf{\Phi}$.

We now introduce the next stronger notion of stability, namely, Harris recurrence.

Definition 3 **(Harris recurrence [32])** The process $\mathbf{\Phi}$ is called Harris recurrent if for some σ-finite measure φ on $(X, \mathcal{B}(X))$, $\mathbb{P}_x(\{\eta_A = \infty\}) = 1$ whenever $\varphi(A) > 0$.

Taking φ as the counting measure one sees that the notion of φ-irreducibility and Harris recurrence coincide with the usual concepts of irreducibility and recurrence for countable state space Markov processes. Sometimes in order to emphasize the special choice of φ, we will say $\mathbf{\Phi}$ is φ-Harris recurrent. It can be shown (see Theorem 2.4 in [31], [20]), that $\mathbf{\Phi}$ is φ-Harris recurrent if and only if there exists a σ-finite measure ν on $(X, \mathcal{B}(X))$ such that $\mathbb{P}_x(\{\tau_A < \infty\}) = 1$ whenever $\nu(A) > 0$. Clearly φ-Harris recurrence implies φ-irreducibility. Furthermore, it can be shown that if $\mathbf{\Phi}$ is Harris recurrent then it has a unique (up to a scalar multiplier) invariant measure π (cf. [15] and p. 491 in [32]). We recall that a σ-finite measure π on $(X, \mathcal{B}(X))$ is called an *invariant measure* for $\mathbf{\Phi}$ if and only if for all $A \in \mathcal{B}(X)$, $x \in X$ and $t \geq 0$, $\pi(A) = \pi P^t(x, A)$.

The next stronger notion of stability is the finiteness of the invariant measure π. If π is finite then it can be normalized to a probability measure and with an abuse of notation we denote the normalized probability measure once more by π. This unique invariant probability measure π plays a central role in the study of asymptotic properties of $\mathbf{\Phi}$.

Definition 4 **(Positive Harris recurrence [32])** Suppose that $\mathbf{\Phi}$ is Harris recurrent with a finite invariant measure π. Then $\mathbf{\Phi}$ is called positive Harris recurrent.

Once positive Harris recurrence is assured, then the next important and practical question of interest is the convergence and speed of convergence to the steady-state. For example, does the transition function P^t converge to invariant measure π as $t \to \infty$, if so, how fast is the convergence? Exponential rate of convergence to the steady-state is perhaps one of the most sought after ergodic properties of a Markov process. For any signed measure ν on $\mathcal{B}(X)$ and $f \geq 1$, define its *f-norm* as $||\nu||_f \doteq \sup_{|g| \leq f} |\nu(g)| = \sup_{|g| \leq f} |\int \nu(dy) g(y)|$. Now we introduce the notion of f-uniform ergodicity from [11].

Definition 5 **(f-uniform ergodicity [11])** Let a Markov process $\mathbf{\Phi}$ be positive Harris recurrent with invariant probability measure π. For a measurable function $f : X \to [1, \infty)$, $\mathbf{\Phi}$ is called f-uniformly ergodic if there exist constants $D \in (0, \infty)$, $\rho \in (0, 1)$ such that for all $t \in \mathbb{R}_+$ and $x \in X$, $||P^t(x, \cdot) - \pi||_f \leq f(x) D \rho^t$.

3 Some stability results

The following moment stability properties are key ingredients in the proofs.

Proposition 2 *Let Q_x be defined by* (8) *with $Q_x(0) \equiv x \in S$. Then there exists a $\delta \in (0, \infty)$ such that,*

$$\lim_{|x| \to \infty} \frac{1}{|x|} \mathbb{E}\left(|Q_x(\delta |x|)|\right) = 0. \tag{9}$$

Before we prove Proposition 2, we present its implication in Proposition 3, which gives practical criteria of positive Harris recurrence in terms of *petite* sets.

For a probability measure a on $(0, \infty)$ and a locally compact, separable metric state space $(X, \mathcal{B}(X))$, define the Markov transition function $K_a : X \times \mathcal{B}(X) \to [0, 1]$ as $K_a(x, A) \doteq \int_0^\infty P^t(x, A)a(dt)$. Note that if a is the distribution of increment for the (undelayed) renewal process $\{t_k\}$, then K_a is the transition function of the Markov chain "sampled" at time-points drawn successfully according to distribution a, that is, $\{\Phi_{t_k} : k \in \mathbb{N}\}$ from a Markov process $\mathbf{\Phi}$.

Definition 6 A non-empty set $A \in \mathcal{B}(X)$ is said to be ν_a*-petite* if ν_a is a non-trivial measure on $(X, \mathcal{B}(X))$ (i.e., $\nu_a(X) > 0$) and a is a probability distribution on $(0, \infty)$ satisfying $K_a(x, \cdot) \geq \nu_a(\cdot)$ for all $x \in A$; ν_a is then called a *petite measure*. The set A will be called simply *petite* when the specific measure ν_a is unimportant.

Petiteness of A can be interpreted as the property that each set in $\mathcal{B}(X)$ is "equally accessible" from any $x \in A$ with respect to the measure ν_a. The following result is established in [9], Theorem 3.1.

Proposition 3 *Let Q_x be defined by* (8) *with $Q_x(0) \equiv x \in S$. If* (9) *holds for $B = \{x \in S : |x| \leq \kappa\}$ for some $\kappa > 0$, then $\sup_{x \in B} \mathbb{E}_x[\tau_B(\delta)] < \infty$, where $\tau_B(\delta) = \inf\{t \geq \delta : Q(t) \in B\}$. In particular, if B is a closed petite set, Q is positive Harris recurrent.*

Remark 5 In order to satisfy petite set requirement, the following two conditions, in addition to i.i.d assumptions, on the distribution of the interarrival times $(\xi(i) : i \geq 1)$ are made in [9] (also see [30]). The first is that $\xi(1)$ is unbounded, that is, $P(\xi(1) \geq t) > 0$ for all $t > 0$. The second is the spread-out condition, that is, there exists some positive function $q(\cdot)$ with $\int_0^\infty q(t)dt > 0$, such that $P(\xi(1) + \cdots + \xi(\ell) \in [c, d]) \geq \int_c^d q(t)dt$ for all $c < d$. Note that even i.i.d. assumption may be violated in our setting due to state-dependent arrival rate. Nevertheless, the lemma below justifies that $B = \{x \in S : |x| \leq \kappa\}$ is indeed a petite set in our model as well. This can be shown in a way similar to the proof of Lemma 3.7 of [30] and the proof is relegated to the Appendix.

Lemma 2 *Let $B = \{x \in S : |x| \leq \kappa\}$ for some $\kappa > 0$. Then B is a closed petite set in S.*

Proof (***Proof of Proposition 2***) Fix $x \in S$. We write (8) as,

$$Q_x(t) \equiv \Gamma\left(x + r(\cdot) + M(\cdot)\right)(t),$$

where

$$r(t) = \int_0^t a(Q(s))ds \equiv \int_0^t b(s)ds, \quad t \geq 0.$$

Define

$$Z_x(t) \doteq \Gamma(x + r(\cdot))(t), \quad t \geq 0. \tag{10}$$

Using the Lipschitz property of Γ (Proposition 1), we have

$$|Q_x(t) - Z_x(t)| \leq L \sup_{0 \leq s \leq t} |M(s)|, \text{ for all } t \geq 0. \tag{11}$$

Next, denoting by $\mathcal{C} \doteq \{v \in \mathbb{R}^K : R^{-1}v \leq 0\}$ we see from condition (S) that there exists a $\beta \in (0, \infty)$ satisfying

$$\inf_{s \geq 0} \operatorname{dist}(b(s), \partial\mathcal{C}) \geq \beta. \tag{12}$$

Thus for $s \geq 0$, $b(s) \in \mathcal{C}_\beta \doteq \{v \in \mathcal{C} : \mathrm{dist}(v, \partial\mathcal{C}) \geq \beta\}$. For $z_0 \in S$ denote by $\mathcal{K}(z_0)$ the collection of all trajectories $\psi : [0,\infty) \to S$ of the form

$$\psi(t) \doteq \Gamma\left(z_0 + \int_0^{\cdot} \varpi(s)ds\right)(t), \quad t \geq 0,$$

where $\varpi : [0,\infty) \to I\!R^K$ is a measurable map satisfying

$$\text{for all } t \in [0,\infty), \quad \int_0^t |\varpi(s)|ds < \infty, \quad \varpi(t) \in \mathcal{C}_\beta.$$

Define the "hitting time to the origin" function as follows,

$$T(z_0) \doteq \sup_{\psi \in \mathcal{K}(z_0)} \inf\{t \in [0,\infty) : \psi(t) = 0\}. \tag{13}$$

Lemma 3.1 of [2] shows that

$$T(z_0) \leq \frac{4L^2}{\beta}|z_0|, \text{ and for all } \psi \in \mathcal{K}(z_0),\ \psi(t) = 0 \text{ for all } t \geq T(z_0). \tag{14}$$

Combining this observation with (12) we now have that $Z_x(t) = 0$, for all $t \geq \delta_0|x|$, where $\delta_0 \doteq \frac{4L^2}{\beta}$. Using this in (11) we now see that

$$|Q_x(t|x|)| \leq L \sup_{0 \leq s \leq t|x|} |M(s)|, \tag{15}$$

for all $t \geq \delta_0$ and for all initial conditions x. Next we obtain an estimate on the first moment of the right side of (15). Since M_i is an $\{\mathcal{F}_t\}$ square integrable martingale, one gets using Doob's inequality that

$$\begin{aligned} I\!E \sup_{0 \leq s \leq t} |M_i(s)| &\leq (I\!E \sup_{0 \leq s \leq t} |M_i(s)|^2)^{1/2} \leq (4I\!E|M_i(t)|^2)^{1/2} \\ &\leq c_1 \left(I\!E[\int_0^t [\lambda_i(Q(s)) + \sum_{j=1}^K \mu_j(Q(s))ds]\right)^{1/2} \leq c_2(1+t)^{1/2}, \end{aligned} \tag{16}$$

for some $c_1, c_2 \in (0,\infty)$ and the last inequality follows from boundedness of $\lambda(\cdot)$ and $\mu(\cdot)$. Applying this estimate in (15) we now have that for all $t \geq \delta_0$ and $x \in S$, $I\!E|Q_x(t|x|)| \leq c_3(1+t|x|)^{1/2}$ for some $c_3 \in (0,\infty)$. The result now follows on choosing $\delta = \delta_0$. □

By Propositions 2 and 3 we establish the positive Harris recurrence of Q.

Theorem 1 *Let Q_x be defined by (8) with $Q_x(0) \equiv x \in S$. Then Q is positive Harris recurrent.*

Henceforth, we denote the unique invariant probability measure of Q by π.

4 V-uniform ergodicity of Q

We will now study V-uniform ergodic properties of Q. Fluid models and fluid limit are useful for stability analysis for underlying queueing systems (cf. [9], [8], [10], [5], [6] among others). Roughly speaking we will show (a) the *fluid* scaled queue length process converges *uniformly* (in a sufficiently strong sense as in [27] assumption (U1)) to a fluid limit model and (b) this fluid limit model is *stable*, i.e., the origin of the coordinates is the absorbing point and fluid limit model eventually reaches the origin. (See Definition 4.1 and Theorem 5.1 in [9].) Consider a sequence of queueing networks indexed by $n = 1, 2, \ldots$, each of which is specified by (1)-(4) and satisfies Main Assumptions (A) in Section 2. A superscript n will be used to indicate the corresponding quantity related to the n^{th} network. Introduce the fluid scaled processes $q^n = \{q^n(t) : t \geq 0\}, n = 1, 2, \ldots$, by $q^n(t) = Q^n(t)/n$. Many works that study fluid approximations for queueing systems consider scaled processes where time is scaled up by a factor of n and space is scaled down by the same factor. The scaling considered here, which is adapted from [24], the time parameter is left unchanged and the factor of n is absorbed in the arrival and service rates. We make the following standing assumptions on the primitives λ^n, μ^n, and $q^n(0)$.

Main Assumptions on Primitives for Fluid Models (B).

(B1) For $\xi \in S$, $\frac{1}{n}\lambda^n(n\xi) \to \lambda(\xi)$, $\frac{1}{n}\mu^n(n\xi) \to \mu(\xi)$ uniformly on compact sets as $n \to \infty$, where λ, μ are given vector-valued locally Lipschitz functions.

(B2) For $\xi \in S$, $\max(|\lambda^n(n\xi)|_\infty, |\mu^n(n\xi)|_\infty) \leq nL_1(1 + |\xi|), n = 1, 2, \ldots$, where $|a|_\infty = \max_k |a_k|$ and L_1 is a given positive constant.

(B3) As $n \to \infty$, $q^n(0) \xrightarrow{P} q(0)$, where $q(0)$ is a given deterministic vector and the sequence $\{I\!\!E|q^n(0)|\}$ is bounded uniformly in n.

The following theorem describes the asymptotic behavior of $\{q^n\}$ (cf. Theorem 4.6 of [24]).

Theorem 2 *The fluid scaled queue length process $\{q^n\}$ converges uniformly on compact sets over $[0, \infty)$ in probability, as $n \to \infty$, to a deterministic absolutely continuous function q. This q is the unique solution to the following differential equation:*

$$q(t) = q(0) + \int_0^t a(q(s))ds + Ry(t) \geq 0, \quad t \geq 0, \tag{17}$$

$$y \text{ is non-decreasing in each coordinate, } y(0) = 0, \tag{18}$$

$$\int_0^\infty e^T[q(t) > 0]dy(t) = 0, \text{ where } e_i = 1 \text{ for each } i \in I\!\!K. \tag{19}$$

Hereafter, q will be referred to as the *fluid limit* model, which can be also expressed as

$$\frac{d}{dt}q(t) = a(q(t)) + Ry(t).$$

We now present the main result of this work.

Theorem 3 *The Markov process Q is V-uniformly ergodic; i.e., there exist constants $D \in (0, \infty)$, $\rho \in (0, 1)$ such that for all $t \in I\!\!R_+$ and $x \in S$, $||P^t(x, \cdot) - \pi||_V \leq V(x)D\rho^t$.*

Proof We start with constructing a Lyapunov function following the perturbation technique in [29] (see also Sections 4.9 and 8.4 in [28]). Define $V_0 : I\!\!R^K \to I\!\!R$

as $V_0(x) \doteq e^T R^{-1} x$, then whenever $q \in S^\circ$, $\frac{d}{dt}V_0(q(t)) = e^T R^{-1} a(q(t)) \leq -K\theta$, where $\theta \in (0, \infty)$ is as in (S). To deal with the boundaries of S, consider the following perturbation of V_0 through a change of variables: For fixed $\gamma \geq 1$, we denote $\tilde{x}_i \doteq x_i + \gamma(e^{-x_i/\gamma} - 1)$ for $i \in I\!K$ and $x \in I\!R^K$, and let $\tilde{x}$ denote the corresponding vector $\tilde{x} \doteq (\tilde{x}_1, \ldots, \tilde{x}_K)^T \in S$. Set $V_1(x) \doteq V_0(\tilde{x})$ for $x \in S$. We then have by chain rule

$$\frac{d}{dt}V_1(q(t)) = \sum_{i,j \in I\!K} R^{-1}_{ij}([a(q(t))]_j + [Ry(t)]_j)(1 - e^{-q_j(t)/\gamma}).$$

From (19) we get $y_j(t)(1 - e^{-q_j(t)/\gamma}) \equiv 0$ and hence

$$\frac{d}{dt}V_1(q(t)) = \sum_{i,j \in I\!K} R^{-1}_{ij}[a(q(t))]_j(1 - e^{-q_j(t)/\gamma}) \leq \sum_{i,j \in I\!K} R^{-1}_{ij}[a(q(t))]_j.$$

It follows from (S) that

$$\frac{d}{dt}V_1(q(t)) \leq -K\theta, \tag{20}$$

which establishes condition (ii) of Theorem 4 in [27]. (Condition (i) of cited theorem is fulfilled by Assumptions (B) and Theorem 2 above.) Given the uniform convergence of $\{q^n\}$ and the inequality (20) for the limit, V_1 also acts as a Lyapunov function for the network. Applying Theorem 4 of [27] (see also Proposition A.5.7 of [28], Theorem 16.3.1 of [34]) it follows that the process is V-uniformly ergodic with $V \equiv V_\epsilon = e^{\epsilon V_1}$, for sufficiently small $\epsilon > 0$. $\square$

Theorem 3 asserts that expected value of unbounded (and possibly exponentially growing) functionals of the state process converges to the expectation under the invariant measure, at an exponential rate. In addition to the very strong total variation norm convergence that V-uniformly ergodic processes satisfy by definition, a strong form of large deviations principle and mixing result may be obtained for these stochastic processes (cf. [3], [21], [22], Section 16.1.2 of [34]). Furthermore, one can also establish functional central limit result for $S_t \doteq \int_{[0,t)} F(Q(s))ds$, as $t \to \infty$, for a broad family of measurable functions $F : S \to I\!R$ (cf. [16]). The next result is a Strassen-type functional law of the iterated logarithm, which is lesser known in the literature. It follows directly from Theorem 2.7 in [4] and Theorem 6 in [39].

Theorem 4 *Let $F : S \to I\!R$ be a measurable function such that $F^2(x) \leq V(x)$ for all $x \in S$ and suppose $Q(0)$ is distributed according to π. Then the sequence of random functions*

$$\left\{ \frac{1}{\sqrt{2n \log \log n}} \int_0^{nt} F(Q_s)ds : 0 \leq t \leq 1 \right\}, \quad n = 2, 3, \ldots,$$

is relatively compact in $C[0,1]$ and the set of limit points is the set of all absolutely continuous functions ζ on $[0,1]$ satisfying $\int_0^1 \zeta'(t)^2 dt \leq \sigma_F^2$, $\zeta(0) = 0$.

Remark 6 The above conclusion holds for every initial condition x, since we have the V-norm convergence of the transition kernel $P^t(x, \cdot)$ to π (cf. Theorem 2.7 (b) in [4]).

Appendix

Proof of Lemma 2. We start by observing that any non-empty measurable subset of a petite set is also petite. Let $[\kappa]$ denote the integer part of $\kappa > 0$. To prove $B = \{x \in S : |x| \leq \kappa\}$ is petite, it suffices to show that for arbitrary $\kappa \in (0,\infty)$ the set $G_\kappa \doteq \{0,1,\ldots,[\kappa]+1\}^K$ is petite. This can be shown in a way similar to the proof of Lemma 3.7 of [30], where the authors made unboundedness and spread-out assumptions (cf. Remark 5) in addition to the i.i.d. condition on the interarrival times distribution. In our setting, i.i.d. assumption is invalid due to state-dependent arrival rate. Nevertheless, by recalling Assumption (A3) we can think of the non-homogeneous process $\{A(t), t \geq 0\}$ in (2) as being a random sample from a homogeneous Poisson process (see, e.g., Section 2.4 in [36]). Specifically, let $\bar{\lambda} \in (0,\infty)$ be such that $K \sup_{x\in S, i\in I\!\!K} \lambda_i(x) \leq \bar{\lambda}$ and consider a Poisson process with rate $\bar{\lambda}$. Now if we suppose that an event of the Poisson process that occurs at time t is counted with probability $\lambda_i(Q(t))/\bar{\lambda}$, then the process of counted events is a non-homogeneous Poisson process $A_i(t) \doteq N_i\left(\int_0^t \lambda_i(Q(s))ds\right)$. The following argument is adapted from the proof of Lemma 3.7 in [30]. For arbitrary $\kappa \in (0,\infty)$, we estimate, for large $n \in I\!\!N$, the probability $P^n(x,\{0\})$ for $x \in G_\kappa$ as follows. Let v_r denote the r^{th} interarrival time to the network. For large n we have,

$$\begin{aligned} P^n(x,\{0\}) &\geq I\!\!P_x\left\{Q_n = 0, v_1 + \cdots + v_{k_0} \leq \frac{n}{2}, v_{k_0+1} \geq 2n\right\} \\ &\geq I\!\!P\left\{E_{L,[\kappa]+1}, v_1 + \cdots + v_{k_0} \leq \frac{n}{2}, v_{k_0+1} \geq 2n\right\}, \end{aligned}$$

where $E_{L,[\kappa]+1}$ denotes the event that no customer is routed to the same queue twice and that each of the first $K(K([\kappa]+1)+k_0)$ services take no more than L units of time and L is chosen large enough so that $\epsilon_0 \doteq I\!\!P(E_{L,[\kappa]+1}) > 0$. We have from Assumption (A2) that

$$\begin{aligned} P^n(x,\{0\}) &\geq \epsilon_0 I\!\!P\left\{v_1 + \cdots + v_{k_0} \leq \frac{n}{2}, v_{k_0+1} \geq 2n\right\} \\ &= \epsilon_0 \sum_{M_2=1}^{\infty}\sum_{M_1=k_0}^{\infty} I\!\!P\left\{\sum_{i=1}^{\eta_{k_0}} X_i \leq \frac{n}{2}, \sum_{i=1}^{\eta_{k_0+1}} Y_i \geq 2n \middle| \eta_{k_0} = M_1, \eta_{k_0+1} = M_2\right\} \\ &\quad \times I\!\!P\left\{\eta_{k_0} = M_1, \eta_{k_0+1} = M_2\right\}, \end{aligned}$$

where $\eta_{k_0}, \eta_{k_{0+1}}$ are non-negative integer-valued random variables counting total number of arrivals based on Poisson process with rate $\bar{\lambda}$ and $\{X_i\}_{i\geq 1}, \{Y_i\}_{i\geq 1}$ denote independent exponential random variables with parameter $\bar{\lambda}$. Then one gets

$$\begin{aligned} P^n(x,\{0\}) &\geq \epsilon_0 \sum_{M_2=1}^{\infty}\sum_{M_1=k_0}^{\infty} \int_0^\infty\int_0^\infty 1(s \leq \frac{n}{2})1(r \geq 2n) p_{M_1}(ds) q_{M_2}(dr) \\ &\quad \times I\!\!P\left\{\eta_{k_0} = M_1, \eta_{k_0+1} = M_2\right\}, \end{aligned}$$

where $p_{M_1}(\cdot), q_{M_2}(\cdot)$ denote probability density functions of exponential random variables with parameters $M_1\bar{\lambda}$ and $M_2\bar{\lambda}$, respectively. Define

$$\begin{aligned} \bar{\nu}_{[\kappa]}(\{0\}) \doteq\ & \epsilon_0 \sum_{M_2=1}^{\infty}\sum_{M_1=k_0}^{\infty} \int_0^\infty\int_0^\infty 1(s \leq \frac{n}{2})1(r \geq 2n) p_{M_1}(ds) q_{M_2}(dr) \\ & \times I\!\!P\left\{\eta_{k_0} = M_1, \eta_{k_0+1} = M_2\right\}. \end{aligned}$$

By construction we have for any set $G \in \mathcal{B}(S)$ and $x \in G_\kappa$ that

$$P^n(x, G) \geq \delta_0(G)\bar{\nu}_{[\kappa]}(\{0\}),$$

where δ_0 is the unit mass concentrated on $0 \in S$. For $\kappa \in (0, \infty)$ and $G \in \mathcal{B}(S)$, define $\nu_{[\kappa]}(x, G) \doteq 1\{x \in G_\kappa\}\delta_0(G)\bar{\nu}_{[\kappa]}(\{0\})$, then we see that the measure $\nu_{[\kappa]}(x, \cdot)$ is non-trivial for $x \in G_\kappa$. For $l \in I\!N$, let n_l denote an integer time at which P^{n_l} admits a non-trivial measure ν_l on $G_{l-1} \doteq \{0, 1, \ldots, l\}^K$. Finally, we define the distribution a on $(0, \infty)$ as $a \doteq \sum_{l=1}^{\infty} 2^{-l}\delta_{n_l}$ and the measure ν_a as $\nu_a \doteq \sum_{l=1}^{\infty} 2^{-l}\nu_l$. Then ν_a is a non-trivial measure on $(S, \mathcal{B}(S))$ and a is a probability distribution satisfying $K_a(x, \cdot) \geq \nu_a(\cdot)$ for all $x \in G_\kappa$. This completes the proof. $\square$

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References

1. S. Asmussen and P. W. Glynn. *Stochastic simulation: algorithms and analysis*, volume 57 of *Stochastic Modelling and Applied Probability*. Springer, New York, 2007.
2. R. Atar, A. Budhiraja, and P. Dupuis. On positive recurrence of constrained diffusion processes. *Ann. Probab.*, 29(2):979–1000, 2001.
3. S. Balaji and S. P. Meyn. Multiplicative ergodicity and large deviations for an irreducible Markov chain. *Stochastic Process. Appl.*, 90(1):123–144, 2000.
4. R. N. Bhattacharya. On the functional central limit theorem and the law of the iterated logarithm for Markov processes. *Z. Wahrsch. Verw. Gebiete*, 60(2):185–201, 1982.
5. M. Bramson. Stability of two families of queueing networks and a discussion of fluid limits. *Queueing Systems Theory Appl.*, 28(1-3):7–31, 1998.
6. M. Bramson. *Stability of queueing networks*, volume 1950 of *Lecture Notes in Mathematics*. Springer, Berlin, 2008. Lectures from the 36th Probability Summer School held in Saint-Flour, July 2–15, 2006.
7. P. Brémaud. *Point processes and queues*. Springer-Verlag, New York, 1981. Martingale dynamics, Springer Series in Statistics.
8. H. Chen. Fluid approximations and stability of multiclass queueing networks: work-conserving disciplines. *Ann. Appl. Probab.*, 5(3):637–665, 1995.
9. J. G. Dai. On positive Harris recurrence of queueing networks: A unified approach via fluid limit models. *Ann. Appl. Probab.*, 5:49–77, 1995.
10. J. G. Dai and S. P. Meyn. Stability and convergence of moments for multiclass queueing networks via fluid limit models. *IEEE Transactions on Automatic Control*, 40:1889–1904, 1995.
11. D. Down, S. P. Meyn, and R. L. Tweedie. Exponential and uniform ergodicity of Markov processes. *Ann. Probab.*, 23:1671–1691, 1996.
12. P. Dupuis and H. Ishii. On Lipschitz continuity of the solution mapping to the Skorohod problem, with applications. *Stochastics*, 35:31–62, 1991.
13. S. N. Ethier and T. G. Kurtz. *Markov processes: Characterization and Convergence*. Wiley, New York, 1986.
14. D. Gamarnik and S. Meyn. On exponential ergodicity of multiclass queueing networks. *Asymptotic Analysis of Stochastic Systems, invited session at the INFORMS Annual Meeting, November 13-16, 2005*, Available at http://arXiv.org/abs/math/0612544.
15. R. K. Getoor. Transience and recurrence of Markov processes. In *Seminar on Probability, XIV (Paris, 1978/1979) (French)*, volume 784 of *Lecture Notes in Math.*, pages 397–409. Springer, Berlin, 1980.
16. P. W. Glynn and S. P. Meyn. A Lyapunov bound for solutions of the Poisson equation. *Ann. Appl. Probab.*, 24(2):916–931, 1996.
17. J. M. Harrison and M. I. Reiman. Reflected Brownian motion on an orthant. *Ann. Probab.*, 9(2):302–308, 1981.

18. J. R. Jackson. Jobshop-like queueing systems. *Management Sci.*, 10(1):131–142, 1963.
19. I. Karatzas and S. E. Shreve. *Brownian Motion and Stochastic Calculus, Second Edition.* Graduate Texts in Mathematics 113. Springer, New York, 1991.
20. H. Kaspi and A. Mandelbaum. On Harris recurrence in continuous time. *Math. Oper. Res.*, 19(1):211–222, 1994.
21. I. Kontoyiannis and S. P. Meyn. Spectral theory and limit theorems for geometrically ergodic Markov processes. *Ann. Appl. Probab.*, 13(1):304–362, 2003.
22. I. Kontoyiannis and S. P. Meyn. Large deviations asymptotics and the spectral theory of multiplicatively regular Markov processes. *Electron. J. Probab.*, 10:no. 3, 61–123 (electronic), 2005.
23. H. J. Kushner and L. F. Martins. Heavy traffic analysis of a data transmission system with many independent sources. *SIAM J. Appl. Math.*, 53(4):1095–1122, 1993.
24. A. Mandelbaum and G. Pats. State-dependent queues: approximations and applications. In *Stochastic networks*, volume 71 of *IMA Vol. Math. Appl.*, pages 239–282. Springer, New York, 1995.
25. A. Mandelbaum and G. Pats. State-dependent stochastic networks. I. Approximations and applications with continuous diffusion limits. *Ann. Appl. Probab.*, 8(2):569–646, 1998.
26. W. A. Massey and W. Whitt. Networks of infinite-server queues with nonstationary Poisson input. *Queueing Systems Theory Appl.*, 13(1-3):183–250, 1993.
27. S. P. Meyn. Sequencing and routing in multiclass queueing networks. I. Feedback regulation. *SIAM J. Control Optim.*, 40(3):741–776 (electronic), 2001.
28. S. P. Meyn. *Control techniques for complex networks.* Cambridge University Press, Cambridge, 2008.
29. S. P. Meyn. Stability and asymptotic optimality of generalized maxweight policies. *SIAM J. Control Optim.*, 47(6):3259–3294, 2009.
30. S. P. Meyn and D. Down. Stability of generalized Jackson networks. *Ann. Appl. Probab.*, 4(1):124–148, 1994.
31. S. P. Meyn and R. L. Tweedie. Generalized resolvents and Harris recurrence of Markov processes. *Proceedings of the Doeblin Conference held November 2-7, at the University of Tubingen's Heinrich Fabri Institut*, 1991.
32. S. P. Meyn and R. L. Tweedie. Stability of Markovian processes II: Continous-time processes and sampled chains. *Adv. in Appl. Probab.*, 25:497–517, 1993.
33. S. P. Meyn and R. L. Tweedie. State-dependent criteria for convergence of Markov chains. *Ann. Appl. Probab.*, 4(1):149–168, 1994.
34. S. P. Meyn and R. L. Tweedie. *Markov chains and stochastic stability.* Cambridge University Press, Cambridge, second edition, 2009. With a prologue by Peter W. Glynn.
35. P. E. Protter. *Stochastic Integration and Differential Equations*, volume 21 of *Applications of Mathematics (New York).* Springer-Verlag, second edition, 2004.
36. S. M. Ross. *Stochastic processes.* Wiley Series in Probability and Statistics: Probability and Statistics. John Wiley & Sons Inc., New York, second edition, 1996.
37. R. F. Serfozo. Queueing networks with dependent nodes and concurrent movements. *Queueing Systems Theory Appl.*, 13(1-3):143–182, 1993.
38. K. Sigman. The stability of open queueing networks. *Stochastic Process. Appl.*, 35(1):11–25, 1990.
39. A. Touati. Loi fonctionnelle du logarithme itéré pour les processus de Markov récurrents. *Ann. Probab.*, 18(1):140–159, 1990.
40. W. Whitt. Queues with service times and interarrival times depending linearly and randomly upon waiting times. *Queueing Systems Theory Appl.*, 6(4):335–351, 1990.
41. K. Yamada. Diffusion approximation for open state-dependent queueing networks in the heavy traffic situation. *Ann. Appl. Probab.*, 5(4):958–982, 1995.