On the return time for a reflected fractional Brownian motion process on the positive orthant

Chihoon Lee*
Department of Statistics
Colorado State University
Fort Collins, CO 80523, USA

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Abstract

We consider a d-dimensional reflected fractional Brownian motion process (rfBm) on the positive orthant $S = \mathbb{R}_+^d$, with drift $r^0 \in \mathbb{R}^d$ and Hurst parameter $H \in (1/2, 1)$. Under a natural stability condition on the drift vector r^0 and reflection directions, we establish a return time result for rfBm process Z; that is, for some $\delta, \kappa > 0$,

$$\sup_{x \in B} \mathbb{E}_x[\tau_B(\delta)] < \infty,$$

where $B = \{x \in S : |x| \leq \kappa\}$ and $\tau_B(\delta) = \inf\{t \geq \delta : Z(t) \in B\}$. Similar results are known for reflected processes driven by standard Brownian motions, and our result can be viewed as their fBm counterpart. Motivation for this study is that rfBm appears as a limiting workload process for fluid queueing network models fed by a large number of heavy-tailed ON/OFF sources in heavy traffic.

Keywords: Reflected fractional Brownian motion, heavy traffic theory, return times.

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*E-mail address: chihoon@stat.colostate.edu

1 Introduction

This paper is devoted to the study of a multi-dimensional reflected fractional Brownian motion process (rfBm) on the positive orthant $S := \mathbb{R}^d_+$, with drift $r^0 \in \mathbb{R}^d$ and Hurst parameter $H \in (1/2, 1)$. Recently, Delgado [5] showed that the workload process of multi-station fluid queueing network models with feedback and non-deterministic arrival process, generated by a large number of heavy-tailed ON/OFF sources, can be approximated under suitable heavy traffic conditions by a multi-dimensional rfBm with Hurst parameter $H \in (1/2, 1)$. This model was further studied in subsequent papers [6, 7]. Besides [5], the multi-dimensional rfBm process has also been obtained as an approximating model in [14], in which a single-class queueing network with long-range dependent arrival and service processes is considered. It was shown therein that the normalized queue-length process converges to a d-dimensional rfBm, with d being the number of nodes or servers. Long time asymptotics and stability analysis of such class of models are of fundamental interest. However, since a fBm is neither a semi-martingale nor a Markov process, many techniques from the classical theory of stochastic calculus are inapplicable to its analysis.

In this work, we establish a uniform moment estimate on expected return times of the rfBm process $\{Z(t) : t \geq 0\}$ to a compact set; that is, we show for some $\delta, \kappa \in (0, \infty)$,

$$\sup_{x \in B} \mathbb{E}_x[\tau_B(\delta)] < \infty, \tag{1.1}$$

where $B = \{x \in S : |x| \leq \kappa\}$, $\tau_B(\delta) = \inf\{t \geq \delta : Z(t) \in B\}$ and $\mathbb{E}_x$ denotes the expectation conditional on the process Z starting from $x \in S$. This result is reminiscent of a necessary and sufficient condition for positive Harris recurrence of Harris recurrent Markov processes (see, e.g. [18, 16]). More precisely, for a wide class of Markov processes, the condition (1.1), combined with a *petite* set requirement for B, implies the positive Harris recurrence of the process. Similar results as in (1.1) are known for reflected processes driven by standard Brownian motions; indeed, the positive Harris recurrence result of [10] (see also [13]) together with Theorem 4.4 of [15] implies (1.1) for semi-martingale reflecting Brownian motions. We also refer the reader to Theorem 4.7 and Corollary 4.14 of [3] for more refined such results. In this regard, results in this paper can be viewed as a significant step towards the further analysis of rfBm with the aim of establishing similar recurrent properties for reflected processes driven by non-Markovian processes.

The organization of the paper is as follows. In Section 2, we carefully describe our model in Definition 2.1 and make a standard assumption on reflection matrix (see **(HR)** in Section 2), which is used in heavy traffic analysis for invoking a functional central limit theorem in [5, 6]. In addition, similar to [13], we assume a natural stability condition (see **(S)** in Section 2) on the rfBm process. Our proof is based on uniform stability estimates (see proof of Theorem 3.1, in particular (3.4)) on a family of certain deterministic dynamical systems obtained from a fluid limit analysis of the underlying rfBm process. This result, together with a maximal inequality for fBm with $H \in (1/2, 1)$, leads to uniform time estimates (Theorem 3.1) on the p-th ($p \geq 1$) moment of the process in terms of its initial condition. In order to connect this result with moments of return times to a compact set, we deduce a drift inequality (see (3.11)) leading to control of the expected overall hitting time, and establish the main result in Theorem 3.3.

We use the following notation. The set of positive integers is denoted by $I\!N$, the set of real numbers by $I\!R$ and non-negative real numbers by $I\!R_+$. Let $I\!R^d$ be the d-dimensional Euclidean space and for $x \in I\!R^d$ the L_1 norm of x, i.e., $\sum_{i=1}^d |x_i|$, will be denoted by $|x|$. Let $I\!R^{d\times m}$ be the space of real $(d \times m)$-matrices with the norm $||A|| = \max\limits_{1\leq j\leq m} \sum_{i=1}^d |a_{ij}|$ for $A \in I\!R^{d\times m}$. For a given matrix M, denote by M^T its transpose and by M_i the i-th row of M. Let $\mathbb{I} = \mathbb{I}_{d\times d}$ denote the identity matrix for some d. When it is clear from the context, we will omit the subscript. For a set $A \subseteq I\!R^d$, denote its interior and boundary by A° and ∂A, respectively. For sets $A, B \subseteq I\!R^d$, $\mathrm{dist}(A, B)$ will denote the distance between two sets, i.e., $\inf\{|x - y| : x \in A, y \in B\}$. Let $C(X, Y)$ denote the space of continuous functions from X to Y, endowed with the topology of uniform convergence on compact intervals. Inequalities for vectors are interpreted componentwise.

2 Model and assumptions

We begin with the definitions of multi-dimensional fBm and reflected fBm. Let $d \in I\!N$. A stochastic process $B_H = \{B_H(t) = (B_H^{(1)}(t), \ldots, B_H^{(d)}(t))^T, t \geq 0\}$, defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, I\!P)$, is called a d-dimensional fBm of (Hurst) parameter $H \in (0, 1)$, starting from

$B_H(0) \in I\!R^d$, and associated matrix Λ, if it is a continuous Gaussian process with initial condition $B_H(0)$ $I\!P$-a.s. and with covariance function given by

$$\mathrm{Cov}(B_H(t), B_H(s)) = I\!E\left((B_H(t) - B_H(0))(B_H(s) - B_H(0))^T\right) = \Lambda_H(s,t)\Lambda,$$

for any $s, t \geq 0$, where Λ is a $d \times d$ positive definite matrix and

$$\Lambda_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right).$$

Also, it is assumed that B_H is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. We will say that B_H is a d-dimensional fBm with associated data $(B_H(0), H, \Lambda)$.

Fix column vectors $r^0, r^1, \ldots, r^d \in I\!R^d$ and let $R := \left[r^1, \ldots, r^d\right]_{d\times d}$. We call the quintuple $(B_H(0), H, \Lambda, r^0, R)$ as the *data* for a reflected fBm. The following definition is similar to that of [5].

Definition 2.1. (rfBm) For $x \in S$, a reflected fractional Brownian motion associated with the data (x, H, Λ, r^0, R) is a continuous d-dimensional process Z, defined on some probability space $(\Omega, \mathcal{F}, I\!P)$, such that

(i) $Z(t) = x + B_H(t) + r^0 t + RY(t) \in S \quad \text{for all} \quad t \geq 0, \quad I\!P\text{-}a.s.,$

(ii) B_H is a d-dimensional fBm with data $(0, H, \Lambda)$,

(iii) Y is a d-dimensional process such that $Y_i(0) = 0$ for $i = 1, \ldots, d$, $I\!P$-a.s. For each $i = 1, \ldots, d$, Y_i is continuous, non-decreasing and Y_i can increase only when $Z(\cdot)$ is on the face $F^i := \{x \in S : x_i = 0\}$, *i.e.*, $\int_0^t 1_{\{Z_i(s)\neq 0\}} dY_i(s) = 0$ for all $t \geq 0$.

For $y \in \partial S$, the set of directions of reflections is defined as:

$$r(y) := \left\{\sum_{i=1}^d q_i r^i : \sum_{i=1}^d q_i = 1,\ q_i \geq 0,\ \text{and } q_i > 0 \text{ only if } y_i = 0\right\}.$$

To get an idea of a rfBm introduced in the above definition, we note that it behaves like a fBm in the interior of the orthant S and it is confined to the orthant by instantaneous "reflection" at the boundary ∂S. For each i, the i-th column of the *reflection matrix* R gives the direction of the reflection on the i-th face F^i. Specifically, if the boundary F^i is hit, it is Y_i that increases, the direction of displacement is given by r^i, the i-th column of R, and the magnitude of the displacement is the minimal amount required to keep Z_i nonnegative. We refer the reader to [21] and references therein for the related definition and properties of semimartingale reflecting Brownian motions in an orthant.

Remark 2.2. *We call a square matrix R is completely-$\mathcal{S}$ if for every $k \times k$ principal submatrix G of R, there is a k-dimensional vector v_G such that $v_G \geq 0$ and $Gv_G > 0$. The completely-$\mathcal{S}$ condition on the reflection matrix R ensures that for every $x \in \partial S$, there exists a convex combination of vectors in $r(x)$ which points into S° from x. Also, the completely-$\mathcal{S}$ property is sufficient to ensure the existence of a pair (Z, Y) satisfying (i) and (iii) in Definition 2.1 (cf. Theorem 2 in [2]). However, this property does not ensure the adaptness of process Y to a filtration to which B_H is adapted. This problem is overcome under a stronger assumption on R, that we quote below (cf. Proposition 4.2 in [21] and Section 2 of [5]).*

We will impose throughout that matrix R verifies the following assumption:

(HR) (i) The reflection matrix R can be expressed as $\mathbb{I} + \Theta$, with Θ a $d \times d$ matrix such that $|\Theta|$, that is the matrix obtained from Θ by replacing all the entries in Θ by their absolute values, has spectral radius $r(|\Theta|)$ strictly less than 1. (ii) Moreover, the matrix $\Theta = (\theta_{ij})$ satisfies $\theta_{ij} \leq 0$ and $\theta_{ii} = 0$ for $1 \leq i, j \leq d$.

Part (i) of assumption (HR) implies that R is a completely-$\mathcal{S}$ matrix. Also, it can be shown (cf. [21, 5]) that if B_H is adapted to some filtration $\{\mathcal{F}_t : t \geq 0\}$, then (Z, Y) is adapted to filtration $\{\mathcal{G}_t : t \geq 0\}$, with $\mathcal{G}_t = \mathcal{F}_t \vee \mathcal{N}$, where $\mathcal{N}$ denotes the collection of $I\!\!P$-null sets in $\mathcal{F}$. Furthermore, (HR) part (i) is a sufficient condition for strong pathwise uniqueness of a solution of the *Skorohod Problem* described below. Henceforth, with an abuse of notation, we will assume that (Z, Y) is adapted to filtration $\{\mathcal{F}_t : t \geq 0\}$.

We also notice that the spectral radius of $-\Theta$ in (HR) is strictly less than 1, since $r(-\Theta) = r(\Theta) \leq r(|\Theta|)$, where the inequality is due to recalling the Gelfand's Theorem on spectral radius [11] (that is, the spectral radius of a square matrix A is given by $r(A) = \lim_{k\to\infty} ||A^k||^{1/k}$ with consistent matrix norm $||\cdot||$ on the space of matrices). Hence, $\mathbb{I} + (-\Theta) + (-\Theta)^2 + \cdots$ is convergent and therefore, assumption (HR) part (i) implies the existence of R^{-1}. Part (ii) of (HR), together with part (i), verifies the so-called Harrison-Reiman condition in [12]. Therefore, (HR) will imply that a solution to the Skorohod Problem exists, and moreover the Skorohod map is Lipschitz continuous in the sense of Proposition 2.4 below. (See also pages 163–165 and 200–208 in [9] for the generalized Harrison-Reiman condition for Lipschitz continuity of the Skorohod map.)

Definition 2.3. (Skorohod Problem) Let $\psi \in C([0,\infty), \mathbb{R}^d)$ be given with $\psi(0) \in S$. Then $(\phi, \eta) \in C([0,\infty), \mathbb{R}^d) \times C([0,\infty), \mathbb{R}^d)$ solves the Skorohod problem for ψ with respect to S and R if and only if the following hold:

(i) $\phi(t) = \psi(t) + R\eta(t) \in S$, for all $t \geq 0$;

(ii) η satisfies, for $1 \leq i \leq d$, (a) $\eta_i(0) = 0$, (b) η_i is non-decreasing, and (c) η_i can increase only when ϕ is on the i-th face of S, that is, $\int_0^\infty 1_{\{\phi_i(s)\neq 0\}} d\eta_i(s) = 0$.

Let $C_S([0,\infty), \mathbb{R}^d) := \{\psi \in C([0,\infty), \mathbb{R}^d) : \psi(0) \in S\}$. On the domain $D \subset C_S([0,\infty), \mathbb{R}^d)$ on which there is a unique solution to the Skorohod problem, we define the Skorohod map Γ as

$$\Gamma(\psi) := \phi,$$

if $(\phi, R^{-1}[\phi - \psi])$ is the unique solution of the Skorohod problem posed by ψ. Equivalent form of rfBm in Definition 2.1 in terms of the Skorohod map can now be written as follows:

$$Z = \Gamma(x + B_H + r^0\imath), \quad Z - (x + B_H + r^0\imath) = RY,$$

where $\imath : [0,\infty) \to [0,\infty)$ is the identity map.

The following Proposition 2.4 gives the regularity of the Skorohod map, which is a consequence of Assumption (HR). We refer the reader to the proof of (10) in page 305 of Harrison and Reiman [12] and arguments in pages 164–165 of [9] for its proof; although the Lipschitz continuity is not stated explicitly in [12], it follows easily from the method used to prove existence of solutions and continuity of the Skorohod map (map $\phi(\cdot)$ in [12]). See also [8, 9] for more sufficient conditions under which this regularity property holds.

Proposition 2.4. *The Skorohod map is well-defined on all of $C_S([0,\infty), \mathbb{R}^d)$, i.e. $D = C_S([0,\infty), \mathbb{R}^d)$, and the Skorohod map is Lipschitz continuous in the following sense: There exists a constant $L \in (1,\infty)$ such that for all $\psi_1, \psi_2 \in C_S([0,\infty), \mathbb{R}^d)$ and $t \geq 0$,*

$$\sup_{0\leq s\leq t} |\Gamma(\psi_1)(s) - \Gamma(\psi_2)(s)| < L \sup_{0\leq s\leq t} |\psi_1(s) - \psi_2(s)|.$$

Finally, we introduce the condition on the drift vector r^0 and the matrix R that will be assumed throughout this paper:

(S) There exists a $\theta > 0$, such that $\sup_{1\leq i\leq d}[R^{-1}r^0]_i < -\theta$.

Remark 2.5. *For a model driven by regular Brownian motion (i.e. $H = 1/2$), the assumption (S) is known as a necessary and sufficient condition for the existence of a unique stationary distribution. See [13] and references therein for the related asymptotic stability conditions of reflected Brownian motions on a positive orthant.*

3 Main results

Let Z_x be defined by Definition 2.1 with $Z_x(0) = x \in S$ and the Hurst parameter $H \in (1/2, 1)$. In what follows, note that $\mathbb{E}|Z_x(t)|$ can be alternatively written as $\mathbb{E}_x|Z(t)|$. The following moment stability properties are key ingredients in the proofs.

Theorem 3.1. *There exists a $\delta \in (0,\infty)$ such that for all $p \geq 1$,*

$$\lim_{|x|\to\infty} \frac{1}{|x|^p} \mathbb{E}\left(|Z_x(\delta|x|)|^p\right) = 0. \tag{3.1}$$

Proof. Fix $x \in S$ and $p \geq 1$. We write the rfBm Z in Definition 2.1 as,

$$Z_x(t) = \Gamma\left(x + r^0\imath + B_H(\cdot)\right)(t), \quad t \geq 0,$$

where $\imath : [0,\infty) \to [0,\infty)$ is the identity map. Define the deterministic trajectory

$$z_x(t) = \Gamma(x + r^0 \imath)(t), \quad t \geq 0.$$

Using the Lipschitz property of Γ (Proposition 2.4), we have

$$|Z_x(t) - z_x(t)| \leq L \sup_{0 \leq s \leq t} |B_H(s)|, \text{ for all } t \geq 0. \tag{3.2}$$

Next, denoting by $\mathcal{C} := \{v \in I\!\!R^d : R^{-1}v \leq 0\}$ we see from condition (S) that there exists a $\beta \in (0,\infty)$ satisfying

$$\begin{aligned} \operatorname{dist}(r^0, \partial\mathcal{C}) &= \inf\left\{ \left|r^0 - v\right| : R^{-1}v = 0, v \in I\!\!R^d \right\} \\ &\geq \frac{1}{||R^{-1}||} \inf\left\{ \left|R^{-1}r^0 - R^{-1}v\right| : R^{-1}v = 0, v \in I\!\!R^d \right\} \\ &= \frac{1}{||R^{-1}||} \left|R^{-1}r^0\right| \geq \frac{1}{||R^{-1}||}\theta d =: \beta > 0, \end{aligned} \tag{3.3}$$

where $\theta \in (0,\infty)$ is a constant as in condition (S), and the first inequality is owing to the fact $|Ax| \leq ||A|| \cdot |x|$ for a matrix $A_{d\times d}$ and a vector $x_{d\times 1}$. Thus,

$$r^0 \in \mathcal{C}_\beta := \{v \in \mathcal{C} : \operatorname{dist}(v, \partial\mathcal{C}) \geq \beta\}.$$

For $x_0 \in S$, denote by $\mathcal{K}(x_0)$ the collection of all trajectories $\psi : [0,\infty) \to S$ of the form

$$\psi(t) = \Gamma\left(x_0 + \varpi\imath\right)(t), \quad t \geq 0,$$

where ϖ ranges over all of $\mathcal{C}_\beta$. Define the "hitting time to the origin" function as follows,

$$T(x_0) := \sup_{\psi \in \mathcal{K}(x_0)} \inf\{t \in [0,\infty) : \psi(t) = 0\}.$$

Then, owing to Lemma 3.1 of [1] we have that

$$T(x_0) \leq \frac{4L^2}{\beta}|x_0|, \text{ and } \text{ for all } \psi \in \mathcal{K}(x_0),\ \psi(t) = 0 \text{ for all } t \geq T(x_0). \tag{3.4}$$

Combining this observation with (3.3), we now have that $z_x(t) = 0$, for all $t \geq \delta_0|x|$, where $\delta_0 := \frac{4L^2}{\beta}$. Using this in (3.2) we now see that

$$|Z_x(t|x|)| \leq L \sup_{0 \leq s \leq t|x|} |B_H(s)|, \tag{3.5}$$

for all $t \geq \delta_0$ and for all initial conditions x. Next we obtain an estimate on the p-th moment of the right side of (3.5). For $H \in (1/2, 1)$, it is known from Theorem 1.2 of [19] (see also, Ex. 5.1.5 in [20]) that

$$I\!E \left(\sup_{0 \leq s \leq t} |B_H^{(i)}(s)| \right)^p \quad \leq \quad C(p,H) t^{pH},$$

where $i = 1, \ldots, d$ and $C(p,H) \in (0,\infty)$ is a constant which depends only on p and H. Applying this estimate in (3.5), we now have that for all $t \geq \delta_0$ and $x \in S$,

$$I\!E|Z_x(t|x|)|^p \leq C(t|x|)^{pH} \tag{3.6}$$

for some constant $C \in (0,\infty)$, which depends only on p and H. The result now follows on choosing any $\delta \geq \delta_0$ since $H < 1$. ■

Remark 3.2. *We note that* (3.1) *with* $p = 1$ *was established in [4] for a wide class of Markov processes. Combined with a certain petite set requirement, it was shown in [4] that* (3.1) *with* $p = 1$ *implies the positive Harris recurrence of underlying Markov processes.*

We now present the main result on return times to a compact set for rfBm process. Its proof is along similar lines to those presented in Theorem 3.1 of [4] and Theorem 2.1(ii) of [17].

Theorem 3.3. *Let* $B = \{x \in S : |x| \leq \kappa\}$ *for some* $\kappa \geq 1$, *then*

$$\sup_{x \in B} I\!E_x[\tau_B(\delta)] < \infty, \tag{3.7}$$

where $\tau_B(\delta) := \inf\{t \geq \delta : Z(t) \in B\}$ *with some* $\delta > 0$.

Proof. By Theorem 3.1 (with $p = 1$), there exists a $\kappa \geq 1$ such that with $B = \{x \in S : |x| \leq \kappa\}$,

$$\mathbb{E}|Z_x(\delta|x|)| \leq \frac{1}{2}|x|, \quad \forall x \in B^c, \tag{3.8}$$

where δ is as in Theorem 3.1. Analogously to (3.6), we have that $\mathbb{E}|Z_x(t)|^p \leq Ct^{pH}$ for any $t \geq \delta_0|x|$ and some constant $C > 0$. Take $\delta = \delta_0\kappa$ ($\geq \delta_0$). Then, with $p = 1$ and $t = \delta$ ($\geq \delta_0|x|$ for $x \in B$), one has for some constant $b > 0$

$$\mathbb{E}|Z_x(\delta)| \leq b, \quad \forall x \in B. \tag{3.9}$$

For $x \in S$, let

$$n(x) = \begin{cases} \delta|x|, & \text{if } x \in B^c, \\ \delta, & \text{if } x \in B. \end{cases} \tag{3.10}$$

Since $\kappa \geq 1$, $n(x) \geq \delta$ for all $x \in S$. It follows from (3.8) and (3.9) that

$$\mathbb{E}|Z_x(n(x))| \leq \frac{1}{2}|x| + b1_B(x) \leq |x| - \frac{1}{2\delta}n(x) + \tilde{b}1_B(x) \tag{3.11}$$

for some $\tilde{b} \geq \frac{1}{2} + b > 0$ and all $x \in S$.

In order to prove (3.7), we work with an "embedded" process $\breve{Z}$. Notice that $n(x)$ in (3.10) is a (trivial) stopping time. For $k \geq 1$, let $s(k)$ denote its iterates; that is, along any sample paths,

$$s(0) = 0, \quad s(1) = n(x), \text{ and } s(k+1) = s(k) + n(Z_x(s(k))).$$

Define

$$\breve{Z}(k) := Z_x(s(k)), \quad \breve{\mathcal{F}}_k := \mathcal{F}_{s(k)}, \text{ and } \quad \breve{\tau}_B(\delta) := \inf\{k \geq \delta : \breve{Z}(k) \in B\}.$$

Then $s(\breve{\tau}_B(\delta))$ denotes the time of the first return to B by the original process along an embedded path; that is,

$$s(\breve{\tau}_B(\delta)) = \sum_{k=0}^{\breve{\tau}_B(\delta)-1} n(\breve{Z}(k)) \tag{3.12}$$

and so we have a.s.

$$s(\breve{\tau}_B(\delta)) \geq \tau_B(\delta). \tag{3.13}$$

We claim that

$$\mathbb{E}_x\left[\sum_{k=0}^{\breve{\tau}_B(\delta)-1} n(\breve{Z}(k))\right] \leq 2\delta(|x|+\tilde{b}). \tag{3.14}$$

Then, we have for each $x \in S$ by adding the lengths of the embedded time $n(x)$ along any sample path, and from (3.12)–(3.14), that

$$\mathbb{E}_x(\tau_B(\delta)) \leq \mathbb{E}_x\left[\sum_{k=0}^{\breve{\tau}_B(\delta)-1} n(\breve{Z}(k))\right] \leq 2\delta(|x|+\tilde{b}).$$

Hence,

$$\begin{aligned}\sup_{x\in B}\mathbb{E}_x\left(\tau_B(\delta)\right) &\leq 2\delta\left(\sup_{x\in B}|x|+\tilde{b}\right)\\ &= 2\delta(\kappa+\tilde{b}) < \infty.\end{aligned}$$

Thus it only remains to prove the claim in (3.14). For $n \geq 1$, define

$$\tau^n := \min\left\{n, \breve{\tau}_B(\delta), \inf\{k \geq 0 : \breve{Z}(k) \geq n\}\right\}.$$

Note that since $\{\tau^n \geq i\} \in \breve{\mathcal{F}}_{i-1}$ we have

$$\mathbb{E}_x[\breve{Z}(\tau^n)] = \mathbb{E}_x[\breve{Z}(0)] + \mathbb{E}_x\left[\sum_{i=1}^{\tau^n}\left(\mathbb{E}_x\left[\breve{Z}(i)|\breve{\mathcal{F}}_{i-1}\right] - \breve{Z}(i-1)\right)\right]. \tag{3.15}$$

Fix $N > 0$. From (3.11) and (3.15) we see that

$$0 \leq \mathbb{E}_x[\breve{Z}(\tau^n)] \leq |x| + \mathbb{E}_x\left[\sum_{i=1}^{\tau^n}\left(\tilde{b}1_B(\breve{Z}(i-1)) - \frac{1}{2\delta}n(\breve{Z}(i-1)) \wedge N\right)\right]. \tag{3.16}$$

Hence, by adding a finite term to each side of (3.16), we get

$$\begin{aligned}\mathbb{E}_x\left[\sum_{i=1}^{\tau^n}\frac{1}{2\delta}n(\breve{Z}(i-1)) \wedge N\right] &\leq |x| + \mathbb{E}_x\left[\sum_{i=1}^{\tau^n}\tilde{b}1_B(\breve{Z}(i-1))\right]\\ &\leq |x| + \mathbb{E}_x\left[\sum_{i=1}^{\breve{\tau}_B(\delta)}\tilde{b}1_B(\breve{Z}(i-1))\right]\\ &\leq |x| + \tilde{b}.\end{aligned}$$

Letting $n \to \infty$ and then $N \to \infty$ gives the result by the monotone convergence theorem. ■

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References

[1] R. Atar, A. Budhiraja, and P. Dupuis. On positive recurrence of constrained diffusion processes. *Ann. Probab.*, 29(2):979–1000, 2001.

[2] A. Bernard and A. el Kharroubi. Régulations déterministes et stochastiques dans le premier "orthant" de $\mathbf{R}^n$. *Stochastics Stochastics Rep.*, 34(3-4):149–167, 1991.

[3] A. Budhiraja and C. Lee. Long time asymptotics for constrained diffusions in polyhedral domains. *Stochastic Process. Appl.*, 117(8):1014–1036, 2007.

[4] J. G. Dai. On positive Harris recurrence of queueing networks: A unified approach via fluid limit models. *Ann. Appl. Probab.*, 5:49–77, 1995.

[5] R. Delgado. A reflected fBm limit for fluid models with ON/OFF sources under heavy traffic. *Stochastic Process. Appl.*, 117(2):188–201, 2007.

[6] R. Delgado. State space collapse for asymptotically critical multi-class fluid networks. *Queueing Syst.*, 59(2):157–184, 2008.

[7] R. Delgado. On the reflected fractional Brownian motion process on the positive orthant: Asymptotics for a maximum with application to queueing networks. *Stochastic Models*, 26(2):272–294, 2010.

[8] P. Dupuis and H. Ishii. On Lipschitz continuity of the solution mapping to the Skorohod problem, with applications. *Stochastics*, 35:31–62, 1991.

[9] P. Dupuis and K. Ramanan. Convex Duality and the Skorokhod Problem. I, II. *Probab. Theory Related Fields*, 115(2):153–195, 197–236, 1999.

[10] P. Dupuis and R. J. Williams. Lyapunov functions for semimartingale reflecting Brownian motions. *Ann. Probab.*, 22(2):680–702, 1994.

[11] I. Gelfand. Normierte Ringe. *Rec. Math. [Mat. Sbornik] N. S.*, 9 (51):3–24, 1941.

[12] J. M. Harrison and M. I. Reiman. Reflected Brownian motion on an orthant. *Ann. Probab.*, 9(2):302–308, 1981.

[13] J. M. Harrison and R. J. Williams. Brownian models of open queueing networks with homogeneous customer populations. *Stochastics*, 22(2):77–115, 1987.

[14] T. Konstantopoulos and S. Lin. Fractional Brownian approximations of queueing networks. In *Stochastic networks (New York, 1995)*, volume 117 of *Lecture Notes in Statist.*, pages 257–273. Springer, New York, 1996.

[15] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes II: Continous-time processes and sampled chains. *Adv. in Appl. Probab.*, 25:497–517, 1993.

[16] S. P. Meyn and R. L. Tweedie. Stability of Markovian processes III: Foster-Lyapunov criteria for continous-time processes. *Adv. in Appl. Probab.*, 25:518–548, 1993.

[17] S. P. Meyn and R. L. Tweedie. State-dependent criteria for convergence of Markov chains. *Ann. Appl. Probab.*, 4(1):149–168, 1994.

[18] S. P. Meyn and R. L. Tweedie. *Markov chains and stochastic stability.* Cambridge University Press, Cambridge, second edition, 2009. With a prologue by Peter W. Glynn.

[19] A. Novikov and E. Valkeila. On some maximal inequalities for fractional Brownian motions. *Statist. Probab. Lett.*, 44(1):47–54, 1999.

[20] D. Nualart. *The Malliavin calculus and related topics.* Probability and its Applications (New York). Springer-Verlag, Berlin, second edition, 2006.

[21] R. J. Williams. An invariance principle for semimartingale reflecting Brownian motions in an orthant. *Queueing Systems Theory Appl.*, 30(1-2):5–25, 1998.