Long time asymptotics for constrained diffusions in polyhedral domains

Amarjit Budhiraja and Chihoon Lee
Department of Statistics and Operations Research
University of North Carolina
Chapel Hill, NC 27599-3260, USA

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Abstract

We study long time asymptotic properties of constrained diffusions that arise in the heavy traffic analysis of multiclass queueing networks. We first consider the classical diffusion model with constant coefficients namely a semimartingale reflecting Brownian motion (SRBM), in a d-dimensional positive orthant. Under a natural stability condition on a related deterministic dynamical system [9] showed that an SRBM is ergodic. We strengthen this result by establishing geometric ergodicity for the process. As consequences of geometric ergodicity we obtain finiteness of the moment generating function of the invariant measure in a neighborhood of zero, uniform time estimates on polynomial moments of all orders, of the process, and functional central limit results. Similar long time properties are obtained for a broad family of constrained diffusion models with state dependent coefficients under a natural condition on the drift vector field. Such models arise from heavy traffic analysis of queueing networks with state dependent arrival and service rates.

Keywords. Semimartingale reflecting Brownian motion, φ-irreducibility, V-uniform ergodicity, geometric ergodicity, Constrained diffusions, Heavy traffic, Poisson equation, Moment stability, Functional central limit theorems.

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1 Introduction

Stochastic networks is an active area of research in applied probability with diverse applications arising from computer, telecommunications, and complex manufacturing systems. The study of stability of such network models is of great importance. Excepting special cases, the networks of interest are too complex to be analyzed directly and thus one seeks tractable approximate models. In this respect, constrained diffusion processes which arise as scaling limits of critically loaded queueing networks are key. In this work, we study long time asymptotic properties of constrained diffusions that arise in the heavy traffic analysis of multiclass queueing networks.

We begin our study with semimartingale reflecting Brownian motions (SRBMs) in a positive orthant $\mathbb{R}^d_+$, $d \in \mathbb{N}$. Such Markov processes commonly arise in the heavy traffic analysis of multiclass open queueing networks and have been extensively studied [13, 12, 19, 20, 21, 9, 5, 22]. The study of stability properties of an SRBM is an important problem in the analysis of stochastic networks. The main paper on stability properties of an SRBM is [9], where it is established that if the "fluid trajectories" associated with the SRBM are attracted to the origin then the SRBM is positive recurrent and admits a unique invariant probability measure. The goal of the current work is to study the rate of convergence of the transition probability kernel to the invariant distribution and other refined long term asymptotics for an SRBM. Under exactly the same conditions of [9], we identify a suitable exponentially growing Lyapunov function V and establish that the process is V-uniformly ergodic. This result is then used to prove that the unique invariant measure of the SRBM admits a finite moment generating function in a neighborhood of zero. As other consequences we establish uniform (in time and initial condition in a compact set) estimates on exponential moments of an SRBM. Growth estimates of polynomial moments of the process as a function of the initial condition are obtained. Finally we establish a functional central limit theorem for functionals of an SRBM and characterize the asymptotic variance in this limit result via the solution of the related Poisson equation.

We next consider a family of diffusion models with state dependent coefficients, constrained to take values in some convex polyhedral cone in $\mathbb{R}^d$ with the vertex at the origin. Positive recurrence for such constrained diffusions under suitable conditions on the drift coefficient was established in [1]. In this work we strengthen this result by establishing V-uniform ergodicity for a function V that grows exponentially. As consequences of this result we establish, as in the constant coefficients case, exponential moment bounds, moment stability results and functional central limit theorems.

Our proofs make critical use of Lyapunov function methods developed in [16, 6]. At the heart of the proofs for the SRBM is Theorem 4.7 which obtains suitable bounds on exponential moments of hitting times of compact sets. Once these estimates are available, the results of [6](cf. Theorem 4.4) yield a Lyapunov function V for which the inequality (4.14) holds and as a consequence the process is V-uniformly ergodic. Lemma 4.8 establishes that V has exponential upper and lower bounds. From these estimates one immediately obtains finiteness of exponential moments of invariant measure (Theorem 4.11) and convergence of expected value of unbounded (exponentially growing) functionals of the state process to the expectation under the invariant measure, at an exponential rate. Furthermore, these estimates are key in proving stability results (Corollary 4.14 and Theorem 4.15) for polynomial moments of the process. Finally we obtain, as a consequence of results in [11], functional central limit theorems for processes $\xi_n(t) \doteq \frac{1}{\sqrt{n}} \left(\int_0^{nt} [F(Z_s) - \pi(F)] ds \right)$, where Z is the underlying Markov process, π the unique invariant measure and F is allowed to have exponential growth. In the state dependent case (see Section 5) although one can prove similar bounds on exponential moments of hitting times as in the constant coefficients case, we are unable to establish an exponential lower bound (4.10) as in Lemma 4.8. The main obstacle to such a result is that, in Section 5, the drift vector field of the underlying diffusion is allowed to have linear growth, and as a result estimate (4.13) which critically uses the boundedness of the drift coefficients fails. In view of this difficulty we proceed by making a different choice of a Lyapunov function V

(cf. Lemma 5.9) that, from results in [1], is known to have exponential upper and lower bounds. We show that this Lyapunov function satisfies the multiplicative drift condition (5.6) for a sampled Markov chain. The results of [16] can then be brought to bear to establish V-uniform ergodicity and as a consequence one obtains similar exponential moment estimates, moment stability results and functional central limit theorems as in the constant coefficients case.

The paper is organized as follows. We begin, in Section 2, by collecting some standard Markov processes terminology that is used in this paper. Sections 3 and 4 are devoted to the study of an SRBM. We give basic definitions and present the key result of [9] which gives sufficient conditions for ergodicity of an SRBM. In Theorem 4.11, by identifying a suitable Lyapunov function, we show that the invariant measure has a finite moment generating function in a neighborhood of zero. This result is then used to establish uniform (in time and initial condition in a compact set) estimates on exponential moments of an SRBM. Growth of polynomial moment of the process as a function of the initial condition is investigated in Corollary 4.14 and Theorem 4.15. Finally in Theorem 4.17 we establish a functional central limit theorem for functionals of an SRBM and characterize the asymptotic variance in this limit result via the solution of the corresponding Poisson equation. In Section 5 we consider constrained diffusions with state dependent coefficients. In Lemma 5.9 we establish the key drift inequality for the 1-skeleton chain of the process. The stability results that follow as a consequence of this lemma are summarized in Corollary 5.10 and Corollary 5.11. Finally in the Appendix of this paper we provide a proof of the fact that the law of the constrained diffusion (under the conditions of Section 5) at any time instant $t > 0$, is mutually absolutely continuous with respect to the Lebesgue measure on the state space. Although such a result is a folklore in the literature, we give a self contained proof based on arguments in [2] and [14].

The following notation is used in this paper. For a metric space X, let $BM(X)$ denote the space of real bounded measurable functions on X and $\mathcal{B}(X)$ be the Borel σ-field on X. By convention all measures on $(X, \mathcal{B}(X))$ will be nontrivial. For a real valued measurable function f on X and a measure μ on $\mathcal{B}(X)$, let $\mu(f) \doteq \int_X f d\mu$. The Dirac measure at the point x is denoted by δ_x. The set of natural numbers is denoted by $I\!N$ and let $I\!N_0 \doteq I\!N \bigcup \{0\}$. Denote the set of real numbers by $I\!R$ and non-negative real numbers by $I\!R_+$. Let $I\!R^d$ be the d-dimensional Euclidean space with the usual Euclidean norm, and $I\!R^{d\times m}$ the space of real $d \times m$-matrices with the norm $|A| \doteq (\sum_{i=1}^d \sum_{j=1}^m A_{ij}^2)^{1/2}$ for $A \in I\!R^{d\times m}$. For a given matrix M denote by M' its transpose and $tr(M)$ its trace. For a set $A \subseteq I\!R^d$, denote its interior, closure, boundary by A°, $\bar{A}$, and ∂A, respectively. For sets $A, B \subseteq I\!R^d$, $dist(A, B)$ will denote the distance between two sets, i.e., $\inf\{|x - y| : x \in A, y \in B\}$. The class of continuous functions $f : X \to Y$ is denoted by $C(X, Y)$ and real continuous bounded functions on X by $C_b(X)$. Denote the class of real continuous and twice differentiable functions defined on X by $C^2(X)$. Finally, $D(X, Y)$ denote the class of right continuous functions with having left limit defined from X to Y, equipped with the usual Skorohod topology.

2 Setting and preliminary results

In this section we will collect standard Markov processes terminology and definitions that will be used in this paper. Our main sources are [16] (for discrete time) and [17] (for continuous time). Let

S be a complete, locally compact, separable metric space. By an S-valued strong Markov process $(Z, \{P_x\}_{x\in S})$ given on some filtered measurable space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0})$ we will mean a collection of probability measures $\{P_x\}_{x\in S}$ defined on $(\Omega, \mathcal{F})$ and an $\{\mathcal{F}_t\}$-adapted stochastic process $\{Z_t\}_{t\geq 0}$ which has right continuous with left limit (RCLL) paths P_x-a.s. for all $x \in S$ and satisfies:

- $x \mapsto I\!\!E_x f(Z_t)$ is a measurable map from S to $I\!\!R$ for all $f \in BM(S)$,
- For all $\{\mathcal{F}_t\}$ stopping times τ such that $P_x[\tau < \infty] = 1$, $P_x[Z_{t+\tau} \in A | \mathcal{F}_\tau] = P^t(Z_\tau, A)$, P_x-a.s., for all $t \geq 0$ and $A \in \mathcal{B}(S)$, where $P^t(x, A) \doteq I\!\!E_x[Z_t \in A]$ is referred to as the transition probability kernel of Z.

Frequently, when the family $\{P_x\}$ is clear, we will suppress it from the notation and refer to Z as the (strong) Markov process. We will occasionally need to refer to discrete time Markov processes. An S-valued discrete time stochastic process $\breve{Z} = \{\breve{Z}_n : n \in I\!\!N_0\}$ along with a family of probability measure $\{\breve{P}_x\}_{x\in S}$ defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_n\}_{n\in I\!\!N_0})$ is called a Markov family if the map $y \mapsto \breve{P}(y, A) \doteq \breve{P}_y[\breve{Z}_1 \in A]$, is a measurable function from S to $[0, 1]$ for all $A \in \mathcal{B}(S)$ and under each measure $\breve{P}_x$, $(\breve{Z}_n)$ is a Markov chain (with respect to $\{\mathcal{F}_n\}$) with initial distribution δ_x and transition probability kernel $\breve{P}(y, A)$. Once more the reference to $\{\breve{P}_x\}$ will be omitted when clear from the context. Throughout the paper the symbols Z_t and $Z(t)$ will be used interchangeably.

For the rest of this section we will fix a strong Markov process Z and a discrete time Markov chain $\breve{Z}$ as described above. The following notion of irreducibility plays an important role in our analysis.

Definition 2.1. *Let φ be a σ-finite measure on $(S, \mathcal{B}(S))$. For $A \in \mathcal{B}(S)$ let $\eta_A \doteq \int_0^\infty 1_{\{Z_t\in A\}}dt$. The Markov process Z is called φ-irreducible if whenever $A \in \mathcal{B}(S)$ is such that $\varphi(A) > 0$, we have $I\!\!E_x[\eta_A] > 0, \quad \forall x \in S$. The measure φ is called an irreducibility measure for the process Z. For the Markov chain $\breve{Z}$, φ-irreducibility is defined in a similar way on setting $\eta_A \doteq \sum_{n=0}^\infty 1_{\{\breve{Z}_n\in A\}}$.*

For a σ-finite measure μ and transition kernel Q given on $(S, \mathcal{B}(S))$, define for $A \in \mathcal{B}(S)$, $\mu Q(A) \doteq \int_S \mu(dx)Q(x, A)$. A probability measure μ on $(S, \mathcal{B}(S))$ is called invariant for the Markov process Z if $\mu P^t = \mu$ for all $t \geq 0$. It is invariant for the chain $\breve{Z}$ if $\mu\breve{P} = \mu$. For a signed measure μ on $(S, \mathcal{B}(S))$, define the total variation of μ, $||\mu||$, as $||\mu|| \doteq \sup_{f:|f|\leq 1} |\mu(f)|$. The process Z will be called ergodic if it has a unique invariant probability measure π and

$$\lim_{t\to\infty} ||P^t(x, \cdot) - \pi|| = 0, \quad \forall x \in S.$$

Ergodicity for the chain $\breve{Z}$ is defined similarly. Ergodicity ensures the convergence of the expectation $I\!\!E[f(Z_t)]$ to the steady state value $\pi(f)$ for bounded measurable functions f, as $t \to \infty$. To investigate convergence for an unbounded function f, one typically considers the following norm. Fix a measurable function $f : S \to [1, \infty)$. For any signed measure μ on $(S, \mathcal{B}(S))$, define its f-norm as

$$||\mu||_f \doteq \sup_{|g|\leq f} |\mu(g)| = \sup_{|g|\leq f} |\int \mu(dy)g(y)|.$$

Note that f-norm is the same as the total variation norm if $f \equiv 1$. For a measurable function $f : S \to [1, \infty)$, the Markov process Z will be called f-ergodic if $P^t(x, f) < \infty$ for all t, x; the process is ergodic with the invariant probability measure π satisfying $\pi(f) < \infty$; and

$$\lim_{t \to \infty} ||P^t(x, \cdot) - \pi||_f = 0, \quad \forall x \in S.$$

The f-ergodicity of $\breve{Z}$ is defined similarly.

In order to study the rate of convergence to steady state we introduce the notion of f-uniform ergodicity from [6].

Definition 2.2. *Let the Markov process Z be f-ergodic with invariant probability measure π. We say Z is f-uniformly ergodic if there exist constants $D \in (0, \infty)$, $\rho \in (0, 1)$ such that for all $t \in \mathbb{R}_+$ and $x \in S$,*

$$||P^t(x, \cdot) - \pi||_f \leq f(x) D \rho^t.$$

In our analysis of continuous time Markov process Z, we will consider some Markov chains derived from Z. For a probability measure a on $\mathbb{R}_+$, define the Markov transition function $K_a : S \times \mathcal{B}(S) \to [0, 1]$ as

$$K_a(x, A) \doteq \int_0^\infty P^t(x, A) a(dt).$$

The discrete time Markov chain with one step transition kernel $K_a(\cdot, A)$ will be referred to as the K_a-chain for the Markov process Z. If a is an exponential distribution with parameter β we will denote K_a by $\mathfrak{R}_\beta$, i.e.,

$$\mathfrak{R}_\beta(x, A) \doteq \int_0^\infty P^t(x, A) \beta \exp(-\beta t) dt$$

and call the corresponding sampled chain as the $\mathfrak{R}_\beta$-chain; if $\beta = 1$, we write $\mathfrak{R}_\beta$ merely as $\mathfrak{R}$. Finally, if a is degenerate at $\Delta \in (0, \infty)$, we will call the associated Markov chain $\{Z(j\Delta) : j \in \mathbb{N}_0\}$ as the Δ-skeleton chain.

3 Semimartingale reflecting Brownian motion

This section recalls some basic definitions [21] and the main stability result [9] for an SRBM. We begin with the following definitions.

3.1 Definitions and background

Let $d \in \mathbb{N}$ and let S denote the d-dimensional positive orthant, i.e., $S = \{x = (x_1, \ldots, x_d)' \in \mathbb{R}^d : x_i \geq 0, i = 1, \ldots, d\}$. Fix column vectors $r^0, r^1, \ldots, r^d \in \mathbb{R}^d$ and let $R \doteq \left[r^1, \ldots, r^d\right]_{d \times d}$. For $x \in \partial S$, the set of directions of constraints is defined as:

$$r(x) \doteq \left\{ \sum_{i=1}^d q_i r^i : \sum_{i=1}^d q_i = 1, q_i \geq 0, \text{ and } q_i > 0 \text{ only if } x_i = 0 \right\}. \tag{3.1}$$

Let Σ be a $d \times d$ strictly positive definite matrix. We call the quadruple (S, r^0, Σ, R) as the *data* for an SRBM. The following definition is taken from [21].

Definition 3.1. *For $x \in S$, an SRBM associated with the data (S, r^0, Σ, R) that starts from x is a continuous, $\{\mathcal{F}_t\}$-adapted d-dimensional process Z, defined on some filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0}, P)$ such that*

(i) $Z(t) = B(t) + r^0 t + RY(t) \in S$ *for all* $t \geq 0$, P*-a.s.,*

(ii) B *is a d-dimensional* $\{\mathcal{F}_t\}$*-Brownian motion with covariance matrix* Σ *such that* $B(0) = x$, P*-a.s.,*

(iii) Y *is an* $\{\mathcal{F}_t\}$*-adapted d-dimensional process such that* $Y_i(0) = 0$ *for* $i = 1, \ldots, d$, P*-a.s. For each* $i = 1, \ldots, d$, Y_i *is* continuous, nondecreasing *and* Y_i *can increase only when* $Z(\cdot)$ *is on the face* $F^i \doteq \{x \in S : x_i = 0\}$, i.e., $\int_0^t 1_{\{Z_i(s) \neq 0\}} dY_i(s) = 0$ *for all* $t \geq 0$, P*-a.s.*

We will assume throughout that R is completely-$\mathcal{S}$ (cf. [20]), namely, for every $k \times k$ principle submatrix G of R, there is a k-dimensional vector v_G such that $v_G \geq 0$ and $Rv_G > 0$. In [20] Theorem 2, it was shown that a necessary condition for the existence of an SRBM is that the reflection matrix R is a completely-$\mathcal{S}$ matrix. The paper [21] shows that the condition is sufficient as well for (weak) existence and uniqueness of SRBM to hold. As a consequence of this result it follows that there exists a filtered measurable space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t \geq 0})$ on which are given a family of probability measures $\{P_x\}_{x \in S}$ and continuous stochastic processes Z, B and Y such that for every $x \in S$ under P_x, $(i), (ii)$, and (iii) of Definition 3.1 hold and $(Z, \{P_x\})$ is an S-valued strong Markov process. For rest of this section and Section 4 we will fix such a filtered space along with process (Z, B, Y) and the family $\{P_x\}_{x \in S}$. We will denote the Markov family $(Z, \{P_x\})$ merely as Z and refer to it as the SRBM. We will denote the transition kernel of Z by P^t, namely for $x \in S$, $A \in \mathcal{B}(S)$, $P^t(x, A) = P_x[Z_t \in A]$.

We now formulate the key condition, introduced in [9], for positive recurrence of an SRBM in terms of the associated "fluid limit" trajectories.

Definition 3.2. *Let* $\psi \in C([0, \infty), \mathbb{R}^d)$ *with* $\psi(0) \in S$. *Then* $(\phi, \eta) \in C([0, \infty), \mathbb{R}^d) \times C([0, \infty), \mathbb{R}^d)$ *solves the Skorohod problem (SP) for* ψ *(with respect to* S *and* R*) if the following hold:*

(i) $\phi(t) = \psi(t) + R\eta(t) \in S$, *for all* $t \geq 0$;

(ii) η *is such that, for* $i = 1, \ldots, d$, *(a)* $\eta_i(0) = 0$, *(b)* η_i *is nondecreasing, and (c)* η_i *can increase only when* ϕ *is on* F^i, *that is,* $\int_0^t 1_{\{\phi_i(s) \neq 0\}} d\eta_i(s) = 0$, *for all* $t \geq 0$.

We say that a path $\phi \in C([0, \infty), \mathbb{R}^d)$ is "attracted to the origin" if for any $\epsilon > 0$ there exists $T < \infty$ such that $t \geq T$ implies $|\phi(t)| \leq \epsilon$.

Condition 3.3. *The* ϕ *component of every solution of the SP for paths* $\psi(\cdot)$ *of the form* $\psi(t) = x + r^0 t$, $t \geq 0, x \in S$, *is attracted to the origin.*

In [9] the authors showed that Condition 3.3 implies that the SRBM has a unique invariant distribution. The main objective of this work is to strengthen this result by establishing uniform f-ergodicity of the SRBM for an exponentially growing Lyapunov function f. In preparation for this result we first observe in the following lemma that an SRBM is φ-irreducible. Let λ denote the Lebesgue measure on S.

Lemma 3.4. *For every $A \in \mathcal{B}(S)$ with $\lambda(A) > 0$, $P^t(x, A) > 0$ for all $t > 0$ and $x \in S$. In particular, Z is λ-irreducible.*

Remarks on Proof: Note that the second statement in the lemma is immediate from the first. The proof of the first statement is outlined in Lemma 9 of [14] for a family of SRBMs arising from open queueing networks. The current paper considers a much broader family of SRBMs, nevertheless the basic arguments in [14] are quite general and the only feature of an SRBM, in addition to the strong Markov property, continuity of sample paths and nondegeneracy of Σ, is the following boundary property:

$$I\!\!E_x \left[\int_0^\infty 1_{\partial S}(Z(s)) ds \right] = 0.$$

The above property in the complete generality of an SRBM has been established in Lemma 2.1 of [21]. For details on the proof we refer the reader to the appendix where we provide a proof of a similar result for the state dependent model of Section 5.

4 Geometric ergodicity

The main result of this section is the V-uniform ergodicity (cf. Definition 2.2) of an SRBM under Condition 3.3 where V is an exponentially growing function. We begin by introducing the following Lyapunov function $W(\cdot)$, which was constructed in [9]. For function $f \in C^2(I\!\!R^d)$, let $Df(x)$ and $D^2f(x)$ denote the gradient and Hessian, respectively, of f at x. Let $\langle \cdot, \cdot \rangle$ denote the standard inner product on $I\!\!R^d$.

Theorem 4.1. [9] *Suppose that Condition 3.3 holds. Then there exists a continuous map $W : S \to I\!\!R$ such that the following hold.*

$(P1)$ $W(\cdot) \in C^2(S \backslash \{0\})$.

$(P2)$ *Given $N < \infty$, there is an $M < \infty$ such that $x \in S$ and $|x| \geq M$ imply $W(x) \geq N$.*

$(P3)$ *Given $\epsilon > 0$, there is an $M < \infty$ such that $x \in S$ and $|x| \geq M$ imply $|D^2W(x)| \leq \epsilon$.*

$(P4)$ *There exists a $c > 0$ such that*

$$\begin{aligned} \langle DW(x), r^0 \rangle &\leq -c, \quad \textit{for all} \quad x \in S \backslash \{0\}, \\ \langle DW(x), r \rangle &\leq -c, \quad \textit{for all} \quad r \in r(x), \quad x \in \partial S \backslash \{0\}. \end{aligned}$$

$(P5)$ *$W(\cdot)$ is radially homogeneous: $W(\alpha x) = \alpha W(x)$ for $\alpha \geq 0$, $x \in S$.*

Some consequences of properties $(P1) - (P5)$ are the following.

$(P6)$ For every $M \in (0, \infty)$ there exists a $\gamma \equiv \gamma(M) \in (0, \infty)$ such that $\sup_{|x| \leq M} W(x) \leq \gamma$.

$(P7)$ $\Gamma \doteq \sup_{x \in S \setminus \{0\}} |DW(x)| < \infty$.

$(P8)$ There exist $c_1, c_2 \in (0, \infty)$ such that $c_1|x| \leq W(x) \leq c_2|x|$, for all $x \in S$.

Proof of $(P8)$ is immediate from $(P5)$ and $(P2)$ and similarly $(P7)$ follows from $(P5)$ and $(P1)$ on noting that for $x \neq 0$, $DW(x) = DW(x/|x|)$. For a proof of the following elementary lemma we refer the reader to Lemma 4.3 in [1]. The main ingredient in the proof is the property $(P7)$ of the Lyapunov function W. Although W may not be differentiable at 0, with an abuse of notation, we set $DW(0) = 0$ and $D^2W(0) = 0$.

Lemma 4.2. *Let $x \in S$ and $\Delta \in (0, \infty)$ be fixed. For $m \in I\!N$ let ν_m be defined as follows:*

$$\nu_m \doteq \sup_{(m-1)\Delta \leq t \leq m\Delta} \left| \int_{(m-1)\Delta}^{t} \langle DW(Z(s)), dB(s) \rangle \right| .$$

Then for any $\kappa \in (0, \infty)$ and $m, n \in I\!N; m \leq n$,

$$I\!E_x (e^{\kappa \sum_{i=m}^{n} \nu_i}) \leq [2\sqrt{2} e^{\kappa^2 \Gamma^2 \gamma \Delta}]^{(n-m+1)},$$

where $\gamma \in (0, \infty)$ depends only on the norm of the covariance matrix Σ^{-1} and Γ is as in (P7).

By $(P3)$, there exists an $M_0 > 1$ such that $x \in S$ and $|x| \geq M_0$ imply $tr\left[D^2W(x)\Sigma\right] < c$, where c is as in $(P4)$. Fix $M \in (M_0, \infty)$ and define $B_1 \doteq \{x : |x| \leq M\}$. Choose $L \in (0, \infty)$ large enough so that $B_2 \doteq \{x : W(x) \leq L\} \supseteq B_1$. Let $\sigma_1 \doteq \inf\{t \geq 0 : Z(t) \in B_2\}$.

Theorem 4.3. *There exist $\beta \in (0, \infty)$ and $\alpha_1, \alpha_2 \in (0, \infty)$ such that for all $x \in S$, $I\!E_x[e^{\beta\sigma_1}] < \alpha_1 e^{\alpha_2 |x|}$. In particular, for any compact set $K \subseteq S$, $\sup_{x \in K} I\!E_x[e^{\beta\sigma_1}] < \infty$.*

Proof. Fix $\Delta \in (0, \infty)$ and $m \in I\!N$. By applying Itô's formula to $W(\cdot)$, we have

$$\begin{aligned} W(Z(m\Delta) \wedge \sigma_1) &= W(Z(m-1)\Delta) \wedge \sigma_1) + \int_{(m-1)\Delta\wedge\sigma_1}^{m\Delta\wedge\sigma_1} \left(\frac{1}{2} tr\left[D^2W(Z(s))\Sigma\right]\right) ds \\ &\quad + \int_{(m-1)\Delta\wedge\sigma_1}^{m\Delta\wedge\sigma_1} \langle DW(Z(s)), r^0 \rangle ds + \int_{(m-1)\Delta\wedge\sigma_1}^{m\Delta\wedge\sigma_1} \langle DW(Z(s)), dB(s) \rangle \\ &\quad + \sum_{i=1}^{d} \int_{(m-1)\Delta\wedge\sigma_1}^{m\Delta\wedge\sigma_1} \langle DW(Z(s)), r^i \rangle dY_i(s). \end{aligned} \tag{4.1}$$

For $n \in I\!N$, define $A_n \doteq \{\omega \in \Omega : \inf_{s \in [0, n\Delta]} W(Z(s)) > L\}$. For $\omega \in A_n$ and $m \leq n$ we have from

(4.1) that

$$
\begin{aligned}
W(Z(m\Delta)) &= W(Z((m-1)\Delta)) + \int_{(m-1)\Delta}^{m\Delta} \left(\frac{1}{2} tr\left[D^2 W(Z(s))\Sigma\right]\right) ds \\
&\quad + \int_{(m-1)\Delta}^{m\Delta} \langle DW(Z(s)), r^0 \rangle ds + \int_{(m-1)\Delta}^{m\Delta} \langle DW(Z(s)), dB(s)\rangle \\
&\quad + \sum_{i=1}^{d} \int_{(m-1)\Delta}^{m\Delta} \langle DW(Z(s)), r^i \rangle dY_i(s) \\
&\equiv T_1 + T_2 + T_3 + T_4 + T_5.
\end{aligned}
$$

On A_n, $T_2 \leq \frac{c}{2}\Delta$ and by $(P4)$ we have $T_3 \leq -c\Delta$ and $T_5 \leq 0$. As a result, on the set A_n and for $m \leq n$

$$
\begin{aligned}
W(Z(m\Delta)) &\leq W(Z((m-1)\Delta)) - \frac{c}{2}\Delta + \int_{(m-1)\Delta}^{m\Delta} \langle DW, dB\rangle \\
&\leq W(Z((m-1)\Delta)) - \frac{c}{2}\Delta + \sup_{(m-1)\Delta \leq t \leq m\Delta} \left| \int_{(m-1)\Delta}^{t} \langle DW, dB\rangle \right|,
\end{aligned}
$$

where in the above display, $DW(Z(s))$, $dB(s)$ are abbreviated as DW and dB, respectively. Thus for $1 \leq m \leq n$ and on the set A_n,

$$
L < W(Z(m\Delta)) \leq W(Z((m-1)\Delta)) - \frac{c}{2}\Delta + \nu_m, \tag{4.2}
$$

where ν_m is as in Lemma 4.2. Iterating inequality (4.2) we have that, on A_n, $L < W(Z(n\Delta)) \leq W(x) - \frac{c}{2}n\Delta + \sum_{j=1}^{n} \nu_j$.

An application of Lemma 4.2 and Markov's inequality yield after some simplifications (cf. proof of Theorem 4.1 in [1])

$$
\begin{aligned}
P_x(A_n) &\leq P_x\left(\sum_{j=1}^{n} \nu_j \geq \frac{c}{2}n\Delta - W(x) + L\right) \\
&\leq \exp\left(\alpha W(x)\right) \exp\left(n\Delta\left(\frac{\log 8}{2\Delta} + \alpha^2\Gamma^2\gamma - \frac{\alpha c}{2}\right)\right),
\end{aligned}
$$

where $\alpha > 0$ is arbitrary. Let $-\eta \doteq (\frac{\log 8}{2\Delta} + \alpha^2\Gamma^2\gamma - \frac{\alpha c}{2})$ and choose sufficiently large $\Delta > 0$ and sufficiently small $\alpha > 0$ so that $\eta > 0$. Let $t \in (0, \infty)$ be arbitrary and pick $n_0 \in I\!N$ such that $t \in [n_0\Delta, (n_0+1)\Delta]$. Then $P_x(\sigma_1 > t) \leq P_x(A_{n_0}) \leq C \cdot e^{\alpha W(x)} e^{-\eta t}$, where $C \doteq \exp(\eta\Delta)$.

Finally, for $\beta \in (0, \eta)$

$$
I\!E_x[e^{\beta\sigma_1}] = 1 + \int_0^\infty \beta e^{\beta t} P[\sigma_1 > t] dt \leq 1 + C\beta e^{\alpha W(x)} \int_0^\infty e^{(\beta-\eta)t} dt = 1 + \frac{C\beta}{\eta - \beta} e^{\alpha W(x)}.
$$

The result now follows from the above estimates on recalling $(P8)$. ■

The starting point of our study of geometric ergodicity properties of Z is the following result of [6]. (See Theorems 6.2 and 5.1 therein.) For $\delta \in (0,\infty)$ and a compact set $C \subseteq S$, let $\tau_C(\delta) \doteq \inf\{t \geq \delta : Z(t) \in C\}$. Denote by $\mathcal{D}(\mathcal{A})$ the set of all measurable functions $v : S \to I\!R$ for which for all $x \in S$, $I\!E_x|v(Z_t)| < \infty$ and there exists a measurable function $w : S \to I\!R$ satisfying

$$I\!E_x[v(Z_t)] = v(x) + I\!E_x\left[\int_0^t w(Z_s)ds\right],$$

$$\int_0^t I\!E_x[|w(Z_s)|]ds < \infty,$$

for $t > 0$. For $v \in \mathcal{D}(\mathcal{A})$ and a corresponding w, we write $(v,w) \in \mathcal{A}$ (or sometimes with an abuse of notation $w = \mathcal{A}v$). We refer to the (multivalued) map $\mathcal{A}$ as the extended generator of Z and $\mathcal{D}(\mathcal{A})$ the domain of the extended generator. The following result of [6] is key in our analysis.

Theorem 4.4. *Suppose that for some compact set $C \subseteq S$ and $\eta, \delta \in (0,\infty)$ we have $I\!E_x e^{\eta\tau_C(\delta)} < \infty$ for all $x \in S$. Let*

$$V_0(x) \doteq \frac{1}{\eta}[I\!E_x e^{\eta\tau_C(\delta)} - 1] + 1 \tag{4.3}$$

and suppose that $\sup_{x\in C} V_0(x) < \infty$. Then for all $\beta > 0$, $(V_\beta, W_\beta) \in \mathcal{A}$, where $V_\beta \doteq \mathfrak{R}_\beta V_0$ and $W_\beta \doteq \beta V_\beta - V_0$. Furthermore, there exist $b, c \in (0,\infty)$ such that

$$\mathcal{A}V_\beta(x) \leq -cV_\beta(x) + b1_C(x), \quad \text{for all} \quad x \in S.$$

Proof. The result follows on taking $f \equiv 1$ in Theorem 6.2 of [6] and using the (b) part of the cited theorem along with (a) part of Theorem 5.1 in the same paper. We note that the cited results in [6] are formulated in terms of petite sets. However, since the Markov process Z is Feller and from Lemma 3.4 the support of the maximal irreducibility measure for Z is all of S, we have from Theorem 6.2.9 and Theorem 6.2.5 (ii) of [16] that all compact subsets of S are petite for Z. ■

Let $B_3 \doteq \{x \in S : W(x) \leq \tilde{L}\}$, where $\tilde{L} \geq L$ is chosen large enough so that $dist(B_2, \partial B_3) > 1$. Note that $B_1 \subseteq B_2 \subseteq B_3$. Let $\sigma_0 \doteq \inf\{t \geq 0 : Z(t) \in \partial B_3\}$.

Lemma 4.5. *For each fixed $\delta \in (0,\infty)$ there exists an $\epsilon_0 \equiv \epsilon_0(\delta) \in (0,1)$ such that*

$$\sup_{x\in B_3} P_x(\sigma_0 > \delta) < \epsilon_0, \tag{4.4}$$

$$\sup_{x\in \partial B_3} P_x(\sigma_1 < \delta) < \epsilon_0. \tag{4.5}$$

Proof. We will only prove (4.4), the proof of (4.5) is similar and is omitted. We will argue via the method of contradiction. Fix $\delta \in (0,\infty)$ and suppose that (4.4) does not hold for any $\epsilon_0 \in (0,1)$. Then there exist sequences $\{x_n\}, \{\epsilon_n\}$ such that $P_{x_n}[\sigma_0 > \delta] \geq \epsilon_n$, where $\{x_n\} \subseteq B_3$, $\epsilon_n \in (0,1)$ and $\epsilon_n \uparrow 1$ as $n \to \infty$. From the Feller property of SRBM it follows that if $\{x_{n_k}\}_{k\geq 1}$ is a subsequence of $\{x_n\}$ such that $x_{n_k} \to x$ as $k \to \infty$ then

$$\overline{\lim}\, P_{x_{n_k}}\left[\sup_{0\leq s\leq \delta} W(Z(s)) < \tilde{L}\right] \leq P_x\left[\sup_{0\leq s\leq \delta} W(Z(s)) \leq \tilde{L}\right].$$

Hence $P_x[W(Z(\delta)) \leq \tilde{L}] = 1$. However this contradicts Lemma 3.4 in view of $(P8)$ and hence the result follows. ∎

Let $\tau_1 \doteq \inf\{t \geq \sigma_0 : Z(t) \in \partial B_2\}$. The following lemma is the key step in the proof of Theorem 4.7.

Lemma 4.6. *Under Condition 3.3, there exists* $\beta_1 \in (0, \infty)$ *and* $A \in (0, \infty)$ *such that*

$$\sup_{x \in B_2} I\!\!E_x[e^{\beta_1 \tau_1}] < A.$$

Proof. In view of Theorem 4.3 and strong Markov property of SRBM it suffices to show that there exists a $r \in (0, \infty)$ such that $\sup_{x \in B_2} I\!\!E_x[e^{r\sigma_0}] < \infty$. This will follow if we show that there exists a $\theta_0 \in (0, \infty)$ such that for all $k \in I\!\!N$ and $x \in B_2$,

$$P_x[\sigma_0 > k] < e^{-\theta_0 k}. \tag{4.6}$$

Next note that $P_x[\sigma_0 > k] = I\!\!E_x(I\!\!E_x[1_{\sigma_0 > k} | \mathcal{F}_{k-1}] 1_{\sigma_0 > k-1})$. Furthermore,

$$I\!\!E_x[1_{\sigma_0 > k} | \mathcal{F}_{k-1}] 1_{\sigma_0 > k-1} \leq \sup_{x \in B_3} P_x[\sigma_0 > 1] 1_{\sigma_0 > k-1} < \tilde{\epsilon}_0 1_{\sigma_0 > k-1},$$

where $\tilde{\epsilon}_0 = \epsilon_0(1) \in (0, 1)$ is as in Lemma 4.5. Inequality (4.6) now follows on iterating the above conditioning argument k times. This completes the proof of the lemma. ∎

The following result is a key ingredient in our construction of a suitable Lyapunov function.

Theorem 4.7. *Suppose Condition 3.3 holds. Fix* $\delta \in (0, \infty)$ *and let* $\beta_1 \in (0, \infty)$ *be as in Lemma 4.6. Then* $\sup_{x \in B_2} I\!\!E_x[e^{\beta_1 \tau_{B_2}(\delta)}] < \infty$.

Proof. Define sequences of stopping times $\{\hat{\tau}_i\}, \{\hat{\sigma}_i\}, i \in I\!\!N$ as follows. Let $\hat{\tau}_0 \doteq 0$ and $\hat{\sigma}_n \doteq \inf\{t \geq \hat{\tau}_{n-1} : Z(t) \in \partial B_3\}$, $\hat{\tau}_n \doteq \inf\{t \geq \hat{\sigma}_n : Z(t) \in B_2\}$, $n = 1, 2, \ldots$. Let $\theta \doteq \epsilon_0(\delta)$, where $\epsilon_0(\cdot)$ is as in Lemma 4.5. Also for $n \in I\!\!N$, set $m_n \doteq \hat{\tau}_n - \hat{\tau}_{n-1}$ and let $m_0 \doteq 0$. From the second inequality in Lemma 4.5, $P_x[\hat{\tau}_n - \hat{\tau}_{n-1} > \delta | \mathcal{F}_{\hat{\tau}_{n-1}}] \geq 1 - \theta$ for all $n \geq 1$. Thus by the second Borel Cantelli lemma (cf. p.253 in [10]) $\hat{\tau}_n \to \infty$ a.s. as $n \to \infty$. Therefore, for $x \in B_2$

$$I\!\!E_x[e^{\beta_1 \tau_{B_2}(\delta)}] = e^{\beta_1 \delta} I\!\!E_x \sum_{n=1}^{\infty} 1_{\delta \in (\hat{\tau}_{n-1}, \hat{\tau}_n]} e^{\beta_1(\tau_{B_2}(\delta) - \delta)}. \tag{4.7}$$

Observing that $1_{\delta \in (\hat{\tau}_{n-1}, \hat{\tau}_n]} e^{\beta_1(\tau_{B_2}(\delta) - \delta)} \leq 1_{\delta \in (\hat{\tau}_{n-1}, \hat{\tau}_n]} e^{\beta_1(\hat{\tau}_n - \hat{\tau}_{n-1})}$ we have for $n \geq 1$,

$$\begin{aligned} I\!\!E_x 1_{\delta \in (\hat{\tau}_{n-1}, \hat{\tau}_n]} e^{\beta_1(\tau_{B_2}(\delta) - \delta)} &\leq I\!\!E_x 1_{m_1 < \delta} \cdots 1_{m_{n-1} < \delta} e^{\beta_1(\hat{\tau}_n - \hat{\tau}_{n-1})} \\ &= I\!\!E_x 1_{m_1 < \delta} \cdots 1_{m_{n-1} < \delta} I\!\!E_x[e^{\beta_1(\hat{\tau}_n - \hat{\tau}_{n-1})} | \mathcal{F}_{\hat{\tau}_{n-1}}] \\ &\leq A I\!\!E_x 1_{m_1 < \delta} \cdots 1_{m_{n-1} < \delta}, \end{aligned} \tag{4.8}$$

where the last inequality is a consequence of the strong Markov property of SRBM and Lemma 4.6. Next note that from Lemma 4.5, for $n \geq 2$, $I\!\!E_x[1_{m_{n-1} < \delta} | \mathcal{F}_{\hat{\tau}_{n-2}}] < \theta$. By a successive conditioning argument we now have that the right side of (4.8) is bounded by $A\theta^{n-1}$. The result now follows on substituting this bound in (4.7):

$$\sup_{x\in B_2} I\!E_x[e^{\beta_1 \tau_{B_2}(\delta)}] \le Ae^{\beta_1\delta}\sum_{n=1}^{\infty}\theta^{n-1} = \frac{Ae^{\beta_1\delta}}{1-\theta}. \qquad \blacksquare$$

The following inequality which follows from the oscillation result, Theorem 5.1 of [23], will be used several times in what follows: There is a $C \in (0,\infty)$ such that for all $x \in S$ and $0 \le t_1 < t_2 < \infty$

$$\sup_{t_1\le s\le t\le t_2} |Z_s - Z_t| \le C\left[\sup_{t_1\le s\le t\le t_2} |B(t)-B(s)| + (t_2-t_1)\right], \quad P_x\text{-}a.s. \tag{4.9}$$

Lemma 4.8. *Let V_0 be as in (4.3) with C there replaced by B_2, then under Condition 3.3 there exist $a_1, a_2, A_1, A_2 \in (0,\infty)$, such that*

$$a_1 e^{a_2|x|} \le V_0(x) \le A_1 e^{A_2|x|} \quad \text{for each} \quad x \in S. \tag{4.10}$$

Furthermore, there is a $A_3 \in (0,\infty)$ such that for every $\beta \in (A_3,\infty)$ there are $\tilde a_1, \tilde a_2, \tilde A_1, \tilde A_2 \in (0,\infty)$, such that

$$\tilde a_1 e^{\tilde a_2|x|} \le V_\beta(x) \le \tilde A_1 e^{\tilde A_2|x|} \quad \text{for each} \quad x \in S, \tag{4.11}$$

where V_β is as in Theorem 4.4.

Proof. We begin by showing the first inequality of (4.10). Note that $V_0(x) \ge 1$, so in order to prove the inequality it suffices to show that there exist $M \in (1,\infty)$, $a_1, a_2 \in (0,\infty)$ such that for all $|x| \ge M$, $V_0(x) \ge a_1 e^{a_2|x|}$. By Jensen's inequality $I\!E_x[e^{\beta_1\tau_{B_2}(\delta)}] \ge e^{\beta_1 I\!E_x \tau_{B_2}(\delta)}$. For the lower bound on $I\!E_x\tau_{B_2}(\delta)$, let M be large enough so that for all $|x| \ge M$, $d(x, B_2) \ge \frac{1}{2}|x|$. Then for $\vartheta \in (0,1)$ and $|x| \ge M$,

$$\begin{aligned} I\!E_x\tau_{B_2}(\delta) &\ge I\!E_x\tau_{B_2}(\delta) 1\left\{\sup_{0\le s\le \vartheta|x|} |Z_s - x| \le \frac{1}{2}|x|\right\} \\ &\ge \vartheta|x| P_x\left[\sup_{0\le s\le\vartheta|x|} |Z_s - x| \le \frac{1}{2}|x|\right]. \end{aligned} \tag{4.12}$$

By Markov inequality and (4.9), we have that

$$\begin{aligned} P_x\left[\sup_{0\le s\le\vartheta|x|} |Z_s - x| \ge \frac{1}{2}|x|\right] &\le \frac{2I\!E_x \sup_{0\le s\le\vartheta|x|} |Z_s - x|}{|x|} \\ &\le \frac{4C\left[I\!E_x \sup_{0\le s\le\vartheta|x|} |B_s| + \vartheta|x|\right]}{|x|}, \end{aligned}$$

where C is as in (4.9). Thus there is a $\tilde C \in (0,\infty)$ such that for all $x \in S$ and $\vartheta \in (0,1)$,

$$P_x\left[\sup_{0\le s\le\vartheta|x|} |Z_s - x| \ge \frac{1}{2}|x|\right] \le \tilde C \vartheta^{1/2}. \tag{4.13}$$

Choosing $\vartheta < (\frac{1}{2\bar{C}})^2$, we now have from (4.12) that for all $|x| \geq M$, $I\!\!E_x \tau_{B_2}(\delta) \geq \frac{\vartheta}{2}|x|$. The desired lower bound in (4.10) now follows. Next recalling that $V_\beta(x) \doteq \mathfrak{R}_\beta V_0 = \int_0^\infty I\!\!E_x V_0(Z_t)\beta e^{-\beta t}dt$, we have

$$\begin{aligned} V_\beta(x) &\geq I\!\!E_x \int_0^\infty a_1 e^{a_2|Z_t|}\beta e^{-\beta t}dt \geq a_1 e^{a_2|x|} I\!\!E_x \int_0^\infty e^{-a_2|Z_t - x|}\beta e^{-\beta t}dt \\ &\geq a_1 e^{a_2|x|} I\!\!E_x \int_0^\infty e^{-a_2 \sup_{0\leq s\leq t}|Z_s - x|}\beta e^{-\beta t}dt. \end{aligned}$$

Applying (4.9) once more and a similar argument as leading to (4.13), we can find $b \in (0,\infty)$ such that for all $x \in S$ and $\beta \in (0,\infty)$

$$V_\beta(x) \geq a_1 e^{a_2|x|} \int_1^\infty \beta e^{-bt} e^{-\beta t}dt.$$

This proves the lower bound in (4.11).

For the second inequality of (4.10) recall the stopping time σ_1 introduced above Theorem 4.3. By a conditioning argument and the strong Markov property,

$$\begin{aligned} I\!\!E_x e^{\beta_1 \tau_{B_2}(\delta)} &\leq I\!\!E_x e^{\beta_1\sigma_1} + I\!\!E_x[e^{\beta_1\tau_{B_2}(\delta)} 1_{\sigma_1 \leq \delta}] \\ &= I\!\!E_x e^{\beta_1\sigma_1} + I\!\!E_x[1_{\sigma_1\leq\delta} I\!\!E_x(e^{\beta_1\tau_{B_2}(\delta)}|\mathcal{F}_{\sigma_1})] \\ &\leq I\!\!E_x e^{\beta_1\sigma_1} + \sup_{x\in B_2} I\!\!E_x[e^{\beta_1\tau_{B_2}(\delta)}]. \end{aligned}$$

The desired upper bound in (4.10) now follows on using Theorem 4.3 and 4.7 in the above display. Finally,

$$V_\beta(x) \leq I\!\!E_x \int_0^\infty A_1 e^{A_2|Z_t|}\beta e^{-\beta t}dt \leq A_1 e^{A_2|x|} I\!\!E_x \int_0^\infty \beta e^{A_2|Z_t - x|} e^{-\beta t}dt.$$

Once more from (4.9), we can find $A_3 \in (0,\infty)$ such that for all $\beta \in (0,\infty)$ and $x \in S$

$$V_\beta(x) \leq A_1 e^{A_2|x|} \int_0^\infty \beta e^{A_3 t} e^{-\beta t}dt.$$

The upper bound in (4.11) now follows on choosing $\beta \in (A_3, \infty)$. ■

Henceforth we will fix a $\beta > A_3$ and denote the corresponding V_β by V.

Corollary 4.9. *Under Condition 3.3, the SRBM satisfies the following "drift criteria": There exist $b, c \in (0,\infty)$, some compact set $C \subseteq S$ such that*

$$\mathcal{A}V(x) \leq -cV(x) + b1_C(x), \quad \textit{for all} \quad x \in S. \tag{4.14}$$

Proof. From Lemma 4.8 $\sup_{x\in B_2} V_0(x) < \infty$, $I\!\!E_x e^{\beta_1\tau_{B_2}(\delta)} < \infty$ for all $x \in S$. Result follows on combining these two facts and applying Theorem 4.4. ■

Corollary 4.10. *Suppose that Condition 3.3 holds. Let π be the unique invariant probability distribution of Z. Then $\pi(V) < \infty$.*

Proof. Theorem 5.1 (c) of [6] along with equation (4.14) implies the existence of $\lambda \in (0,1)$ and $\tilde{b} \in (0,\infty)$ such that $\mathfrak{R}V(x) \le V(x) - (1-\lambda)V(x) + \tilde{b}1_C(x)$ for all $x \in S$. Combining the above drift condition with Theorem 3.1 of [15] and Theorem 14.3.7 of [16], we get $\pi(V) \le b\pi(C)/(1-\lambda) < \infty$. ■

Theorem 4.11. *Suppose that Condition 3.3 holds and let π be the unique invariant distribution for Z. Let $\tilde{a}_2 \in (0,\infty)$ be as in Lemma 4.8. Then for all $c \in I\!\!R^d$ with $|c| \le \tilde{a}_2$ we have $\int_S e^{c\cdot x}\pi(dx) < \infty$.*

Proof. Let $\tilde{a}_1 \in (0,\infty)$ be as in Lemma 4.8. Then

$$\tilde{a}_1 \int_S e^{c\cdot x}\pi(dx) \le \tilde{a}_1 \int_S e^{\tilde{a}_2|x|}\pi(dx) \le \int_S V(x)\pi(dx) < \infty,$$

where the last inequality follows from Corollary 4.10. In particular, from the proof of Corollary 4.10 we see that $\int_S e^{c\cdot x}\pi(dx) \le \frac{b\pi(C)}{\tilde{a}_1(1-\lambda)}$. ■

Now we present the main result of this section.

Theorem 4.12. *Under Condition 3.3, Z is V-uniformly ergodic; i.e., there exist constants $D \in (0,\infty)$, $\rho \in (0,1)$ such that for all $t \in I\!\!R_+$ and $x \in S$, $||P^t(x,\cdot) - \pi||_V \le V(x)D\rho^t$.*

Proof. We recall that Z is λ-irreducible and aperiodic (in the sense of [6]). Theorem 5.2 (c) of [6] shows that for such a Markov process, if a function V satisfies (4.14) then the process is V-uniformly ergodic. Thus the result follows from Corollary 4.9. ■

For a function $U : S \to [1,\infty)$, let L^U_∞ be the vector space of functions $h : S \to I\!\!R$ such that $||h||_U \doteq \sup_{x\in S} \frac{|h(x)|}{U(x)} < \infty$.

Theorem 4.13. *Suppose that Condition 3.3 holds. Then for every $g \in L^V_\infty$, there exists $\tilde{D} \in (0,\infty)$ such that for all $x \in S$ and $t \ge 0$,*

$$I\!\!E_x g(Z_t) \le \tilde{D}[1 + V(x)\rho^t],$$

where $\rho \in (0,\infty)$ is as in Theorem 4.12. In particular, for a suitable $\tilde{D} \in (0,\infty)$, $I\!\!E_x e^{\tilde{a}_2|Z_t|} \le \tilde{D}[1 + V(x)\rho^t]$, where $\tilde{a}_2 \in (0,\infty)$ is as in Lemma 4.8, and for every compact set $K \subseteq S$ we have $\sup_{t\ge 0}\sup_{x\in K} I\!\!E_x e^{\tilde{a}_2|Z_t|} < \infty$.

Proof. For $g \in L^V_\infty$, let $\tilde{g} \doteq \frac{g}{||g||_V}$. Then $\tilde{g} \le V$ and by Theorem 4.12, we have that for all $t \in I\!\!R_+$ and $x \in S$,

$$\left| I\!\!E_x \left\{ \tilde{g}(Z_t) - \int_S \tilde{g}(y)\pi(dy) \right\} \right| \le DV(x)\rho^t,$$

where $D \in (0,\infty)$ is as in Theorem 4.12. So

$$I\!\!E_x \tilde{g}(Z_t) \le \int_S \tilde{g}(y)\pi(dy) + DV(x)\rho^t.$$

Since $\int_S \tilde{g}(y)\pi(dy) \le \pi(V) < \infty$, there is a $\tilde{D} \in (0,\infty)$ such that $\mathbb{E}_x g(Z_t) \le \tilde{D}[1 + V(x)\rho^t]$. Choosing $g(x)$ to be $\tilde{a}_1 e^{\tilde{a}_2|x|}$ as in Lemma 4.8 so that $|g| \le V$, yields that $\mathbb{E}_x e^{\tilde{a}_2|Z_t|} \le \tilde{D}[1+V(x)\rho^t]$, and hence from Lemma 4.8, for every compact set $K \subseteq S$, we have $\sup_{t\ge 0}\sup_{x\in K}\mathbb{E}_x e^{\tilde{a}_2|Z_t|} < \infty$. ■

Corollary 4.14. *Suppose that Condition 3.3 holds. Then there exists a* $t_0 > 0$ *such that for all* $p > 0$, $\lim_{|x|\to\infty}\sup_{t\ge t_0}\frac{1}{|x|^{p+1}}\mathbb{E}_x\left(|Z(t|x|)|^{p+1}\right) = 0$.

Proof. Fix $p > 0$ and choose $\theta_{p+1} \in (0,\infty)$ small enough so that $\theta_{p+1}x^{p+1} \le e^{\tilde{a}_2 x}$ for $x \in \mathbb{R}_+$, where $\tilde{a}_2$ is as in Theorem 4.13. Then from Theorem 4.13, for $0 < t_0 \le t < \infty$,

$$\begin{aligned}\frac{1}{|x|^{p+1}}\mathbb{E}_x\left(|Z(t|x|)|^{p+1}\right) &\le \frac{1}{|x|^{p+1}\theta_{p+1}}\mathbb{E}_x e^{\tilde{a}_2|Z(t|x|)|}\\ &\le \frac{\tilde{D}}{|x|^{p+1}\theta_{p+1}}[1+V(x)\rho^{t_0|x|}] \le \frac{\tilde{D}}{|x|^{p+1}\theta_{p+1}}[1+\tilde{A}_1 e^{\tilde{A}_2|x|}\rho^{t_0|x|}],\end{aligned}$$

where the last inequality follows from Lemma 4.8. The result now follows on taking t_0 large enough so that $\rho^{t_0} \le e^{-\tilde{A}_2}$. ■

Theorem 4.15. *Suppose that Condition 3.3 holds. Then for each* $p > 0$ *there exists a constant* $\kappa_p \in (0,\infty)$ *such that*

$$\frac{1}{t}\int_0^t \mathbb{E}_x\left[|Z(s)|^p\right]ds \le \kappa_p\left\{\frac{1}{t}|x|^{p+1}+1\right\}, \quad t>0, \quad x\in S.$$

Proof. By Corollary 4.14, there exists an $L \in (0,\infty)$ such that with $D \doteq \{x \in S : |x| < L\}$

$$\mathbb{E}_x[Z(t_0|x|)^{p+1}] \le \frac{1}{2}|x|^{p+1}, \quad \forall\, x \in D^c, \tag{4.15}$$

where t_0 is as in Corollary 4.14. Let $\delta \doteq t_0 L$ and set $\tau(\delta) \doteq \inf\{t \ge \delta : |Z(t)| \le L\}$. Define $\hat{V}(x) \doteq \mathbb{E}_x\left[\int_0^{\tau(\delta)}(|Z(t)|^p+1)dt\right]$, $x \in S$. We will show next that there exists a $d \in (0,\infty)$ such that

$$\hat{V}(x) \le d(|x|^{p+1}+1), \quad \forall x \in S. \tag{4.16}$$

The result will then follow as an immediate consequence of Proposition 5.4 of [4]. Define a sequence of stopping times σ_n as $\sigma_0 \doteq 0$, $\sigma_n = \sigma_{n-1} + t_0[|Z(\sigma_{n-1})| \vee L]$, $n \in \mathbb{N}$. Also, let $n_0 \doteq \min\{n \ge 1 : |Z(\sigma_n)| \le L\}$. Then

$$\begin{aligned}\hat{V}(x) &\le \mathbb{E}_x\left[\int_0^{\sigma_{n_0}}(|Z(t)|^p+1)dt\right]\\ &= \sum_{k=0}^{\infty}\mathbb{E}_x\left[\int_{\sigma_k}^{\sigma_{k+1}}(|Z(t)|^p+1)dt 1_{k<n_0}\right].\end{aligned}\tag{4.17}$$

An application of strong Markov property and (4.9) shows that there exists a $d_1 \in (0,\infty)$ such that

$$\mathbb{E}_x\left[\int_{\sigma_k}^{\sigma_{k+1}}(|Z(t)|^p+1)dt\Big|\mathcal{F}_{\sigma_k}\right]1_{k<n_0} \le d_1\left(|Z(\sigma_k)|^{p+1}+1\right)1_{k<n_0}. \tag{4.18}$$

Using this estimate in (4.17) we get by suitable conditioning

$$\hat{V}(x) \leq d_1 I\!E_x \left[\sum_{k=0}^{n_0-1} \left(|Z(\sigma_k)|^{p+1} + 1 \right) \right]. \tag{4.19}$$

Next note that $\{Z(\sigma_k)\}_{k \geq 1}$ is a Markov chain with the transition kernel

$$\breve{P}(x, A) \doteq P^{t_0(|x| \vee L)}(x, A), \quad x \in S, \quad A \in \mathcal{B}(S).$$

Using (4.9) once more, and (4.15), one sees that there exists a $\tilde{b} \in (0, \infty)$ such that

$$\int_S \breve{P}(x, dy)|y|^{p+1} \leq |x|^{p+1} - \frac{1}{2}|x|^{p+1} + \tilde{b}1_{[0,L]}(|x|). \tag{4.20}$$

Using Theorem 14.2.2 of [16] we have now that

$$I\!E_x \sum_{k=0}^{n_0-1} \left[|Z(\sigma_k)|^{p+1} + 1 \right] \leq 2 \left\{ |x|^{p+1} + \tilde{b}1_{[0,L]}(|x|) \right\}.$$

The inequality (4.16) now follows on using the above estimate in (4.19). ■

Next, we use the results of [11] and the above geometric ergodicity results to study central limit results for $S_t \doteq \int_{[0,t)} F(Z(s))ds$, as $t \to \infty$, for a broad family of measurable functions $F : S \to I\!R$, allowed to have exponential growths. We begin by considering the Poisson equation, the solution of which characterizes the asymptotic variance in the central limit theorem (CLT) for S_t.

Theorem 4.16. *Suppose that Condition 3.3 holds. Then the following hold.*

(i) For all $F \in L_\infty^V$ and $x \in S$, the limit, as $t \to \infty$, of $I\!E_x[S_t - t\pi(F)]$ exists. Denoting the limit by $\hat{F}(x)$, we have that $\hat{F}(x) \in L_\infty^V$.

(ii) $\hat{F}$ solves the Poisson equation for F, i.e., $\hat{F}(x) \in \mathcal{D}(\mathcal{A})$, where $\mathcal{A}$ is the extended generator of Z (cf. above Theorem 4.3) and

$$\mathcal{A}\hat{F}(x) = \pi(F) - F(x), \quad x \in S. \tag{4.21}$$

(iii) The convergence in (i) is exponentially fast, i.e., denoting $I\!E_x[S_t - t\pi(F)]$ by $F_t^c(x)$, we have that for some $b_0, B_0 \in (0, \infty)$, $||F_t^c - \hat{F}||_V \leq B_0 e^{-b_0 t}$ for all $t \in (0, \infty)$.

Remark 4.1. *Note that from Corollary 4.10 $\pi(|F|) < \infty$ and from Theorem 4.13 $I\!E_x|S_t| < \infty$ for all $F \in L_\infty^V$, thus the statements in the above theorem are meaningful. Also for any $F \in L_\infty^V$, Poisson equation $\mathcal{A}g = \pi(F) - F$ admits at most one solution g, up to a constant additive term, with the property $\pi(|g|) < \infty$. I.e., if $g, \breve{g}$ are two solutions and $\pi(|g| + |\breve{g}|) < \infty$, then $g - \breve{g} = c$ a.s. $[\pi]$ for some $c \in I\!R$. Proof of this statement follows along the lines of Proposition 17.4.1 of [16].*

Proof. (i) Fix $F \in L^V_\infty$. From Theorem 4.12,

$$\int_0^\infty |P^t(x,F) - \pi(F)|dt \le V(x)\int_0^\infty D\rho^t dt \le D_0 V(x),$$

where $\rho \in (0,1)$ is as in Theorem 4.12 and $D_0 \in (0,\infty)$. Thus the limit

$$\lim_{t\to\infty}[\mathbb{E}_x S_t - t\pi(F)] = \lim_{t\to\infty}\int_0^t [P^s(x,F) - \pi(F)]ds$$

exists and denoting the limit by $\hat{F}(x)$, we have $\hat{F}(x) \le D_0 V(x)$ for all $x \in S$. Thus $\hat{F} \in L^V_\infty$.

(ii) Using the exponential bounds on V obtained in Lemma 4.8 one can check that $\mathbb{E}_x \int_0^t |\hat{F}(Z_s)|ds < \infty$. Next for $t > 0$,

$$\begin{aligned}
\mathbb{E}_x \hat{F}(X_t) &= \int_0^\infty \mathbb{E}_x[P^s(X_t,F) - \pi(F)]ds = \int_0^\infty [P^{s+t}(x,F) - \pi(F)]ds \\
&= \int_t^\infty [P^s(x,F) - \pi(F)]ds = \hat{F}(x) - \int_0^t [P^s(x,F) - \pi(F)]ds \\
&= \hat{F}(x) + \int_0^t \mathbb{E}_x[\pi(F) - F(X_s)]ds.
\end{aligned}$$

This establishes that $\hat{F} \in \mathcal{D}(\mathcal{A})$ and $\mathcal{A}\hat{F} = \pi(F) - F$.

(iii) For $0 \le t < T < \infty$,

$$\begin{aligned}
|\mathbb{E}_x[S_t - t\pi(F)] - \mathbb{E}_x[S_T - T\pi(F)]| &\le \int_t^T |P^s(x,F) - \pi(F)|ds \\
&\le V(x)\int_t^T D\rho^s ds \le V(x) D_1 \rho^t,
\end{aligned}$$

where the next to last inequality is a consequence of Theorem 4.12 and $D_1 \in (0,\infty)$. The result now follows on sending $T \to \infty$. ■

We now present a central limit result, the proof of which is an immediate consequence of Theorem 4.4 of [11] and Corollary 4.9. Define for a $F \in L^V_\infty$

$$\xi_n(t) \doteq \frac{1}{\sqrt{n}}\left(\int_0^{nt} \bar{F}(Z_s)ds\right), \quad t \in [0,1], \tag{4.22}$$

where $\bar{F} \doteq F - \pi(F)$. Let $C[0,1]$ denote the set of continuous functions defined from $[0,1]$ to $\mathbb{R}$.

Theorem 4.17. *Suppose that Condition 3.3 holds. Let $F : S \to \mathbb{R}$ be a measurable function such that $F^2(x) \le V(x)$ for all $x \in S$. Define $\gamma_F^2 \doteq 2\int \hat{F}(x)\bar{F}(x)\pi(dx)$, where $\hat{F}$ is a solution of the Poisson equation (4.21). Then, as $n \to \infty$, ξ_n converges weakly to $\gamma_F B$ in $C[0,1]$, where B is a one dimensional standard Brownian motion.*

Remark 4.2. *Note that since any two solutions of the Poisson equation (4.21) differ by a constant and $\int \bar{F}(x)\pi(dx) = 0$, the choice of the solution in the definition of γ_F^2 is immaterial. The nonnegativity and finiteness of the expression defining γ_F^2 under the drift condition (4.14) is established in [11].*

5 Constrained diffusion processes

In Sections 3 and 4 we studied geometric ergodicity properties for a constrained diffusion in $I\!R_+^d$ with constant drift and diffusion coefficients. In the current section we will address stability properties for a class of diffusion processes, with general state dependent coefficients, constrained to take values in a convex polyhedral cone S in $I\!R^d$ with the vertex at the origin. Our assumption on the reflection vector field $r(x)$ here will be somewhat more restrictive than the completely-$\mathcal{S}$ assumption made in Section 3. In particular we assume that the Skorohod map associated with the reflection data is well defined for all RCLL trajectories and it satisfies a Lipschitz property. (Details are given below.) Study of such diffusions is motivated by queueing networks with state dependent arrival and service rates. Under suitable conditions on the drift vector field, existence of a unique invariant probability distribution for this class of diffusions was established in [1]. In this work, we investigate the rate of convergence to steady state by studying geometric ergodicity for such diffusions. Since many arguments are quite similar to the constant coefficients case studied in Sections 3 and 4, only sketches of proofs will be provided.

We now describe the precise model that will be studied in this section. We assume that S is given as the intersection of half spaces S_i, $i = 1, \ldots, N$, $N \geq d$. Let n_i be the unit vector associated with S_i via the relation $S_i \doteq \{x \in I\!R^d : \langle x, n_i \rangle \geq 0\}$. Define F^i to be the face of S corresponding to n_i, i.e., $F^i \doteq \{x \in \partial S : \langle x, n_i \rangle = 0\}$. With each face F^i we associate the *direction of constraint* unit vector r_i satisfying $\langle r_i, n_i \rangle > 0$. At points on the boundary ∂S where more than one faces meet, there are more than one allowed directions of constraint. In general, for $x \in \partial S$ define

$$r(x) \doteq \left\{ r \in I\!R^d : r = \sum_{i \in In(x)} \alpha_i r_i;\ \alpha_i \geq 0;\ |r| = 1 \right\},$$

where $In(x) \doteq \{i \in \{1, 2, \ldots, N\} : \langle x, n_i \rangle = 0\}$. The set $r(x)$ represents the directions of constraint allowed at the point x. Let $D_S([0,\infty) : I\!R^d) \doteq \{\psi \in D([0,\infty) : I\!R^d) : \psi(0) \in S\}$. For $\eta \in D_S([0,\infty) : I\!R^d)$, $T \in (0,\infty)$ let $|\eta|(T)$ denote the total variation of η on $[0,T]$ with respect to the Euclidean norm on $I\!R^d$.

Definition 5.1. (Skorohod Map) *Let $\psi \in D_S([0,\infty) : I\!R^d)$ be given. Then $(\phi, \eta) \in D([0,\infty) : I\!R^d) \times D([0,\infty) : I\!R^d)$ solves the Skorohod problem (SP) for ψ with respect to S and r, if and only if $\phi(0) = \psi(0)$ and for all $t \in [0,\infty)$: (i) $\phi(t) = \psi(t) + \eta(t)$; (ii) $\phi(t) \in S$; (iii) $|\eta|(t) < \infty$; (iv)$|\eta|(t) = \int_{[0,t]} I_{\{\phi(s) \in \partial S\}} d|\eta|(s)$; (v) There exists a Borel measurable function $\gamma : [0,\infty) \to I\!R^d$ such that $\gamma(t) \in r(\phi(t)), d|\eta|$- almost everywhere and $\eta(t) = \int_{[0,t]} \gamma(s) d|\eta|(s)$.*

On the domain $D \subseteq D_S([0,\infty) : I\!R^d)$ on which there is a unique solution to the Skorohod problem we define the Skorohod map (SM) Γ as $\Gamma(\psi) \doteq \phi$ if $(\phi, \psi - \phi)$ is the unique solution of the Skorohod problem posed by ψ. We will make the following assumption on the regularity of the Skorohod map defined by the data $\{(r_i, n_i) : i = 1, 2, \ldots, N\}$.

Condition 5.2. *The Skorohod map is well defined on all of $D_S([0,\infty) : I\!R^d)$, that is, $D = D_S([0,\infty) : I\!R^d)$, and the SM is Lipschitz continuous in the following sense. There exists a constant $K \in (0,\infty)$ such that for all $\phi_1, \phi_2 \in D_S([0,\infty) : I\!R^d)$:*

$$\sup_{0 \leq t < \infty} |\Gamma(\phi_1)(t) - \Gamma(\phi_2)(t)| < K \sup_{0 \leq t < \infty} |\phi_1(t) - \phi_2(t)|. \tag{5.1}$$

We will assume without loss of generality that $K \geq 1$. We refer the reader to [7] (or alternatively cf. [8]) for sufficient conditions under which this regularity property holds.

We now introduce the constrained diffusion process that will be studied in this section. Let $(\Omega, \mathcal{F}, P)$ be a complete probability space on which is given a filtration $\{\mathcal{F}_t\}_{t\geq 0}$ satisfying the usual hypotheses. Let $(B(t), \{\mathcal{F}_t\})$ be a d-dimensional standard Wiener process on the above probability space. We will study the constrained diffusion process given as a solution to equation

$$Z(t) = \Gamma\left(x + \int_0^{\cdot} \sigma(Z(s))dB(s) + \int_0^{\cdot} b(Z(s))ds\right)(t), \tag{5.2}$$

where $\sigma : S \to I\!R^{d\times d}$ and $b : S \to I\!R^d$ are maps satisfying the following condition:

Condition 5.3. *There exists a $\gamma \in (0,\infty)$ such that*

$$|\sigma(x) - \sigma(y)| + |b(x) - b(y)| \leq \gamma|x-y|, \quad \forall x, y \in S, \tag{5.3}$$

$$|\sigma(x)| \leq \gamma, \quad \forall x \in S. \tag{5.4}$$

Under Condition 5.3 equation (5.2) admits a unique strong solution and as a consequence there exists a filtered measurable space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0})$ on which are given a family of probability measures $\{P_x\}_{x\in S}$ and continuous stochastic processes Z and B such that for all $x \in S$, under P_x, $\{B(t), \{\mathcal{F}_t\}_{t\geq 0}\}$ is a d-dimensional standard Wiener process and (Z, B) satisfy (5.2) P_x-a.s. Furthermore, $(Z, \{P_x\}_{x\in S})$ is a strong Markov family. Henceforth, we will refer to this family merely as Z.

We will make the following uniform nondegeneracy assumption on the diffusion coefficient.

Condition 5.4. *There exists a $c \in (0,\infty)$ such that for all $x \in S$ and $\alpha \in I\!R^d$,*

$$\alpha'(\sigma(x)\sigma'(x))\alpha \geq c\alpha'\alpha.$$

We now introduce the main condition on the drift field b for the process Z to be positive recurrent. Define

$$\mathcal{C} \doteq \left\{-\sum_{i=1}^N \alpha_i r_i : \alpha_i \geq 0, i \in \{1,\ldots,N\}\right\}.$$

The cone $\mathcal{C}$ was used to characterize stability of certain semimartingale reflecting Brownian motions in [3]. For $\delta \in (0,\infty)$, define

$$\mathcal{C}(\delta) \doteq \{v \in \mathcal{C} : \text{dist}(v, \partial\mathcal{C}) \geq \delta\}.$$

Our key assumption on the diffusion model stipulates the permissible drift vector field.

Condition 5.5. *There exists a $\delta \in (0,\infty)$ and a bounded set $A \subseteq S$ such that for all $x \in S\backslash A$, $b(x) \in \mathcal{C}(\delta)$.*

The following is the main result of [1] (Theorem 2.2 therein).

Theorem 5.6. *Assume that Conditions 5.2-5.5 hold. Then Z has a unique invariant probability measure π.*

For the rest of this section Conditions 5.2-5.5 will be assumed to hold. We will now study geometric ergodic properties of Z. We begin with the following result on φ-irreducibility of Z.

Lemma 5.7. *For every $A \in \mathcal{B}(S)$ with $\lambda(A) > 0$, $P^t(x, A) > 0$ for all $t > 0, x \in S$. In particular, Z is λ-irreducible.*

The proof is provided in the appendix. The following result from [1] (cf. Lemmas 3.1 and 4.1 therein) will be key in our analysis.

Lemma 5.8. *There is a function $T : S \to [0, \infty)$ such that the following properties hold.*

(1) For some $c_1 \in (0, \infty)$, $|T(x) - T(y)| \le c_1|x - y|$ for all $x, y \in S$.

(2) For some $c_2, c_3 \in (0, \infty)$, $c_2|x| \le T(x) \le c_3|x|$ for all $x \in S$.

(3) For some $c_4 \in (0, \infty)$,

$$T(Z(t \wedge \sigma_A)) \le [T(z) - (t \wedge \sigma_A)]^+ + c_4\eta_t^\star, \tag{5.5}$$

for all $t \ge 0$, P_z-a.s., for all $z \in S$, where A is as in Condition 5.5, $\sigma_A \doteq \inf\{t \ge 0 : Z_t \in A\}$, and $\eta_t^\star \doteq \sup_{0\le s\le t} |\int_0^s \sigma(Z(r))dB_r|$.

We now have the following result.

Lemma 5.9. *The 1-skeleton chain $\{\breve{Z}_n \doteq Z(n)\}_{n\in\mathbb{N}_0}$ satisfies the following drift inequality: There are $\delta, \beta, b_2 \in (0, \infty)$ and a compact set $C_2 \subseteq S$ such that*

$$\mathbb{E}_x V(\breve{Z}_1) \le (1 - \beta)V(x) + b_2 1_{C_2}(x), \quad x \in S, \tag{5.6}$$

with $V(x) \doteq e^{\delta T(x)}$.

Proof. From Lemma 5.8 *(3)*, for $\delta \in (0, \infty)$

$$V(x)^{-1}\mathbb{E}_x V(\breve{Z}_1)1_{\sigma_A>1} \le \mathbb{E}_x e^{\delta\left[[T(x)-1]^+ + c_4\eta_1^\star - T(x)\right]} 1_{\sigma_A>1}. \tag{5.7}$$

Thus for $x \in S_1 \doteq \{x : T(x) > 1\}$,

$$V(x)^{-1}[\mathbb{E}_x e^{\delta[T(\breve{Z}_1)]} 1_{\sigma_A>1}] \le \mathbb{E}_x e^{\delta c_4\eta_1^\star - \delta} 1_{\sigma_A>1} \le e^{-\delta} e^{\delta^2 c_5}, \tag{5.8}$$

where $c_5 \in (0, \infty)$ is an appropriate constant independent of δ and x. Now fix δ small enough so that $e^{-\delta} e^{\delta^2 c_5} \doteq (1 - 2\beta) < 1$. Then for $x \in S_1$,

$$\mathbb{E}_x V(\breve{Z}_1)1_{\sigma_A>1} \le (1 - 2\beta)V(x). \tag{5.9}$$

From the strong Markov property of Z we see that for all $x \in S$,

$$\mathbb{E}_x V(\breve{Z}_1) 1_{\sigma_A \le 1} = \mathbb{E}_x [\mathbb{E}_{Z(\sigma_A)} V(Z_{1-\sigma_A}) 1_{\sigma_A \le 1}].$$

Thus

$$\mathbb{E}_x V(\breve{Z}_1) 1_{\sigma_A \le 1} \le \sup_{y: y \in A} \mathbb{E}_y \sup_{0 \le t \le 1} V(Z_t) \le \sup_{y: y \in A} \mathbb{E}_y \sup_{0 \le t \le 1} e^{\delta c_3 |Z_t|},$$

where the last inequality follows from Lemma 5.8 (*2*). Using the above inequality; the Lipschitz property (5.1); and Condition 5.3 we now see that, for some $\tilde{K} \in (0, \infty)$

$$\mathbb{E}_x V(\breve{Z}_1) 1_{\sigma_A \le 1} \le \tilde{K}, \quad \forall x \in S. \tag{5.10}$$

Choose $M \in (1, \infty)$ such that $\beta V(x) \ge \tilde{K}$ for all $x \in S_M \doteq \{x : T(x) \ge M\}$. Then

$$\mathbb{E}_x V(\breve{Z}_1) 1_{\sigma_A \le 1} \le \beta V(x), \quad \forall x \in S_M. \tag{5.11}$$

Combining (5.9) and (5.11) we have

$$\mathbb{E}_x V(\breve{Z}_1) \le (1 - \beta) V(x), \quad \forall x \in S_M. \tag{5.12}$$

Also for $x \in C_2 \doteq \overline{S_M^c}$, we have from (5.7) and (5.10) that

$$\begin{aligned} \mathbb{E}_x V(\breve{Z}_1) &= \mathbb{E}_x V(\breve{Z}_1) 1_{\sigma_A \le 1} + \mathbb{E}_x V(\breve{Z}_1) 1_{\sigma_A > 1} \\ &\le \tilde{K} + e^{\delta T(x)} \mathbb{E}_x e^{\delta c_4 \eta_1^\star} \le \tilde{K} + e^{\delta M} e^{\delta^2 c_5} \doteq b_2. \end{aligned} \tag{5.13}$$

Combining (5.12) and (5.13) we have the result. ■

Corollary 5.10. *The invariant measure π satisfies $\pi(V) < \infty$. Furthermore, the* 1*-skeleton chain $(\{\breve{Z}_n\}, P_x)$ is V-uniformly ergodic, i.e., there exist $\rho \in (0,1)$ and $D \in (0, \infty)$ such that for all $x \in S$,*

$$||P^n(x, \cdot) - \pi||_V \le D V(x) \rho^n.$$

Proof. The first part of the corollary is an immediate consequence of Theorem 14.0.1 of [16], while the second follows from Theorem 16.0.1 of the same reference. ■

We now summarize the stability results that follow as a corollary to the above result.

Corollary 5.11. *Let π be the unique invariant distribution for Z. Then the following hold.*

1. *Let $\delta \in (0, \infty)$ be as in Lemma 5.9 and c_2 as in Lemma 5.8. Then for all $c \in \mathbb{R}^d$ with $|c| \le c_2 \delta$, $\int_S e^{c \cdot x} \pi(dx) < \infty$.*
2. *Let V be as in Lemma 5.9. Then, Z is V-uniformly ergodic; i.e., there exist constants $D \in (0, \infty)$, $\rho \in (0,1)$ such that for all $t \in \mathbb{R}_+$ and $x \in S$, $||P^t(x, \cdot) - \pi||_V \le V(x) D \rho^t$.*

3. *Let $g \in L_\infty^V$, where L_∞^V is as defined below Theorem 4.12. Then there exists a $\tilde{D} \in (0,\infty)$ such that for all $g \in L_\infty^V$, $x \in S$, and $t \geq 0$,*

$$\mathbb{E}_x g(Z_t) \leq \tilde{D}[1 + V(x)\rho^t],$$

where $\rho \in (0,\infty)$ is as in Corollary 5.10. In particular $\mathbb{E}_x e^{c_2|Z_t|} \leq \tilde{D}[1+V(x)\rho^t]$, where c_2 is as in Lemma 5.8, and for every compact set $K \subseteq S$ we have $\sup_{t\geq 0}\sup_{x\in K}\mathbb{E}_x e^{c_2|Z_t|} < \infty$.

4. *There exists a $t_0 > 0$ such that for all $p > 0$,*

$$\lim_{|x|\to\infty} \frac{1}{|x|^{p+1}} \mathbb{E}_x \left(|Z(t_0|x|)|^{p+1}\right) = 0.$$

5. *For each $p > 0$ there exists a constant $\kappa_p \in (0,\infty)$ such that*

$$\frac{1}{t}\int_0^t \mathbb{E}_x\left[|Z(s)|^p\right] ds \leq \kappa_p \left\{\frac{1}{t}|x|^{p+1} + 1\right\}, \quad t > 0, \quad x \in S.$$

6. *Conclusions $(i), (ii), (iii)$ of Theorem 4.16 hold.*

7. *Let $F : S \to \mathbb{R}$ be a measurable function such that $F^2(x) \leq V(x)$ for all $x \in S$. Define $\gamma_F^2 \doteq 2\int \hat{F}(x)\bar{F}(x)\pi(dx)$, where $\hat{F}$ is a solution of the Poisson equation (4.21). Let ξ_n be as in* (4.22) *with Z as in the current section. Then, as $n \to \infty$, ξ_n converges weakly to $\gamma_F B$ in $C[0,1]$, where B is a one dimensional standard Brownian motion.*

Proof. *1.* This is immediate from Corollary 5.10 and Lemma 5.8 (*2*).

2. Let $|||P^t - \pi|||_V \doteq \sup_{x\in S} \frac{||P^t(x,\cdot)-\pi||_V}{V(x)}$. From Corollary 5.10 we have that $||P^n - \pi||_V \leq D\rho^n$ for all $n \in \mathbb{N}$. It is easy to check (for example, cf. Proposition 16.1.3 of [16]) that for $t \in (0,\infty)$

$$\begin{aligned} |||P^t - \pi|||_V &\leq |||P^{\lfloor t\rfloor} - \pi|||_V \sup_{0\leq r\leq 1} |||P^r - \pi|||_V \\ &\leq D\rho^{\lfloor t\rfloor} \sup_{0\leq r\leq 1}\sup_{x\in S} \frac{\mathbb{E}_x[V(Z(r))] + \pi(V)}{V(x)}, \end{aligned} \tag{5.14}$$

where $\lfloor t\rfloor$ denotes the greatest integer less or equal to t. From arguments similar to those in Lemma 5.9 we see (cf. (5.13)) that for $r \in [0,1]$ and $x \in S$,

$$\mathbb{E}_x[V(Z(r))] \leq \tilde{K} + V(x)e^{\delta^2 c_5},$$

where c_5 is as in (5.8) and $\tilde{K}$ is as in (5.10). Substituting this estimate in (5.14) we now have that

$$|||P^t - \pi|||_V \leq \tilde{D}\rho^t,$$

where $\tilde{D} = \frac{D}{\rho}[\pi(V) + \tilde{K} + e^{\delta^2 c_5}]$. This proves *2*.

3 − 7. The proofs of *3 − 7* are now carried out exactly as in the case of deterministic coefficients studied in Section 4 on noting that V introduced in Corollary 5.10 satisfies

$$\tilde{a}_1 e^{\tilde{a}_2|x|} \leq V(x) \leq \tilde{A}_1 e^{\tilde{A}_2|x|} \quad \text{for each} \quad x \in S,$$

for suitable $\tilde{a}_1, \tilde{a}_2, \tilde{A}_1, \tilde{A}_2 \in (0, \infty)$. The proof of *5* requires minor modifications to the proof of Theorem 4.15. In particular, the analog of (4.18) is obtained from (4.17) by applying (5.5) instead of (4.9) and in obtaining (4.20) one uses (in addition to *4*) the Lipschitz property (5.1), linear growth condition (5.3), boundedness of σ (5.4), and Gronwall's lemma. Details are omitted. ■

Appendix

Proof of Lemma 5.7. The proof is adapted from arguments in [2], [14] and [20]. We begin by observing that for all $x \in S$,

$$I\!E_x \int_0^\infty 1_{\{Z(s)\in\partial S\}} ds = 0. \tag{A1}$$

Although the result is proved in analogous way as Lemma 2.1 of [21], we provide a quick proof for the sake of completeness. For $i = 1, \ldots, N$, let $\xi_i(t) \doteq Z(t) \cdot n_i$, where n_i is the inward unit normal to the face F^i. In order to prove (A1) it suffices to show that for each i,

$$\int_0^t 1_{\{\xi_i(s)=0\}} ds = 0, P_x\text{-}a.s., \tag{A2}$$

for all $t > 0$ and $x \in S$. Note that ξ_i is a continuous $\{\mathcal{F}_t\}$-semimartingale with quadratic variation $\langle \xi_i \rangle_t = \int_0^t n_i' a(Z_s) n_i ds$, where $a = \sigma\sigma'$. From Condition 5.4, we have that

$$\int_0^t 1_{\{\xi_i(s)=0\}} d\langle \xi_i \rangle_s \geq c \int_0^t 1_{\{\xi_i(s)=0\}} ds. \tag{A3}$$

From Corollary 1, p.216 of [18], the left side of (A3) equals $\int_0^\infty L_t^a 1_{\{0\}}(a) da = 0$, where $\{L_t^a\}_{t\geq 0}$ is the local time process (at level a) of the continuous semimartingale ξ_i. (See page 211 [18] for a definition.) This proves (A2) and hence (A1) follows.

We next show that

$$P_x[Z(t) \in \partial S] = 0, \quad \forall x \in S, \quad t > 0. \tag{A4}$$

Suppose first that $x \in S^\circ$. Let $\eta \doteq \inf\{t > 0 : Z(t) \in \partial S\}$. Without loss of generality we can assume that on the filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0})$ we have probability measures $\{Q_x\}_{x\in S}$ such that under Q_x, Z has the same law as the unconstrained diffusion:

$$X(t) = x + \int_0^t b(X(s)) ds + \int_0^t \sigma(X(s)) dB(s).$$

From uniform nondegeneracy of σ (Condition 5.4) we have that the measure $m_x \doteq Q_x \circ (\eta, Z(\eta))^{-1}$ on $[0, \infty) \times \partial S$ is absolutely continuous with respect to the Lebesgue measure. Now from the strong

Markov property,

$$\begin{aligned} P_x[Z(t) \in \partial S] &= P_x[\eta \le t, Z(t) \in \partial S] \\ &= \int_{[0,t]\times\partial S} P_y[Z(t-u) \in \partial S] m_x(du, dy) = 0, \end{aligned}$$

where the last equality is an immediate consequence of (A1).

Finally consider $x \in \partial S$. Then since (A4) holds for $x \in S^\circ$ we have from Markov property that for all $s < t$, $P_x[Z(t) \in \partial S] = P_x[Z(t) \in \partial S, Z(s) \in \partial S]$. Since $s \in (0,t)$ is arbitrary, we get $P_x[Z(t) \in \partial S] = P_x[Z(t) \in \partial S, Z(q) \in \partial S, \quad \forall q \in \mathbb{Q} \cap [0,t)]$, where $\mathbb{Q}$ is the set of rational numbers. Sample path continuity of Z now gives

$$P_x[Z(t) \in \partial S] = P_x[Z(s) \in \partial S, \quad \forall 0 \le s \le t].$$

However the last expression is 0 from (A1). This proves (A4).

We next show that for every $x \in S$ and $t > 0$, $m_{t,x} \doteq P_x \circ Z(t)^{-1}$ is mutually absolutely continuous with respect to the Lebesgue measure λ on $(S, \mathcal{B}(S))$. We begin by noting that from the nondegeneracy of σ it follows that for all $y \in S$, under Q_y, (Z_t, η) has a nowhere vanishing joint density on $I\!R^d \times (0, \infty)$, for every $t > 0$. In particular for all $y \in S$ and $t > 0$,

$$Q_y(Z_t \in A, \eta > t) = 0 \Leftrightarrow \lambda(A) = 0. \tag{A5}$$

We first show that $m_{t,x} \ll \lambda$. Let $A \in \mathcal{B}(S)$ be such that $\lambda(A) = 0$. In proving $m_{t,x}(A) = 0$ we can assume in view of (A4) that A is contained in a compact subset of S°. Introduce sequences of stopping times as follows. Let $\sigma_0 \doteq \inf\{t : Z_t \in A\}$, and for $n \ge 1$, $\tau_n \doteq \inf\{t \ge \sigma_{n-1} : Z_t \in \partial S\}$ and $\sigma_n \doteq \inf\{t \ge \tau_n : Z_t \in A\}$. Then

$$\begin{aligned} m_{t,x}(A) &= P_x[Z_t \in A] = \sum_{n=0}^{\infty} P_x[Z_t \in A, t \in [\sigma_n, \tau_{n+1})] \\ &= \sum_{n=0}^{\infty} \int_{[0,t)\times A} Q_y[Z_{t-r} \in A, \eta \ge t-r] dm^n(r, y), \end{aligned} \tag{A6}$$

where m^n is the joint law of (σ_n, Z_{σ_n}) and the last equality is a consequence of the strong Markov property of $(Z, \{P_x\})$ and the observation that $P_x \circ (Z_{\cdot\wedge\eta}, \eta)^{-1} = Q_x \circ (Z_{\cdot\wedge\eta}, \eta)^{-1}$ for all $x \in S^\circ$. Now since $\lambda(A) = 0$, we have from (A5) that the right side of (A6) is zero and thus we have shown that $m_{t,x} \ll \lambda$. Conversely, let $A \in \mathcal{B}(S)$ and $(t, x) \in (0, \infty) \times S$ be such that $\lambda(A) > 0$. Once more, since $\lambda(\partial S) = 0$, for purposes of establishing $m_{t,x}(A) > 0$, we can assume without loss of generality that A is contained in a compact subset of S°. The desired inequality is now an immediate consequence of (A5) and (A6). ■

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