Bounds on exponential moments of hitting times for reflected processes on the positive orthant

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Abstract

We first consider a multi-dimensional reflected fractional Brownian motion process on the positive orthant with the Hurst parameter $H \in (0, 1)$. In particular, when $H > 1/2$, this model serves to approximate fluid stochastic network models fed by a big number of heavy tailed ON/OFF sources in heavy traffic. Assuming the initial state lies outside some compact set, we establish that the exponential moment of the first hitting time to the compact set has a lower bound with an exponential growth rate in terms of the magnitude of the initial state. We extend this result to the case for reflected processes driven by a class of stable Lévy motions, which arise as approximations to cumulative network traffic over a time period. For the case of $H = 1/2$, under a natural stability condition on the reflection directions and drift vector, we obtain a matching upper bound on exponential moments of hitting times, which grows at an exponential rate in terms of the initial condition of the process. We also show that such an upper bound is valid for reflected processes driven by general light-tailed Lévy processes.

Keywords: Reflected fractional Brownian motion, reflected Lévy process, heavy traffic theory, first hitting times, exponential moments.
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1. Introduction

We first consider a multi-dimensional reflected fractional Brownian motion process $\{Z(t) : t \geq 0\}$ (abbreviated as RFBM hereafter) on the positive orthant $S := \mathbb{R}^d_+$, with drift $r^0 \in \mathbb{R}^d$ and the Hurst parameter $H \in (0, 1)$. For example, when the Hurst parameter H is greater than $1/2$, the RFBM model serves as an approximation for stochastic networks with a large number of heavy-tailed ON/OFF sources (Delgado (2007)), or with long range dependent arrival and service time processes (Majewski (2005)) in the heavy traffic regime (with the dimension d being the number of nodes

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or servers). This model was further studied in the subsequent papers Delgado (2008, 2010). Besides Delgado (2007), the multi-dimensional RFBM process has also been obtained as an approximating model of the queue-length process in Konstantopoulos and Lin (1996), in which a single-class queueing network with long-range dependent arrival and service processes is considered. Recently, stationarity and control problems of a tandem fluid network with fractional Brownian motion input (a two-dimensional RFBM model in the first quadrant) were studied in Lee and Weerasinghe (2011) via a coupling time approach. From the viewpoint of the applications, it is of great importance to study the basic time behavior of the RFBM process. However, relatively little has been known about the properties of RFBM, since a fractional Brownian motion (FBM) is neither a semi-martingale nor a Markov process, many techniques from the classical theory of stochastic calculus are inapplicable to its analysis.

In this work, we first establish a lower bound growth estimate on the expected exponential first hitting time for RFBM process $\{Z(t) : t \geq 0\}$ on the orthant S; more precisely, it is proven that for any $\beta \in (0, \infty)$ there exists a $\gamma \in (0, \infty)$ such that

$$\mathbb{E}_x[e^{\beta \tau_B}] \geq e^{\gamma |x|} \quad \text{for all } x \in S \backslash D, \tag{1}$$

where $\tau_B := \inf\{t \geq 0 : Z(t) \in B\}$ for some compact sets B and D in S, centered at the origin, satisfying $B \subset D$ and a constant $\gamma = \gamma(H) > 0$, which depends on the Hurst parameter $H \in (0, 1)$. Here (and throughout the paper), $\mathbb{E}_x$ denotes the expectation conditional on the process Z starting from $x \in S$. The result (1) implies that the expectation of the hitting time random variable τ_B has *at least* a linear growth rate, proportional to the distance of the initial state from the origin. We refer the reader to Lee (2011b,a) for the related uniform return time (to some compact sets) results and a basic geometric drift inequality result of the RFBM process on the positive orthant. Our analysis reveals that such a lower bound result can be extended to the case for reflected processes driven by a class of stable Lévy motions, which arise as approximations to cumulative network traffic over a time period (cf. Mikosch et al. (2002) and the references therein).

Next, when $H = 1/2$, we obtain a matching upper bound on exponential moments of hitting times, which grows also at an exponential rate in terms of the initial condition of the process. Namely, we prove under a natural stability condition on the reflection directions and drift vector that there exist $\beta \in (0, \infty)$ and $\alpha_1, \alpha_2 \in (0, \infty)$ such that

$$\mathbb{E}_x[e^{\beta \tau_B}] \leq \alpha_1 e^{\alpha_2 |x|} \quad \text{for all } x \in S \backslash D, \tag{2}$$

where the sets B and D satisfy the same conditions as before. We were unable to establish an upper bound estimate (2) for a general $H \in (0, 1)$, mainly due to the lack of (strong) Markov and independent incremental properties of FBM when $H \neq 1/2$ (see Remark 3.8). However, we establish that such an upper bound is valid for reflected processes driven by general light-tailed Lévy processes; see Corollary 3.9.

The study of the first hitting times of reflected processes is important both in its own right and for the sake of its applications to several areas such as engineering and finance; we refer the reader to Chapter 3 in Harrison (1985), Chapter 7 in Karlin and Taylor (1975), and Budhiraja and Lee (2007)) for several examples of related applications. However, the computation of the moments of hitting times functionals driven by FBMs, seems to be quite challenging in general. For example, commonly used numerical techniques using suitable Markov chain approximations (cf. Chapter 7, Section 4 of Ethier and Kurtz (1986)) for the reflected processes (and then using the transition matrix of the chain to approximate the probability distribution, and hence the moments of the hitting time) are inapplicable, since FBM is not Markov. Also, a linear programming method introduced in Helmes et al. (2001), which is based on the martingale characterization of wide range of stochastic processes via its generator, does not apply to here either, since FBM is not a martingale.

The organization of the paper is as follows. In Section 2, we carefully describe our model in Definition 2.1 and make a standard assumption on reflection matrix (see **(HR0)** in Section 2), which is used in heavy traffic analysis for invoking a functional central limit theorem in Delgado (2007, 2008). We present the main results in Section 3. The proof of the lower bound estimate (Theorem 3.1) is based on oscillation inequality for reflected processes together with a maximal inequality for FBM with $H \in (0,1)$, which leads to an appropriate tail probability estimate of the running supremum of the RFBM process (see (10)) in terms of its initial condition. An explicit computation on the lower bound for the case of simple one-dimensional example is then provided. The upper bound estimate (Theorem 3.7) is obtained from a tail probability estimate with exponential decay rates (24), which is proved by making use of a suitable Lyapunov function $T(\cdot) : S \to [0, \infty)$ and its basic properties. Also, an exponential moment estimate (22) on the running supremum of the standard Brownian motion increments is crucially used in the proof.

We use the following notation. The set of positive integers is denoted by $I\!N$, the set of real numbers by $I\!R$ and non-negative real numbers by $I\!R_+$. For $a \in I\!R$, let $a^+ = \max\{a, 0\}$. Let $I\!R^d$ be the d-dimensional Euclidean space and for $x \in I\!R^d$ the max-norm of x, i.e., $\max_{1\le i\le d} |x_i|$, will be denoted by $|x|$. Let $\mathbb{I} = \mathbb{I}_{d\times d}$ denote the identity matrix for some $d \in I\!N$. When it is clear from the context, we will omit the subscript. For a set $A \subseteq I\!R^d$, denote its interior and boundary by A° and ∂A, respectively, and $1_C(\cdot)$ will denote an indicator function on a set C. For sets $A, B \subseteq I\!R^d$, $\mathrm{dist}(A, B)$ will denote the distance between the two sets, i.e., $\inf\{|x - y| : x \in A, y \in B\}$. Let $\mathcal{D}(X, Y)$ denote the class of right continuous functions with having left limit defined from X to Y, equipped with the usual Skorokhod topology. Inequalities for vectors are interpreted componentwise.

2. Model and assumptions

We begin by the definitions of multi-dimensional FBM and RFBM. Let $d \in I\!N$. A stochastic process $B_H = \{B_H(t) = (B_H^{(1)}(t), \ldots, B_H^{(d)}(t))^T, t \ge 0\}$, defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, I\!\!P)$, is called a d-dimensional FBM of Hurst parameter $H \in (0,1)$, starting from $B_H(0) \in I\!\!R^d$, and associated matrix Λ, if it is a continuous Gaussian process with initial condition $B_H(0)$ $I\!\!P$-a.s. and with covariance function given by

$$\mathrm{Cov}(B_H(t), B_H(s)) = I\!\!E\left((B_H(t) - B_H(0))(B_H(s) - B_H(0))^T\right) = \Lambda_H(s,t)\Lambda,$$

for any $s, t \geq 0$. Here, Λ is a $d \times d$ positive definite matrix and

$$\Lambda_H(s,t) := \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

We assume that B_H is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. We will say that B_H is a d-dimensional FBM with associated data $(B_H(0), H, \Lambda)$.

Fix column vectors $r^0, r^1, \ldots, r^d \in I\!\!R^d$ and let $R := \left[r^1, \ldots, r^d\right]_{d\times d}$. We call the quintuple $(B_H(0), H, \Lambda, r^0, R)$ as the *data* for a reflected FBM. The following definition is similar to that of Delgado (2007) (cf. Definition 2 and Theorem 1 of Delgado (2007)). We refer the reader to a recent paper Lee and Weerasinghe (2011) for a precise derivation of the following non-zero drift RFBM process in the context of controlled ON/OFF fluid model with heavy tails under heavy traffic assumptions.

Definition 2.1. (RFBM) For $x \in S$, a reflected fractional Brownian motion associated with the data (x, H, Λ, r^0, R) is a continuous d-dimensional process Z, defined on some probability space $(\Omega, \mathcal{F}, I\!\!P)$, such that

(i) $Z(t) = x + B_H(t) + r^0 t + RY(t) \in S \quad \text{for all} \quad t \geq 0, \quad I\!\!P\text{-}a.s.$,

(ii) B_H is a d-dimensional FBM with data $(0, H, \Lambda)$,

(iii) Y is a d-dimensional process such that $Y_i(0) = 0$ for $i = 1, \ldots, d$, $I\!\!P$-a.s. For each $i = 1, \ldots, d$, Y_i is continuous, non-decreasing and Y_i can increase only when $Z(\cdot)$ is on the face $F^i = \{x \in S : x_i = 0\}$, *i.e.*, $\int_0^t 1_{\{Z_i(s)\neq 0\}} dY_i(s) = 0$ for all $t \geq 0$.

For $y \in \partial S$, the set of directions of reflections is defined as:

$$r(y) := \left\{\sum_{i=1}^d q_i r^i : \sum_{i=1}^d q_i = 1,\ q_i \geq 0,\ \text{and } q_i > 0 \text{ only if } y_i = 0\right\}. \tag{3}$$

Roughly speaking, an RFBM introduced above behaves like a FBM in the interior of the orthant S and it is confined to the orthant by an instantaneous "reflection" (or "pushing") at the boundary ∂S. For each i, the i-th column of the *reflection matrix* R gives the direction of the reflection on the i-th face F^i. Specifically, if the boundary F^i is hit, it is Y_i that increases, the direction of displacement is given by r^i, the i-th column of R, and the magnitude of the displacement is the minimal amount required to keep Z_i nonnegative. At an intersection of faces, the allowed directions of reflection are given by the convex combinations of the reflection directions associated with the faces meeting there. We refer the reader to Williams (1998) and references therein for the related definition and properties of semimartingale reflecting Brownian motions in an orthant.

Next, we introduce an important notion related to the reflection matrix R.

Definition 2.2. *We call a square matrix R is completely-$\mathcal{S}$ if for every $k \times k$ principal submatrix G of R, there is a k-dimensional vector $v_G \geq 0$ such that $G v_G > 0$.*

The completely-$\mathcal{S}$ condition on the reflection matrix R ensures that for every $y \in \partial S$, there exists a convex combination of vectors in $r(y)$ (see (3)) which points into S° from y. Also, the completely-$\mathcal{S}$ property is sufficient to ensure the existence of a pair (Z, Y) satisfying (i) and (iii) in Definition 2.1 (cf. Theorem 2 in Bernard and El Kharroubi (1991)). However, this property does not ensure the adaptness of the "pushing process" Y to a filtration to which B_H is adapted. This problem is overcome under a stronger assumption on R, that we quote below (cf. Proposition 4.2 in Williams (1998) and Section 2 of Delgado (2007)).

We will impose throughout that the matrix R verifies the following assumption:

(HR0) The reflection matrix R can be expressed as $\mathbb{I} + \Theta$, with Θ a $d \times d$ matrix such that $|\Theta|$, that is the matrix obtained from Θ by replacing all the entries in Θ by their absolute values, has spectral radius $r(|\Theta|)$ strictly less than 1.

Remark 2.3. *The assumption (HR0) implies that R is completely-$\mathcal{S}$ and invertible. Also, it can be shown (cf. Williams (1998); Delgado (2007)) that if B_H is adapted to some filtration $\{\mathcal{F}_t : t \geq 0\}$, then (Z, Y) is adapted to a filtration $\{\mathcal{G}_t : t \geq 0\}$, with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{N}$, where $\mathcal{N}$ denotes the collection of $I\!P$-null sets in $\mathcal{F}$. Furthermore, (HR0) is a sufficient condition for strong pathwise uniqueness of a solution of the* Skorokhod Problem *described below. Henceforth, with an abuse of notation, we will assume that (Z, Y) is adapted to the filtration $\{\mathcal{F}_t : t \geq 0\}$.*

Definition 2.4. (Skorokhod Problem) Let $\psi \in \mathcal{D}(I\!R_+, I\!R^d)$ be given with $\psi(0) \in S$. Then $(\phi, \eta) \in \mathcal{D}(I\!R_+, I\!R^d) \times \mathcal{D}(I\!R_+, I\!R^d)$ solves the Skorokhod problem for ψ with respect to S and R if and only if the following hold:

(i) $\phi(t) = \psi(t) + R\eta(t) \in S$ for all $t \geq 0$;

(ii) η satisfies that, for $1 \leq i \leq d$,

   (a) $\eta_i(0) = 0$,
   (b) η_i is non-decreasing, and
   (c) η_i can increase only when ϕ is on the i-th face of S, that is $\int_0^\infty 1_{\{\phi_i(s) \neq 0\}} d\eta_i(s) = 0$.

Let $\mathcal{D}_S(\mathbb{R}_+, \mathbb{R}^d) := \{\psi \in \mathcal{D}(\mathbb{R}_+, \mathbb{R}^d) : \psi(0) \in S\}$. On the domain $D \subset \mathcal{D}_S(\mathbb{R}_+, \mathbb{R}^d)$ on which there is a unique solution to the Skorokhod problem, we define the Skorokhod map Γ as

$$\Gamma(\psi) := \phi, \tag{4}$$

if $(\phi, R^{-1}[\phi - \psi])$ is the unique solution of the Skorokhod problem posed by ψ. Equivalent form of RFBM in Definition 2.1 in terms of the Skorokhod map can now be written as follows:

$$Z = \Gamma(x + B_H + r^0 \imath), \quad Z - (x + B_H + r^0 \imath) = RY,$$

where $\imath : [0, \infty) \to [0, \infty)$ is the identity map, i.e., $\imath(t) := t$ for $t \in [0, \infty)$.

3. Bounds on the exponential moment of hitting times

3.1. Lower bound estimate when $0 < H < 1$

We present our main result about the lower bound on the exponential moment of the first hitting time to some compact set.

Theorem 3.1. *Let Z_x be defined by Definition 2.1 with $Z_x(0) = x \in S$ and the Hurst parameter $H \in (0, 1)$. Then for any $\beta \in (0, \infty)$, there exists a constant $\gamma \in (0, \infty)$ such that*

$$\mathbb{E}_x[e^{\beta \tau_B}] \geq e^{\gamma |x|} \quad \textit{for all } x \in S \backslash D, \tag{5}$$

where $\tau_B := \inf\{t \geq 0 : Z(t) \in B\}$ for some compact sets B and D in S satisfying $B \subset D$, and a constant $\gamma = \gamma(H) > 0$ depends on the Hurst parameter $H \in (0, 1)$.

Proof. By Jensen's inequality, $\mathbb{E}_x[e^{\beta \tau_B}] \geq e^{\beta \mathbb{E}_x[\tau_B]}$. For the lower bound on $\mathbb{E}_x[\tau_B]$, we fix constants $M_0 \in (0, \infty)$ and $\epsilon_0 \in (0, 1)$. Define a set $B := \{x \in S : |x| \leq M_0\}$ and let $M = M(\epsilon_0)$ be large enough so that for all $|x| > M$, $\text{dist}(x, B) \geq \epsilon_0 |x|$. Given the number $M \in (0, \infty)$, define a set $D := \{x \in S : |x| \leq M\}$. Then for all $\vartheta \in (0, 1)$ and $x \in S \backslash D$, we have

$$\begin{aligned} \mathbb{E}_x[\tau_B] &\geq \mathbb{E}_x\left[\tau_B \cdot 1\left\{\sup_{0 \leq s \leq \vartheta |x|} |Z(s) - x| \leq \epsilon_0 |x|\right\}\right] \\ &\geq \vartheta |x| \mathbb{P}_x\left[\sup_{0 \leq s \leq \vartheta |x|} |Z(s) - x| \leq \epsilon_0 |x|\right] \\ &= \vartheta |x| \left(1 - \mathbb{P}_x\left[\sup_{0 \leq s \leq \vartheta |x|} |Z(s) - x| \geq \epsilon_0 |x|\right]\right), \end{aligned} \tag{6}$$

since on the event $A = \left\{\omega \in \Omega : \sup_{0 \leq s \leq \vartheta |x|} |Z(s) - x| \leq \epsilon_0 |x|\right\}$, the process $\{Z(t)\}_{t \geq 0}$ has not yet hit the ball B, and hence τ_B on A is bounded below by $\vartheta |x|$. Here, $\mathbb{P}_x$ denotes the probability measure conditional on the process Z starting from $x \in S$.

Next, we recall the following inequality which follows from the oscillation result, Theorem 5.1 of Williams (1998) (see also Bernard and El Kharroubi (1991)), based on the completely-$\mathcal{S}$ condition on the reflection matrix R: There exists a constant $C_0 \in (0, \infty)$ such that for all $x \in S$ and $0 \leq t_1 < t_2 < \infty$,

$$\sup_{t_1 \leq s \leq t \leq t_2} |Z(s) - Z(t)| \leq C_0 \left[\sup_{t_1 \leq s \leq t \leq t_2} |B_H(t) - B_H(s)| + (t_2 - t_1) \right], \quad I\!\!P_x\text{-}a.s., \tag{7}$$

where the constant $C_0 > 0$ depends only on the reflection matrix R. (We note that the oscillation inequality result given in Theorem 5.1 of Williams (1998) is of deterministic nature and relies mainly on the path properties of reflected processes.) Then, by Markov's inequality and (7), we have the inequalities in

$$I\!\!P_x \left[\sup_{0 \leq s \leq \vartheta|x|} |Z(s) - x| \geq \epsilon_0 |x| \right] \leq \frac{I\!\!E_x \left[\sup_{0 \leq s \leq \vartheta|x|} |Z(s) - x| \right]}{\epsilon_0 |x|} \leq \frac{C_0 I\!\!E_x \left[\sup_{0 \leq s \leq \vartheta|x|} |B_H(s)| \right] + C_0 \vartheta |x|}{\epsilon_0 |x|}, \tag{8}$$

where we have used the fact $B_H(0) = 0$ as in Definition 2.1(ii).

We then obtain an estimate on the first moment of the right-hand side of (8). For $H \in (0, 1)$ and $p > 0$, it is known from Corollary 3.1 of Yan (2004) (see also, Theorem 1.2 of Novikov and Valkeila (1999) and Ex. 5.1.5 in Nualart (2006) for related maximal inequalities) that

$$I\!\!E_x \left[\sup_{0 \leq s \leq t} |B_H^{(i)}(s)| \right]^p \leq b_p t^{pH}, \tag{9}$$

where $i = 1, \ldots, d$ and $b_p \in (0, \infty)$ is a constant which depends only on p. Applying this estimate in (8) with $p = 1$, we now have that for all $x \in S \backslash D$,

$$I\!\!P_x \left[\sup_{0 \leq s \leq \vartheta|x|} |Z(s) - x| \geq \epsilon_0 |x| \right] \leq \frac{d C_0 b_1 \vartheta^H |x|^{H-1} + C_0 \vartheta}{\epsilon_0} \tag{10}$$

$$\leq C_1 \vartheta^H, \tag{11}$$

where $C_1 \in (0, \infty)$ is a constant and the second inequality follows from the fact that $|x| > M$, $H \in (0, 1)$, and $\vartheta \in (0, 1)$. Notice that the constant $C_1 \in (0, \infty)$ is independent of the initial condition $x \in S$, and without loss of generality we can take $C_1 \in (1, \infty)$.

Combining (6) and (10), we obtain

$$I\!\!E_x[\tau_B] \geq \vartheta |x| - C_1 \vartheta^{H+1} |x| \quad \text{for all } x \in S \backslash D. \tag{12}$$

Next step is to maximize the right side of (12) with respect to $\vartheta \in (0, 1)$ to establish the sharper lower bound. To simplify the notation, we introduce the function $g(\vartheta) = \vartheta|x| - C_1\vartheta^{H+1}|x|$ for all $\vartheta \in (0,1)$. Notice that $g(\cdot)$ is differentiable and $g''(\vartheta) = -H(H+1)C_1|x|\vartheta^{H-1} < 0$. Hence, $g(\vartheta)$ is maximized at ϑ_0, where $g'(\vartheta_0) = 0$.

After simple algebraic manipulations, we obtain that

$$\vartheta_0 = \left(\frac{1}{C_1(H+1)}\right)^{\frac{1}{H}} \in (0,1). \tag{13}$$

Hence,

$$\mathbb{E}_x[\tau_B] \geq g(\vartheta_0) = |x|\vartheta_0(1 - C_1{\vartheta_0}^H) > 0,$$

where the positivity follows from the definition of ϑ_0 in (13). By setting

$$\gamma := \beta\vartheta_0(1 - C_1{\vartheta_0}^H) \in (0,\infty),$$

we complete the proof. □

Example 3.2. *For a simple one-dimensional case ($d = 1$), we can explicitly compute the constant $\gamma \in (0,\infty)$ in* (3.1) *as follows. We recall that the Skorokhod map in* (4) $\Gamma : \mathcal{D}(\mathbb{R}_+, \mathbb{R}) \to \mathcal{D}(\mathbb{R}_+, \mathbb{R})$ *is specifically given by*

$$\Gamma(f)(t) = f(t) + \sup_{s\in[0,t]} (-f(s))^+ \quad \textit{for all } t \geq 0$$

from, e.g., Skorokhod (1961). In this case, the constant C_0 in (7) *is known to equal to* $\max(2,1) = 2$ *(see page 15 in Williams (1998)). Also, the constant $b_1 \in (0,\infty)$ in* (10) *is given by $(e + e\cdot 1)^{(1+1)/1} = 4e^2$ from Corollary 3.1 of Yan (2004) (see also page 50 in Novikov and Valkeila (1999) for the case $H \in (1/2,1)$). Putting these together, we can compute the constant $\gamma = \beta\vartheta_0(1 - C_1{\vartheta_0}^H)$ by specifying*

$$C_1 = \frac{8e^2}{\epsilon_0 M^{1-H}} + \frac{2}{\epsilon_0}.$$

For example, if $\epsilon_0 = \frac{1}{2}$ and $M = 1$ then C_1 equals to $4(4e^2+1)$.

Remark 3.3. *Unlike the recent results established in Lee (2011b,a), we note that the proof of Theorem 3.1 does not make use of the Lipschitz continuity of the Skorokhod map $\Gamma(\cdot)$. Therefore, it provides wider applicability, since there is a rich collection of multiclass queueing networks for which the corresponding diffusion limit is not given in terms of the Skorokhod map with the Lipschitz continuity property.*

3.2. Lower bound estimate for reflected stable Lévy motion

We proceed with extending the result of Theorem 3.1 to reflected processes driven by an α-stable Lévy motion, which is a special case of Lévy process. More precisely, we focus on the α-stable Lévy motion $\{X(t) \in \mathbb{R}^d : t \geq 0\}$ with the index of stability

$\alpha \in (1,2)$ and the skewness parameter $\beta = 1$ (i.e, X_i has no negative jumps). Such a process arises as an approximation to cumulative traffic over a time period if connection rates are modest relative to heavy tailed connection length distribution tails; see Section 4, Theorem 1 of Mikosch et al. (2002) for more details. Furthermore, Section 8.5 in Whitt (2002) describes in detail on the functional central limit theorem obtaining convergence of normalized workload processes for multisource ON/OFF fluid queues to reflected stable Lévy motion. We recall that except in the cases when $\alpha = 1$ and $\beta \neq 1$, the α-stable Lévy motion is self-similar with self-similarity index $1/\alpha$.

Corollary 3.4. *For $x \in S$, let $\{Z_x(t) : t \geq 0\}$ be a reflected process on the positive orthant S with a d-dimensional α-stable Lévy motion $\{X(t) = (X_1(t), \ldots, X_d(t))' : t \geq 0\}$ as its driving noise (i.e., replacing FBM B_H by X in Definition 2.1). Assume $X_1, \ldots, X_d$ are mutually independent, centered stable Lévy motions with the index of stability $\alpha \in (1,2)$, the scale parameter $\sigma = 1$ and the skewness parameter $\beta = 1$. Then, the result of Theorem 3.1 holds.*

Proof. All that is needed to complete the proof is the analogous maximal inequality (9) for α-stable Lévy motion $\{X(t)\}$. If we let $M_t := \sup_{0 \leq s \leq t} |X(s)|$, then the self-similarity property of $\{X(t)\}$ implies $\mathbb{E}[M_t] = t^{1/\alpha}\mathbb{E}[M_1]$. It can be checked that $\mathbb{E}[M_1] < \infty$, since $\mathbb{P}(\sup_{0 \leq s \leq 1} |X_i(s)| > u) \sim ku^{-\alpha}$ as u goes to infinity, where $i = 1, \ldots, d$ and $k > 0$ is a constant satisfying $\mathbb{P}(X_i(1) > u) \sim ku^{-\alpha}$ (cf. Exercise VIII.2 of Bertoin (1996)). Then, the estimates in (10)–(13) remain to hold with H being replaced by $1/\alpha \in (1/2, 1)$. Hence, by setting $\gamma := \beta\vartheta_0(1 - C_1{\vartheta_0}^{1/\alpha}) \in (0, \infty)$, we complete the proof. □

3.3. Upper bound estimate when $H = 1/2$

Establishing an upper bound estimate on the hitting times seems to be considerably more challenging problem. This is mainly due to the lack of (strong) Markov property of the driving noise process FBM when $H \neq 1/2$; see Remark 3.8 below for more details. However, for the case of $H = 1/2$ (namely, reflected Brownian motion process), we are able to establish an upper bound estimate which has an exponential growth in terms of the initial condition $x \in S$.

For the upper bound estimate when $H = 1/2$, we need to impose a natural stability condition on the drift vector and directions of reflection. The following assumption (S) is known as a necessary and sufficient condition for the existence of a unique stationary distribution for the reflected Brownian motion process Z. We refer the reader to Harrison and Williams (1987) and references therein for the related asymptotic stability results of reflected Brownian motions (RBMs) on a positive orthant.

(S) Stability assumption on r^0 and R:
There exists a constant $\bar{\theta} > 0$, such that $\max_{1 \leq i \leq d}[R^{-1}r^0]_i \leq -\bar{\theta}$.

Furthermore, we impose the following condition on the matrix R.

(HR1) Assume (HR0) condition on the matrix R. Additionally, assume that the matrix $\Theta = (\theta_{ij})$ in (HR0) satisfies the condition $\theta_{ij} \le 0$ and $\theta_{ii} = 0$ for $1 \le i, j \le d$.

The condition (HR1) verifies the so-called *Harrison-Reiman condition* made in Harrison and Reiman (1981). As a consequence, (HR1) will imply that a solution to the Skorokhod Problem exists, and moreover the Skorokhod map $\Gamma(\cdot)$ is Lipschitz continuous in the following sense: There exists a constant $K \in (0, \infty)$ such that for all $\psi_1, \psi_2 \in \mathcal{D}_S(I\!R_+, I\!R^d)$ and $t \ge 0$,

$$\sup_{0 \le s \le t} |\Gamma(\psi_1)(s) - \Gamma(\psi_2)(s)| < K \sup_{0 \le s \le t} |\psi_1(s) - \psi_2(s)|. \tag{14}$$

We refer the reader to pages 163–165 and 200–208 in Dupuis and Ramanan (1999) for the generalized Harrison-Reiman condition for Lipschitz continuity of the Skorokhod map. Also, we notice that the condition (S) can be translated into the following cone condition: denoting by $\mathcal{C} := \{v \in I\!R^d : R^{-1}v \le 0\}$, it can be seen that there exists a $\delta \in (0, \infty)$ satisfying

$$\mathrm{dist}(r^0, \partial\mathcal{C}) \ge \frac{1}{||R^{-1}||}\bar{\theta} d =: \delta > 0, \tag{15}$$

where $\bar{\theta} \in (0, \infty)$ is as in condition (S). Thus,

$$r^0 \in \mathcal{C}(\delta) := \{v \in \mathcal{C} : \mathrm{dist}(v, \partial\mathcal{C}) \ge \delta\}. \tag{16}$$

For $x \in S$, denote by $\mathcal{A}_\delta(x)$ the collection of all trajectories $\phi_\delta : [0, \infty) \to S$ of the form $\phi_\delta(t) := \Gamma(x + \zeta\imath)(t), t \ge 0$, where ζ ranges over all of $\mathcal{C}(\delta)$ and $\imath : [0, \infty) \to [0, \infty)$ is the identity map, $\imath(t) = t$. For a fixed $x \in S$, we now define the "hitting time to the origin" function $T : S \to [0, \infty)$ as follows:

$$T(x) := \sup_{\phi_\delta \in \mathcal{A}_\delta(x)} \inf\{t \in [0, \infty) : \phi_\delta(t) = 0\}. \tag{17}$$

This function played a key role in the proof of positive recurrence of certain constrained diffusion processes in polyhedral domains studied in Atar et al. (2001). We have the following result from Atar et al. (2001) (cf. Lemma 3.1 and Lemma 4.1 therein), which will be key in our analysis.

Lemma 3.5. *Let Z_x be defined by Definition 2.1 with $Z_x(0) = x \in S$ and $H = 1/2$. Assume (S) and (HR1). Then, the function $T : S \to [0, \infty)$ defined in* (17) *satisfies the following properties.*

(i) For some $c_1, c_2 \in (0, \infty)$, $c_1|x| \le T(x) \le c_2|x|$ for all $x \in S$.

(ii) For some $c_3 \in (0, \infty)$,

$$T(Z_x(t)) \le [T(x) - t]^+ + c_3\xi_t, \tag{18}$$

for all $t \ge 0$, $I\!P_x$-a.s., for all $x \in S$, and $\xi_t := \sup_{0 \le s \le t} |B(s)|$, where $(B(t) : t \ge 0)$

is a d-dimensional standard Brownian motion.

Here, the positive constants c_i $(i = 1, 2, 3)$ *depend only on* $K \in (0, \infty)$ *and* $\delta \in (0, \infty)$, *appeared in* (14) *and* (15), *respectively.*

Remark 3.6. *In Atar et al. (2001), the above results and other long time asymptotic properties are obtained for a broad family of constrained diffusion models (taking values in some polyhedral cone in* $I\!R^d$*) with state dependent coefficients, under a natural condition on the drift vector field such as* (16) *and the Lipschitz condition* (14). *Also, we remark that the set* $A \subset S$ *appearing in Condition 2.3 of Atar et al. (2001) is an empty set in our case in view of the condition* (16).

The proof of the following result is adapted from that of Theorem 4.1 in Atar et al. (2001) by accommodating our basic model assumptions.

Theorem 3.7. *Let* Z_x *be defined by Definition 2.1 with* $Z_x(0) = x \in S$ *and* $H = 1/2$. *Assume (S) and (HR1). Then, there exist* $\beta \in (0, \infty)$ *and* $\alpha_1, \alpha_2 \in (0, \infty)$ *such that*

$$I\!E_x[e^{\beta\tau_B}] \le \alpha_1 e^{\alpha_2 |x|} \quad \text{for all } x \in S \backslash D, \tag{19}$$

where $\tau_B = \inf\{t \ge 0 : Z(t) \in B\}$ *for some compact sets* B *and* D *in* S *satisfying* $B \subset D$.

Proof. Fix a constant $M_0 \in (0, \infty)$ and define a set $B := \{x \in S : |x| \le M_0\}$. In view of Lemma 3.5 (i), we can choose $L \in (0, \infty)$ large enough so that $D := \{x \in S : |T(x)| \le L\} \supset B$. For $n \in I\!N$, define

$$A_n := \left\{ \omega \in \Omega : \inf_{s \in [0, n\Delta]} |T(Z(s))| > L \right\},$$

where $\Delta \in (0, \infty)$ is a fixed constant. For $m \in I\!N$, let ν_m be defined as follows:

$$\nu_m := \sup_{(m-1)\Delta \le t \le m\Delta} |B(t) - B((m-1)\Delta)|,$$

where $(B(t) : t \ge 0)$ is a d-dimensional standard Brownian motion. Let $x \in S \backslash D$. Then for $1 \le m \le n$, and on the set A_n, it follows that

$$L < T(Z_x(m\Delta)) \le T(Z_x((m-1)\Delta)) - \Delta + c_3 \nu_m, \tag{20}$$

where a constant $c_3 \in (0, \infty)$ is as in (18). For $m = 1, \ldots, n$ iterating inequality (20) we have that, on A_n,

$$L < T(Z_x(n\Delta)) \le T(x) - n\Delta + c_3 \sum_{j=1}^{n} \nu_j.$$

Hence we have for an arbitrary $\alpha > 0$,

$$\begin{aligned} I\!P_x(A_n) &\le I\!P_x\left(L < T(Z_x(n\Delta)) \le T(x) - n\Delta + c_3 \sum_{j=1}^n \nu_j\right) \\ &\le I\!P_x\left(c_3 \sum_{j=1}^n \nu_j \ge n\Delta - T(x) + L\right) \\ &\le \frac{I\!E_x\left(\exp\left\{\alpha c_3 \sum_{i=1}^n \nu_i\right\}\right)}{\exp\left\{\alpha(n\Delta - T(x) + L)\right\}} \le \frac{\left[2\sqrt{2}\exp\left\{\alpha^2 c_3^2 d^2 \Delta\right\}\right]^n}{\exp\left\{\alpha(n\Delta - T(x) + L)\right\}}, \end{aligned} \tag{21}$$

where the last inequality follows from the Doob's maximal inequality for submartingales and successive conditioning arguments. Namely, it can be verified that for any $\kappa \in (0,\infty)$ and $m, n \in I\!N$ such that $m \le n$,

$$I\!E_x\left(\exp\left\{\kappa \sum_{i=m}^n \nu_i\right\}\right) \le \left[2\sqrt{2}\exp\{\Delta d^2 \kappa^2\}\right]^{(n-m+1)} \tag{22}$$

(see, e.g., Lemma 4.3 in Atar et al. (2001)). We rewrite the expression in (21) as

$$\exp\{\alpha(T(x) - L)\} \cdot \exp\left\{\left(\frac{\log 8}{2\Delta} - \alpha + \alpha^2 c_3^2 d^2\right) n\Delta\right\}. \tag{23}$$

Let $-\varpi := (\frac{\log 8}{2\Delta} - \alpha + \alpha^2 c_3^2 d^2)$ and choose sufficiently large $\Delta > 0$ and sufficiently small $\alpha > 0$ so that $\varpi > 0$.

Next, let $t \in (0,\infty)$ be arbitrary and pick $n_0 \in I\!N$ such that $t \in [n_0\Delta, (n_0+1)\Delta]$. Then from the definition of the set A_{n_0}, we have

$$I\!P_x(\tau_B > t) \le I\!P_x(A_{n_0}) \le C \cdot e^{\alpha T(x)} e^{-\varpi t}, \quad \text{where } C := \exp(\varpi\Delta). \tag{24}$$

Finally, for any $\beta \in (0, \varpi)$, we obtain

$$\begin{aligned} I\!E_x[e^{\beta\tau_B}] = 1 + \int_0^\infty \beta e^{\beta t} I\!P_x[\tau_B > t]dt &\le 1 + C\beta e^{\alpha T(x)} \int_0^\infty e^{(\beta-\varpi)t}dt \\ &= 1 + \frac{C\beta}{\varpi - \beta} e^{\alpha T(x)} \le 1 + \frac{C\beta}{\varpi - \beta} e^{\alpha c_2 |x|}, \end{aligned} \tag{25}$$

where the last inequality follows from Lemma 3.5 (i). Choosing $L \in (0,\infty)$ large enough (recall $x \in S \backslash D$) and using Lemma 3.5 (i) once more, the desired result (19) follows from (25). □

Remark 3.8. *We point out that the estimate in* (22) *is the only missing ingredient in the proof of the case* $H \neq 1/2$. *A suitable estimate of the following conditional* *expectation*

$$\mathbb{E}_x\left[\sup_{(m-1)\Delta\le t\le m\Delta}|B_H(t)-B_H((m-1)\Delta)|\,\Big|\,\sigma\left(B_H(s):0\le s\le (m-1)\Delta\right)\right]$$

will be needed to carry out an analysis for an upper bound when $H\neq 1/2$. To the best of our knowledge, obtaining such an estimate seems to be a quite challenging open problem due to FBM's innate dependence structure on the past information.

3.4. Upper bound estimate for reflected light-tailed Lévy motion

We observe that the result of Theorem 3.7 can be readily extended to the more general reflected processes driven by centered, light-tailed Lévy processes. By a *light-tailed* Lévy process $\{X(t)\in\mathbb{R}^d : t\ge 0\}$, we mean the existence of an exponential moment $\mathbb{E}[e^{\beta_0\cdot X(t)}]$ for some $\beta_0\in\mathbb{R}^d_+$ at any fixed time $t\ge 0$, which is guaranteed if and only if

$$\int_{|x|\ge 1} e^{\beta_0\cdot x}\Pi(dx)<\infty,\quad \text{for some } \beta_0\in\mathbb{R}^d_+. \tag{26}$$

Here, the measure Π is the Lévy (characteristic) measure on $\mathbb{R}^d$ (cf. Theorem 3.6 in Kyprianou (2006)). We refer the reader to Konstantopoulos et al. (2004) for the detailed description on a class of Lévy stochastic networks and their asymptotic properties.

For the sake of simplicity of the presentation and as a canonical example, we consider only the following case of a reflected process driven by a compensated Poisson process.

Corollary 3.9. *For $x\in S$, let $\{Z_x(t):t\ge 0\}$ be a reflected process on the d-dimensional positive orthant S with a d-dimensional unit rate compensated Poisson process $\{N(t)=(N_1(t),\ldots,N_d(t))':t\ge 0\}$ as its driving noise (i.e., replacing FBM B_H by N in Definition 2.1). Assume the conditions (S), (HR1) and $N_1,\ldots,N_d$ are mutually independent. Then, the result of Theorem 3.7 holds.*

Proof. It suffices to establish a similar estimate as in (22) for obtaining tail probability estimate (24). For $m\ge 1$, let

$$\hat{\nu}_m := \sup_{(m-1)\Delta\le s\le m\Delta}|N(t)-N((m-1)\Delta)|$$

and apply Doob's maximal inequality for submartingales to get $\mathbb{E}_x e^{\kappa\hat{\nu}_1}\le 4\mathbb{E}_x e^{\kappa|N(\Delta)|}$ for all $\kappa\ge 0$. Then, a straightforward calculation (owing to the moment generating function of a random vector $N(\Delta)$) shows that

$$\mathbb{E}_x e^{\kappa\hat{\nu}_1}\le 4e^{\frac{1}{2}\Delta d^2\kappa^2+o(\kappa^2)}.$$

By taking successive conditional expectations and using independent increment property of N, one gets for $\kappa \in (0, \infty)$ and $m \leq n$ that

$$\mathbb{E}_x \left(\exp \left\{ \kappa \sum_{i=m}^{n} \hat{\nu}_i \right\} \right) \leq \left[4 \exp \left\{ \frac{1}{2} \Delta d^2 \kappa^2 + o(\kappa^2) \right\} \right]^{(n-m+1)} \quad \text{as } \kappa \to 0. \tag{27}$$

Then we can obtain an expression similar to (23) as

$$\exp\{\alpha(T(x) - L)\} \cdot \exp \left\{ \left(\frac{\log 4}{\Delta} - \alpha + \frac{1}{2}\alpha^2 c_3^2 d^2 + o(\alpha^2) \right) n\Delta \right\} \quad \text{as } \alpha \to 0.$$

Let $-\hat{\varpi} := (\frac{\log 4}{\Delta} - \alpha + \frac{1}{2}\alpha^2 c_3^2 d^2 + o(\alpha^2))$ and then one can choose sufficiently large interval size $\Delta > 0$ and sufficiently small factor $\alpha > 0$ so that $\hat{\varpi} > 0$. The desired result follows on choosing $\beta \in (0, \hat{\varpi})$. □

Remark 3.10. *The proof of the general mean-zero light-tailed Lévy process case (in the sense of* (26)*) is essentially the same as those of Theorem 3.7 and Corollary 3.9. Indeed, one can employ the well known Lévy-Itô decomposition of a Lévy process: a (centered) Lévy process can be identified as the independent sum of Brownian motion* $X^{(1)}$*, a compound Poisson process* $X^{(2)}$*, and a square integrable martingale* $X^{(3)}$ *with an almost surely countable number of jumps on each finite time interval, which are of magnitude less than unity (cf. Theorem 2.1 in Kyprianou (2006)).*

We point out that the third process $X^{(3)}$ *of Lévy-Itô decomposition can be taken as the superposition of independent compound Poisson processes (cf. Section 2.5 in Kyprianou (2006)). A mutual independence of these processes and an independent increment property, along with an application of Doob's maximal inequality to each component, will yield similar exponential moment bound as illustrated in* (22) *and* (27)*. Then, the desired result will follow on choosing a sufficiently small* $\alpha \in (0, \infty)$ *as in the two preceding proofs.*

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