Stationarity and control of a tandem fluid network with fractional Brownian motion input

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Abstract

We consider a stochastic control model for a queueing system driven by a two-dimensional fractional Brownian motion with the Hurst parameter $0 < H < 1$. In particular, when $H > 1/2$, this model serves to approximate a controlled two-station tandem queueing model with heavy tailed ON/OFF sources in heavy traffic. We establish the weak convergence results for the distribution of the state process and construct an explicit stationary state process associated with given controls. Based on suitable coupling arguments, we show that each state process couples with its stationary counterpart and we use it to represent the long-run average cost functional in terms of the stationary process. Finally, we establish the existence result of an optimal control, which turns out to be independent of the initial data.

Keywords: Stochastic control, controlled queueing systems, heavy traffic theory, fractional Brownian motion, tandem queue, long-range dependence, self-similarity.

AMS Subject Classifications: primary 60K25, 68M20, 90B22; secondary 90B18.

1 Introduction

Empirical evidence of long-range dependence and self-similarity of the underlying data in several queueing systems has been observed and analyzed [11, 33, 32, 30]. One simple concrete explanation for this kind of phenomena is the behavior of superposition of many ON/OFF sources (also known as "packet trains" [18]) with strictly alternating ON- and OFF-periods. It has been shown that long-range dependence and self-similarity signatures of network traffic are successfully described by stochastic models associated with fractional Brownian motion, abbreviated as fBm hereafter, with the Hurst parameter H greater than $1/2$ (see [20, 29, 30, 26, 15]). It is well known that such models exhibit both of these statistical features and therefore, understanding the behavior and control of these stochastic models are of significant interest. However, the highly non-Markovian nature of fBm makes it more difficult to analyze the control problems related to such models.

In this paper, we focus on a controlled queueing system driven by a two-dimensional fBm with the Hurst parameter $0 < H < 1$, and it serves to approximate a controlled two-station tandem queueing model with ON/OFF sources (when $H > 1/2$). Tandem systems can be seen in

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many applications such as storage systems and high-speed communication networks, from router architectures to protocol stacks [14, 24]. Our work is motivated by the recent article of Delgado [6], which obtained a reflected fBm model as a limiting process for fluid models with heavy tailed ON/OFF sources in heavy traffic. We are interested in the optimal control of such reflected fBm models (we describe our connection to the work of [6] in detail in Section 3). To this end, we introduce the notion of "thin control" related to such models with ON/OFF sources in heavy traffic and obtain a limiting controlled fluid queue driven by a two-dimensional fBm. For ordinary Brownian networks, this notion was used in [1]. This leads us to consider a drift rate control problem of a tandem fluid queueing network fed by a fBm at each station. Our analysis allows these fBm's to be correlated with a constant correlation coefficient. For a related one-dimensional controlled queue with a fBm input, we refer to [10] and our work extends their work to two-dimensional situation. The probability estimates for maximum workload of a one-dimensional queue fed by a fBm were obtained in [35, 9].

Our contributions are two-fold. We consider a state process represented by a two-dimensional reflected fBm model with the Hurst parameter $0 < H < 1$. First, we show that under suitable moment conditions on initial data, any state process couples with an explicitly described stationary state process and this coupling time has finite moments. Second, we establish the existence of an optimal control for a related long-term average cost minimization problem. Despite the non-Markovian behavior of the fBm, this optimal control is independent of the initial data. These results are meant as a first step towards the further analysis of networks with general topology, where the nodes are operating under advanced scheduling and routing disciplines in a heavy traffic environment.

There are only a few stochastic control problems for models driven by fBm that are addressed in the literature. The linear quadratic regulator problem is addressed in [17] and [19]. A stochastic maximum principle was developed and applied to several stochastic control problems in [3]. We refer the reader to [17] and to Chapter 9 of [4] for further examples of such control problems. In contrast with the models considered in the aforementioned references, the model described here is motivated by queueing applications in heavy traffic and involves processes with state constraints. In Section 3, we discuss a concrete example of queueing network which leads to our model.

We begin with the definitions of multi-dimensional fBm and reflected fBm. We closely follow the notation of [6]. Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, I\!\!P)$ be a given filtered probability space. A stochastic process $\mathbf{B}_H = \{\mathbf{B}_H(t) = (B_1(t), \ldots, B_J(t))^T, t \geq 0\}$ defined on $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, I\!\!P)$ is called a J-dimensional fBm of the Hurst parameter $H \in (0,1)$, starting from the origin in $I\!\!R^J$, with drift vector $\boldsymbol{\vartheta} \in I\!\!R^J$ and associated matrix Λ, if it is a continuous Gaussian process with $\mathbf{B}_H(0) = \mathbf{0}$ $I\!\!P$-a.s. with $I\!\!E[\mathbf{B}_H(t)] = \boldsymbol{\vartheta} t$ for all $t \geq 0$, and its covariance function is given by

$$\mathrm{Cov}(\mathbf{B}_H(t), \mathbf{B}_H(s)) = I\!\!E\left((\mathbf{B}_H(t) - \boldsymbol{\vartheta} t)(\mathbf{B}_H(s) - \boldsymbol{\vartheta} s)^T\right) = \Upsilon_H(s,t)\Lambda,$$

for all $s, t \geq 0$. Here Λ is a $J \times J$ non-negative definite matrix and

$$\Upsilon_H(s,t) = \frac{1}{2}\left(t^{2H} + s^{2H} - |t-s|^{2H}\right). \tag{1.1}$$

Also, it is assumed that $\mathbf{B}_H$ is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. We will say that $\mathbf{B}_H$ is a J-dimensional fBm with associated data $(\mathbf{0}, H, \boldsymbol{\vartheta}, \Lambda)$.

Next, let $\mathbf{X}_0$ be an $\mathcal{F}_0$-measurable, $I\!\!R^J$-valued random vector with $I\!\!E|\mathbf{X}_0| < \infty$, defined on the filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, I\!\!P)$. We introduce the process $\{\mathbf{X}_H(t) : t \geq 0\}$ by

$$\mathbf{X}_H(t) = \mathbf{X}_0 + \mathbf{B}_H(t) \quad \text{for all } t \geq 0,$$

where $\mathbf{B}_H$ is a J-dimensional fBm with associated data $(\mathbf{0}, H, \boldsymbol{\vartheta}, \Lambda)$. Notice that the process $(\mathbf{X}_H(t))_{t\geq 0}$ is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$, $\mathbf{X}_H(0) = \mathbf{X}_0$ with $I\!\!E[\mathbf{X}_H(t) - \mathbf{X}_H(0)] = \vartheta t$ for all $t \geq 0$, and its covariance matrix $\mathrm{Cov}(\mathbf{X}_H(t), \mathbf{X}_H(s))$ is given by $\Upsilon_H(s,t)\Lambda$ for all $s, t \geq 0$. We will say that $\mathbf{X}_H$ is a J-dimensional fBm with associated data $(\mathbf{X}_0, H, \boldsymbol{\vartheta}, \Lambda)$.

The following definition of a reflected fBm slightly generalizes that of Delgado [6] to allow random initial data. The stationary process $(\mathbf{Z}^*(t))_{t\geq 0}$ we obtain in Theorem 4.3 turned out to be a reflected fBm with random initial data $\mathbf{Z}^*(0)$.

Definition 1.1. *A reflected fBm (abbreviated as rfBm) on* $S = I\!\!R^J_+$ *associated with the data* $(\mathbf{Z}_0, H, \boldsymbol{\vartheta}, \Lambda, R)$*, that starts from* $\mathbf{Z}_0 \in S$ *is a continuous* J*-dimensional process* $\mathbf{Z}$*, defined on some filtered probability space* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, I\!\!P)$ *such that*

(i) $\mathbf{Z}_0$ *is an* $\mathcal{F}_0$*-measurable,* S*-valued random vector with* $I\!\!E|\mathbf{Z}_0| < \infty$*,*

(ii) $\mathbf{X}_H$ *is a* J*-dimensional fBm adapted to* $(\mathcal{F}_t)_{t\geq 0}$ *with associated data* $(\mathbf{Z}_0, H, \boldsymbol{\vartheta}, \Lambda)$*,*

(iii) $\mathbf{Z}(t) = \mathbf{X}_H(t) + R\mathbf{L}(t) \in S$ *for all* $t \geq 0$*,* $I\!\!P$*-a.s. and the process* $(\mathbf{Z}, \mathbf{L})$ *is adapted to* $(\mathcal{F}_t)_{t\geq 0}$*,* $R_{J\times J}$ *is the "reflection matrix", and*

(iv) $\mathbf{L}$ *is a* J*-dimensional process satisfying* $L_j(0) = 0$ *for* $j = 1, \dots, J$*,* $I\!\!P$*-a.s. For each* $j = 1, \dots, J$*,* L_j *is continuous, non-decreasing and* L_j *can increase only when* $\mathbf{Z}(\cdot)$ *is on the face* $F_j = \{\mathbf{x} \in S : x_j = 0\}$, i.e., $\int_0^t 1_{\{Z_j(s)\neq 0\}} dL_j(s) = 0$ *for all* $t \geq 0$*.*

For our model of tandem queueing network with two stations, the reflection matrix R is given by $\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. In this case, given a filtration $(\mathcal{F}_t)_{t\geq 0}$ and a two-dimensional fBm $\mathbf{X}_H$ adapted to $(\mathcal{F}_t)_{t\geq 0}$ with associated data $(\mathbf{Z}_0, H, \boldsymbol{\vartheta}, \Lambda)$, an explicit construction of the process $(\mathbf{Z}, \mathbf{L})$ adapted to $(\mathcal{F}_t)_{t\geq 0}$ is carefully described in (2.3)–(2.6). The pathwise uniqueness of $(\mathbf{Z}, \mathbf{L})$ also follows from these equations and the uniqueness property of the one-dimensional reflection map. For a general reflection matrix R associated with a J-dimensional fBm, suitable assumptions on R to guarantee the strong existence and pathwise uniqueness of such a reflected process $\mathbf{Z}$ satisfying (i)–(iv) above are carefully described in Section 2 of [6]. We also refer to Theorem 2 of [2] and Proposition 4.2 of [31] for related results on existence and pathwise uniqueness of $\mathbf{Z}$.

To get an idea of rfBm introduced in the above definition, we note that it behaves like a fBm in the interior of the orthant S and it is confined to the orthant by instantaneous "reflection" at the boundary ∂S. For each j, the jth column of the *reflection matrix* R gives the direction of the reflection on the jth face F_j.

Here, we consider a tandem fluid queueing network with two stations $j = 1, 2$. At each station, a fBm input with the Hurst parameter $0 < H < 1$ is added and these fBm's are allowed to be correlated. The constant drift rates u_1 and u_2 at these two stations are considered as control terms. Our interest here is to establish the existence of optimal controls which guarantee the minimization of an appropriate long-term average cost functional.

In our analysis, the key ingredient in the proof is a coupling method. It helps us to analyze the behavior of a controlled state process represented by a rfBm with data $(\mathbf{Z}_0, H, -\mathbf{u}, \Lambda, R)$, where $\mathbf{u} = (u_1, u_2)^T$, R is a given 2×2 reflection matrix (see Section 2 for more details) and the Hurst parameter $0 < H < 1$. We show that the two-dimensional rfBm with initial data $\mathbf{Z}_0$ eventually couples with the rfBm with initial data $\mathbf{0}$. Typically, such a coupling argument works with Markov processes. In our case, the main reason for validity of the coupling arguments is based on the uniqueness results related to the reflection map (also known as the Skorokhod map or the regulator

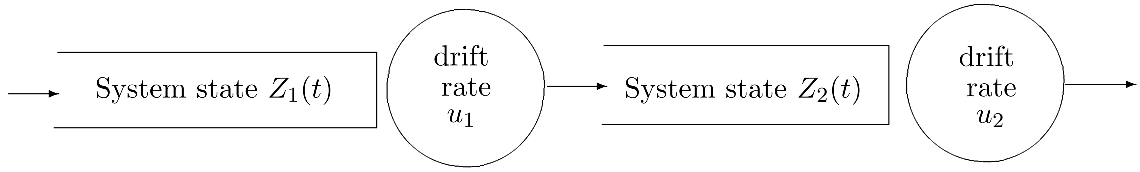


Figure 1: *Two queues in tandem with fBm input to each station and controllable static drift rates.*

map [13, 30]). A similar coupling method was used in the one-dimensional problem addressed in [10]. Our (coupling) techniques are different from those employed in [12, 3, 8, 17].

This coupling argument leads us to establish the existence and uniqueness of a stationary state process for a given control (u_1, u_2). The construction of our stationary state process is explicit (see Theorem 4.3). For a tandem fluid queueing network with a general input (with stationary increments) fed only at the first station, the existence of a stationary state process was established in [5]. In our model, there is a noisy input modeled by fBm at each station and our results complement the work of [5]. We also obtain the estimates for the tail distribution of two-dimensional stationary process. In the discrete setting, when the inter-arrival time and service time sequences are stationary, the stability of a system of queues in series was investigated in [21, 22]. Our stability arguments in Theorem 4.1 (a) complement the results in [21, 22]. We refer to [7, 25, 9] for tail asymptotics of one-dimensional queue length process with fBm input. We use the existence and uniqueness of this stationary process to show that the pay-off from the long-run average cost functional depends only on the control (u_1, u_2) and is independent of the initial data. Further analysis of the cost functional $I(u_1, u_2)$ enables us to establish the existence of an optimal control (u_1^*, u_2^*), which minimizes the cost functional over all available strategies.

The organization of the paper is as follows. In Section 2, we carefully describe our model in (2.3)–(2.6) and introduce the long-run average cost functional in (2.7). In Section 3, we provide a description of a sequence of ON/OFF network models whose limit of suitably scaled workload processes satisfies our model. This example is based on Delgado's work [6]. To obtain the controlled model of Section 2 as limiting model, we also introduce the notion of "thin control" for the ON/OFF queueing network in heavy traffic. Section 4 is devoted to the weak convergence results in Theorem 4.1 for the distribution of the state process with initial data $(0, 0)$. We also construct an explicit stationary state process associated with given control (u_1, u_2) in Theorem 4.3. In Section 5, we introduce the above described coupling method and show that arbitrary state process $\mathbf{Z}$ coalesces with the stationary state process $\mathbf{Z}^*$. We also obtain finite moment bounds for this coalescing time. The main result of this section is given in Theorem 5.5. In Section 6, by combining results in Sections 4 and 5 we represent the long-run average cost functional in terms of stationary state process and establish the existence of an optimal control (u_1^*, u_2^*) in Theorem 6.6. Furthermore, it turned out that this optimal control (u_1^*, u_2^*) is independent of the initial data. We indicate the generalization of our results to a tandem queueing network consists of n stations in Section 7. In particular, we describe the distribution of the stationary process.

The following notation is used. Denote the set of real numbers by $\mathbb{R}$ and non-negative real numbers by $\mathbb{R}_+$. Let $\mathbb{R}^d$ be the d-dimensional Euclidean space endowed with the usual Euclidean norm. For a given matrix M, denote by M^T its transpose and by M_i the i^{th} row of M. Let $\mathbb{I} = \mathbb{I}_{K\times K}$ denote the identity matrix for some K. When it is clear from the context, we will omit the subscript. For a set $A \subseteq \mathbb{R}^d$, denote its boundary by ∂A. When $\sup\limits_{0\le s\le t} |f_n(s) - f(s)| \to 0$

as $n \to \infty$ for all $t \geq 0$, we say that $f_n \to f$ uniformly on compact sets. The symbols "$\stackrel{D}{=}$" and "$\stackrel{D}{\longrightarrow}$" denote the equality and the convergence in distribution, respectively. The class of continuous functions $f : X \to Y$ is denoted by $C(X, Y)$. Inequalities for vectors are interpreted componentwise. We will denote generic constants by $K_1, K_2, \ldots$, and their values may change from one proof to another.

2 Model

Let $\mathbf{W}_H = (W_1, W_2)^T$ be a two-dimensional fBm with data $(\mathbf{0}, H, \mathbf{0}, \Lambda)$, where $\Lambda = \begin{pmatrix} 1 & \rho \\ \rho & 1 \end{pmatrix}$ with $|\rho| < 1$ and the Hurst parameter $0 < H < 1$. It is assumed that there exists a complete right-continuous filtration $(\mathcal{F}_t)_{t\geq 0}$ such that $\mathbf{W}_H$ is adapted to this filtration. We begin with a two-dimensional controlled state process $\{\mathbf{Q}(t) = (Q_1(t), Q_2(t))^T\}_{t\geq 0}$ which is a rfBm. Such a state process satisfying (2.1) below will be obtained as a heavy traffic limit of a controlled ON/OFF network in Section 3. The process $\{\mathbf{Q}(t)\}_{t\geq 0}$ takes values in the state space $S = [0, \infty) \times [0, \infty)$ and it can be written as

$$\mathbf{Q}(t) = \mathbf{Q}(0) + \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \mathbf{W}_H(t) - \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} t + \begin{pmatrix} \nu_1 & 0 \\ -\nu_2 & \nu_3 \end{pmatrix} \mathbf{Y}(t), \tag{2.1}$$

for all $t \geq 0$, where the initial data $\mathbf{Q}(0) = (Q_1(0), Q_2(0))^T \in S$ and $\mathbf{Q}(0)$ is an $\mathcal{F}_0$-measurable random vector such that $\mathbb{E}|\mathbf{Q}(0)| < \infty$. The constant control vector is given by $\boldsymbol{\theta} = (\theta_1, \theta_2)^T$, where $\theta_1 > 0$ and $\theta_2 > 0$ are constants. Also, $\sigma_i > 0$ and $\nu_j > 0$ are constants for $i = 1, 2$ and $j = 1, 2, 3$. The two-dimensional process $\{\mathbf{Y}(t) = (Y_1(t), Y_2(t))^T\}_{t\geq 0}$ satisfies $\mathbf{Y}(0) = \mathbf{0}$, $Y_i(\cdot)$ is non-decreasing with continuous paths and $\int_0^\infty Q_i(t)dY_i(t) = 0$ for $i = 1, 2$. Next, we reduce (2.1) to a simpler model given by (2.3) and (2.4) below for further analysis. Consider the constant matrix $\mathbf{K} = \frac{1}{\sigma_1}\begin{pmatrix} 1 & 0 \\ 0 & \frac{\nu_1}{\nu_2} \end{pmatrix}$ and multiply (2.1) by $\mathbf{K}$ to obtain

$$\mathbf{KQ}(t) = \mathbf{KQ}(0) + \mathbf{K}\begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_2 \end{pmatrix} \mathbf{W}_H(t) - \mathbf{K}\begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} t + \mathbf{K}\begin{pmatrix} \nu_1 & 0 \\ -\nu_2 & \nu_3 \end{pmatrix} \mathbf{Y}(t). \tag{2.2}$$

We label

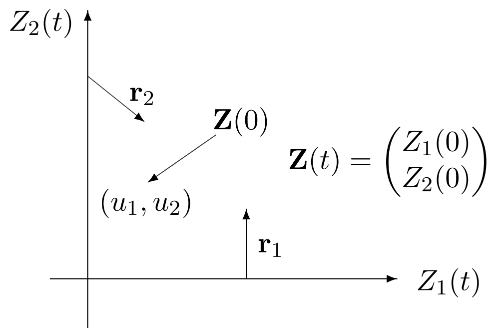
$$\mathbf{Z}(t) = \mathbf{KQ}(t), \quad \mathbf{Z}(0) = \mathbf{KQ}(0), \quad L_1(t) = \frac{\nu_1}{\sigma_1}Y_1(t), \quad L_2(t) = \frac{\nu_1\nu_3}{\nu_2\sigma_1}Y_2(t)$$

and the new constant control vector $\mathbf{u} = \mathbf{K}\boldsymbol{\theta}$. Then our model (2.2) can be written in the form

$$Z_1(t) = Z_1(0) + W_1(t) - u_1 t + L_1(t), \tag{2.3}$$

$$Z_2(t) = Z_2(0) + \sigma W_2(t) - u_2 t - L_1(t) + L_2(t), \tag{2.4}$$

where $\sigma = \frac{\nu_1\sigma_2}{\nu_2\sigma_1} > 0$. Thus the process $\{\mathbf{Z}(t) = (Z_1(t), Z_2(t))^T\}_{t\geq 0}$ also takes values in the two-dimensional orthant S. The process $\mathbf{Z}$ is adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. Notice that for $i = 1, 2$, $L_i(0) = 0$ and $L_i(\cdot)$ is non-decreasing with continuous paths and satisfies $\int_0^\infty Z_i(t)dL_i(t) = 0$. The picture depicted in Figure 2 is useful for a visualization of the two-dimensional state process $\mathbf{Z} = (Z_1, Z_2)^T$ in (2.3) and (2.4). Since $\mathbf{Z}(0) = \mathbf{KQ}(0)$, the random vector $\mathbf{Z}(0)$ is $\mathcal{F}_0$-measurable, $\mathbf{Z}(0) \in S$, and $\mathbb{E}|\mathbf{Z}(0)| < \infty$. Hence the process $\mathbf{Z}$ is a rfBm with associated data $(\mathbf{KQ}(0), H, -\mathbf{u}, \Lambda, R)$,



$$\mathbf{Z}(t) = \begin{pmatrix} Z_1(0) \\ Z_2(0) \end{pmatrix} + \begin{pmatrix} W_H^{(1)}(t) \\ \sigma W_H^{(2)}(t) \end{pmatrix} - \begin{pmatrix} u_1 \\ u_2 \end{pmatrix} t + R\mathbf{L}(t)$$

Figure 2: *A reflected fBm in the first quadrant with drift vector* $\mathbf{u} = (u_1, u_2)^T$ *and reflection matrix* $R = [\mathbf{r}_1, \mathbf{r}_2]$, $\mathbf{r}_1 = (1, 0)^T, \mathbf{r}_2 = (1, -1)^T$.

where $\Lambda = \begin{pmatrix} 1 & \rho\sigma \\ \rho\sigma & \sigma^2 \end{pmatrix}$ and $R = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$. In the later sections, we will assume a suitable moment condition on $\mathbf{Z}(0)$. Using the properties of the Skorokhod map (see e.g. [30]), we can write

$$L_1(t) \;=\; \max\left\{0, \max_{s\in[0,t]} \left(u_1 s - W_1(s) - Z_1(0)\right)\right\}, \tag{2.5}$$

$$L_2(t) \;=\; \max\left\{0, \max_{s\in[0,t]} \left(u_2 s - \sigma W_2(s) + L_1(s) - Z_2(0)\right)\right\}. \tag{2.6}$$

Note that for each $j = 1, 2$, $L_j(t)$ represents the cumulative idle-time in the station j during $[0, t]$.

The process $\mathbf{Z} = (Z_1, Z_2)^T$ can be considered as the workload process of a two-station tandem queueing system, where the controlled queue is fed by a fractional Brownian motion to each station. In Section 3 we provide a concrete example based on recent work of Delgado [6]. For a chosen constant control $\mathbf{u} = (u_1, u_2)^T$ with $u_1 > 0, u_2 > 0$ and $\mathcal{F}_0$-measurable initial data $\mathbf{Z}(0) = \mathbf{KQ}(0) \in S$, with $I\!E|\mathbf{Z}(0)| < \infty$, the corresponding state process $\mathbf{Z}$ is a rfBm with associated data $(\mathbf{Z}(0), H, -\mathbf{u}, \Lambda, R)$, where Λ and R are as given in the previous paragraph. Associated with this controlled state process $\mathbf{Z}$, the controller is faced with a cost structure consisting of the following three additive components: during a time interval $[t, t + dt]$,

(i) a control cost $h(\mathbf{u})dt$,

(ii) a state dependent holding cost $C(\mathbf{Z}(t))dt$, and

(iii) a penalty of $p_1 dL_1(t) + p_2 dL_2(t)$ for the idle-times at two stations.

Here $p_1 \geq 0, p_2 \geq 0$ are constants, h and C are non-negative continuous functions satisfying some basic assumptions. In the long-run average cost minimization problem (also known as *ergodic control problem*), the controller's goal is to minimize the cost functional

$$\begin{aligned} I(\mathbf{u}, \mathbf{Z}(0)) \;&\equiv\; \limsup_{T\to\infty} \frac{1}{T} I\!E\left[\int_0^T [h(\mathbf{u}) + C(\mathbf{Z}(t))]dt + \int_0^T [p_1 dL_1(t) + p_2 dL_2(t)]\right] \\ &=\; h(\mathbf{u}) + \limsup_{T\to\infty} \frac{1}{T} I\!E\left[\int_0^T C(\mathbf{Z}(t))dt + p_1 L_1(T) + p_2 L_2(T)\right]. \end{aligned} \tag{2.7}$$

The functions h and C satisfy the following standing assumptions:

(H1) The function $h : S \to [0,\infty)$ is continuous, $h(0,0) = 0$ and is increasing to $+\infty$ in each variable as the variable tends to infinity.

(H2) The function $C : S \to [0,\infty)$ is also continuous with $C(0,0) = 0$ and non-decreasing in each variable, and $\lim_{x+y\to\infty} C(x,y) = \infty$.

(H3) The function C satisfies the following polynomial growth condition:

$$0 \le C(x,y) \le K(1 + |x|^m + |y|^m)$$

for some constants $K > 0$ and $m \ge 1$. These constants are independent of x and y.

The polynomial growth condition in (H3) of running cost function is quite common in stochastic control problem related to Brownian networks.

3 Controlled two station fluid models with ON/OFF sources

Here we provide a brief description of a sequence of concrete network models in which the limit of suitably scaled workload processes satisfies a controlled rfBm model. This example is based on Delgado's work [6] and we use its notation throughout this section. It should be noted that in [6], a more general model is considered, whereas our example in this section is related to a tandem queue with two service stations. The novel feature here is the introduction of a 'thin control' using the heavy traffic condition.

Consider a sequence of controlled queueing networks indexed by (N, r), where $N \ge 1$ is an integer valued parameter and $r > 0$ is a real valued parameter. Each network consists of two stations ($j = 1, 2$) and there is a single server at each station (recall Figure 1). In the (N, r)-th network, there are N input sources for each station (e.g. N users connected to the server) and each user stays connected to the server for a random ON-period with distribution function F_1, and stays off during a random OFF-period of time with distribution function F_2. It is assumed that for each user, these 'ON' periods and 'OFF' periods are independent of each other. For each $i = 1, 2$, assume that $1 - F_i(x) \sim c_i x^{-\beta_i}$ for large x, where $1 < \beta_i < 2$, c_i and β_i are positive constants. Hence, each F_i has finite mean $\tilde{\mu}_i$ and infinite variance. In the j-th station, ON and OFF periods of the n-th user are described by

$$U_j^{(n)}(t) = \begin{cases} 1, & \text{if the } n\text{-th source is 'ON' at time } t, \\ 0, & \text{if the } n\text{-th source is 'OFF' at time } t. \end{cases}$$

Assume that if all the sources are 'ON', then fluid would arrive at station j at a deterministic rate α_j^N for $j = 1, 2$.

Next, let $\mathbb{P} = (p_{k\ell})_{2\times 2}$ represent the "routing matrix" of the network. Here $p_{k\ell}$ is the proportion of fluid leaving station k goes to next to station ℓ and $1 - \sum_{k=1}^{2} p_{k\ell} \ge 0$ is the proportion of fluid that leaves the network after being served at station k. In our tandem queue example, $\mathbb{P} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The quantities $\tilde{\mu}_1, \tilde{\mu}_2, \alpha_1^N, \alpha_2^N$ and $\mathbb{P}$ are considered as system primitives. We let $c = \frac{\tilde{\mu}_1}{\tilde{\mu}_1+\tilde{\mu}_2}$, $\boldsymbol{\alpha}^N = (\alpha_1^N, \alpha_2^N)^T$ and $Q = (\mathbb{I} - \mathbb{P}^T)^{-1}$. First, we assume that

$$\lim_{N\to\infty} \boldsymbol{\alpha}^N = \boldsymbol{\alpha},$$

where $\boldsymbol{\alpha} = (\alpha_1, \alpha_2)^T$ for some $\alpha_1 > 0$ and $\alpha_2 > 0$.

Then, following [6], we can compute the long run fluid rate vector $\boldsymbol{\lambda}^N$ which satisfies the traffic equation $\boldsymbol{\lambda}^N = cQ\boldsymbol{\alpha}^N$. Then $\boldsymbol{\lambda}^N$ is given by $\lambda_1^N = c\alpha_1^N$ and $\lambda_2^N = c(\alpha_1^N + \alpha_2^N)$. Furthermore,

$$\lim_{N\to\infty} \boldsymbol{\lambda}^N = \begin{pmatrix} \lambda_1 \\ \lambda_2 \end{pmatrix} = \begin{pmatrix} c\alpha_1 \\ c(\alpha_1 + \alpha_2) \end{pmatrix}. \tag{3.1}$$

For the (N, r)-th system, it is assumed that the controlled deterministic service rate at station j is given by

$$\mu_j^N(r) = \lambda_j^N \left(1 + \frac{1}{\sqrt{N}} \theta_j(r) \right) \text{ for } j = 1, 2, \tag{3.2}$$

where the control variables $\theta_j(r)$ are positive bounded continuous functions and $\lim_{r\to\infty} \theta_j(r) = 0$. More precisely, we assume that

$$\lim_{r\to\infty} r^{1-H} \boldsymbol{\theta}(r) = \lim_{r\to\infty} \begin{pmatrix} r^{1-H}\theta_1(r) \\ r^{1-H}\theta_2(r) \end{pmatrix} = \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix}, \tag{3.3}$$

where $\theta_1 > 0$ and $\theta_2 > 0$ are constants and $H = \frac{1}{2}(3 - \min\{\beta_1, \beta_2\}) \in (\frac{1}{2}, 1)$. Mean service time at station j is given by $m_j^N(r) = 1/\mu_j^N(r)$ for $j = 1, 2$, and the corresponding mean service time matrix $M^N(r)$ is given by $M^N(r) = \mathrm{diag}(m_1^N(r), m_2^N(r))$. Notice that $\lim_{N\to\infty} \mu_j^N(r) = \lambda_j$ and $\lim_{N\to\infty} M^N(r) = M$, where $M = \mathrm{diag}(\lambda_1^{-1}, \lambda_2^{-1})$ and those limits are uniform on compact sets (in r).

Next, we introduce the fluid-traffic intensity vector of the (N, r)-th network by

$$\boldsymbol{\rho}^N(r) = M^N(r)\boldsymbol{\lambda}^N. \tag{3.4}$$

It is easy to see that $\lim_{N\to\infty} \boldsymbol{\rho}^N(r) = \mathbf{e} \equiv \begin{pmatrix} 1 \\ 1 \end{pmatrix}$. Then we observe the "heavy traffic" condition (HT)

$$\lim_{N\to\infty} \sqrt{N}(\boldsymbol{\rho}^N(r) - \mathbf{e}) = -\boldsymbol{\theta}(r), \tag{3.5}$$

which readily follows from (3.1), (3.2) and (3.4). We remark that the effect of the control $\boldsymbol{\theta}(r)$ in the service rate is of order $\frac{1}{\sqrt{N}}$ and also (3.3) holds. Such controls are called as "thin controls" (see Ata and Kumar [1]). In comparison, the heavy traffic condition in [6] assumes that $\boldsymbol{\theta}(r)$ is identically zero and their network is not controlled.

To compute the workload process (i.e., the amount of time required for server to complete processing of all fluids in the queue (or being served)), next we compute the matrix $R^N(r) = \mathbb{I} - M^N(r)\mathbb{P}^T M^N(r)^{-1}$ as in Lemma 1 of [6]. It can be easily seen that

$$\lim_{N\to\infty} R^N(r) = R \equiv \begin{pmatrix} 1 & 0 \\ -\frac{\lambda_1}{\lambda_2} & 1 \end{pmatrix},$$

which is independent of r. Then, we introduce the cumulative external fluid arrived at station j during $[0, t]$ by $E_j^N(t)$, where

$$E_j^N(t) = \alpha_j^N \int_0^t \frac{1}{N} \left(\sum_{n=1}^N U_j^{(n)}(s) \right) ds \text{ for } j = 1, 2.$$

The aggregated cumulative external fluid traffic process is given by $\{\mathbf{E}^N(t) = (E_1^N(t), E_2^N(t))^T\}_{t\geq 0}$ for $t \geq 0$. The two-dimensional workload process $\{\mathbf{Z}_r^N(t)\}_{t\geq 0}$ and the cumulative idle time process $\{\mathbf{L}_r^N(t)\}_{t\geq 0}$ of the (N,r)-th network satisfy

$$\mathbf{Z}_r^N(t) = R^N(r)M^N(r)Q\mathbf{E}^N(t) - R^N(r)\mathbf{e}t + R^N(r)\mathbf{L}_r^N(t),$$

for $t \geq 0$. Next, we introduce the scaled processes associated with the (N,r)-th network. Let

$$\widehat{\mathbf{Z}}_r^N(t) \equiv \sqrt{N}\frac{\mathbf{Z}^N(rt)}{r^H \mathcal{L}^{1/2}(r)}, \quad \widehat{\mathbf{E}}_r^N(t) \equiv \sqrt{N}\frac{\mathbf{E}^N(rt) - crt\boldsymbol{\alpha}^N}{r^H \mathcal{L}^{1/2}(r)}, \quad \widehat{\mathbf{L}}_r^N(t) \equiv \sqrt{N}\frac{\mathbf{L}^N(rt)}{r^H \mathcal{L}^{1/2}(r)},$$

where $\mathcal{L}(r)$ is a positive slowly varying function at infinity as defined in Section 3.3 of [6]. Then following the discussions in [6], these scaled processes are related by

$$\left.\begin{aligned} &\widehat{\mathbf{Z}}_r^N(t) = \widehat{\mathbf{X}}_r^N(t) + R^N(r)\widehat{\mathbf{L}}_r^N(t), \\ &\text{where } \widehat{\mathbf{X}}_r^N(t) = R^N(r)M^N(r)Q\widehat{\mathbf{E}}_r^N(t) + \frac{\sqrt{N}}{r^H}R^N(r)(\boldsymbol{\rho}^N(r) - \mathbf{e})rt \\ &\qquad \text{and for each } j,\ \widehat{L}_{r,j}^N(0) = 0, \int_0^\infty \widehat{Z}_{r,j}^N(s)d\widehat{L}_{r,j}^N(s) = 0. \end{aligned}\right\} \tag{3.6}$$

Associated with the scaled processes of the (N,r)-th network, we consider a long-run average cost minimization problem with linear control costs and the corresponding cost functional

$$I_r^N(\boldsymbol{\theta}(r), \widehat{\mathbf{Z}}_r^N(0)) = h(r^{1-H}\boldsymbol{\theta}(r)) + \limsup_{T\to\infty}\frac{1}{T}I\!\!E\left[\int_0^T C(\widehat{\mathbf{Z}}_r^N(t))dt + \mathbf{p}\cdot\widehat{\mathbf{L}}_r^N(T)\right], \tag{3.7}$$

where $\mathbf{p} = (p_1, p_2)^T$ is a constant vector. The function h is linear and C satisfies the assumptions in Section 2. Using Theorem 1 in [6], we can approximate the scaled system in (3.6) by rfBm and we can minimize the associated cost functional of the limiting rfBm system. To obtain the limiting workload process, we introduce different types of convergence and their notation as described in [6]. We denote the convergence in distribution in $C[0,\infty)$ by "$\mathcal{D}$-lim" and the convergence of finite dimensional distribution by "$\widetilde{\lim}$". Following the proof of Theorem 1 in [6] together with (3.3), (3.5) and (3.6) we obtain

$$\mathcal{D}\text{-}\lim_{r\to\infty}\widetilde{\lim_{N\to\infty}}\widehat{\mathbf{X}}_r^N(\cdot) = \mathbf{X}(\cdot), \text{ where } \mathbf{X}(t) = RMQ\widetilde{\mathbf{B}}_H(t) - R\boldsymbol{\theta}t \text{ for all } t \geq 0,$$

where $\widetilde{\mathbf{B}}_H$ is a two-dimensional fBm with $H = (3 - \min\{\beta_1, \beta_2\})/2$ and a covariance matrix $\Lambda = \text{diag}(\sigma^2_{\text{lim}}\alpha_1^2, \sigma^2_{\text{lim}}\alpha_2^2)$ and $\sigma^2_{\text{lim}} > 0$ is as in page 196 of [6]. Notice that $\frac{1}{2} < H < 1$. An easy computation shows $RMQ = \text{diag}(\lambda_1^{-1}, \lambda_2^{-1})$. Therefore, using the continuous mapping theorem as in [6], we obtain the limiting system as a rfBm given by

$$\mathbf{Z}(t) = \begin{pmatrix} \frac{\sigma^2\alpha_1^2}{\lambda_1} & 0 \\ 0 & \frac{\sigma^2\alpha_2^2}{\lambda_2} \end{pmatrix}\mathbf{B}_H(t) - \begin{pmatrix} u_1 t \\ u_2 t \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ -\frac{\lambda_1}{\lambda_2} & 1 \end{pmatrix}\mathbf{L}(t), \tag{3.8}$$

where $\mathbf{B}_H(\cdot)$ is a standard two-dimensional fBm, $u_1 = \theta_1, u_2 = \theta_2 - \frac{\lambda_1}{\lambda_2}\theta_1$, $\mathbf{L}(\cdot)$ represents the idle time process, and for each j, $L_j(\cdot)$ is non-decreasing, $L_j(0) = 0$ and $\int_0^t Z_j(t)dL_j(t) = 0$. Hence, we see that with a super-imposed ON-OFF input sources and controllable services times for the queueing system, a suitably-scaled workload process in the limit satisfies the model in (3.8), which is essentially (2.1). With a cost structure (3.7) for the queueing network problem in mind, we intend to study in this paper a formal fractional Brownian control problem by imposing the cost functional for the limiting model (3.8) as

$$I(\mathbf{u}, \mathbf{Z}(0)) = h(\mathbf{u}) + \limsup_{T\to\infty} \frac{1}{T} I\!E\left[\int_0^T C(\mathbf{Z}(t))dt + \mathbf{p}\cdot\mathbf{L}(T)\right], \tag{3.9}$$

where $\mathbf{p} = (p_1, p_2)^T$ is a constant vector. Notice that $h(\mathbf{u}) = u_1 + u_2 = \frac{\alpha_2}{\alpha_1+\alpha_2}\theta_1 + \theta_2 > 0$. We, however, do not attempt to solve the underlying queueing control problem in this paper. A solution to the limiting control problem with cost functional (3.9) provides useful insights into the queueing network control problem with the associated cost functional (3.7).

4 Weak convergence and stationarity

Recall the model described by (2.3)–(2.6) in Section 2. If the initial data $(Z_1(0), Z_2(0)) = (0, 0)$, then the corresponding processes Z_1^0 and Z_2^0 can be written as

$$Z_1^0(t) = W_1(t) - u_1 t + L_1^0(t), \text{ and} \tag{4.1}$$

$$Z_2^0(t) = \sigma W_2(t) - u_2 t - L_1^0(t) + L_2^0(t), \text{ for } t \geq 0, \tag{4.2}$$

where $\sigma > 0$ is a constant, W_1 and W_2 are correlated fBm's with a constant correlation coefficient $\rho \in [-1, 1]$ and the Hurst parameter $0 < H < 1$. Furthermore, we have

$$L_1^0(t) = \max_{s\in[0,t]} (u_1 s - W_1(s)), \text{ and} \tag{4.3}$$

$$L_2^0(t) = \max_{s\in[0,t]} \left(u_2 s - \sigma W_2(s) + L_1^0(s)\right), \text{ for } t \geq 0. \tag{4.4}$$

We introduce the vector-valued process $\mathbf{Z}^0$ by $(\mathbf{Z}^0(t) = (Z_1^0(t), Z_2^0(t))^T)_{t\geq 0}$ and next we establish the weak convergence of $\mathbf{Z}^0$ and identify its limiting distribution $\mathbf{Z}^0(\infty)$.

Theorem 4.1. *The following results hold:*

(a) Assume $u_1 > 0$ and $u_1 + u_2 > 0$. The process $(\mathbf{Z}^0(t))_{t\geq 0}$ converges weakly to the random vector $\mathbf{Z}^0(\infty) = (Z_1^0(\infty), Z_2^0(\infty))^T$ as $t \to \infty$, where $Z_1^0(\infty)$ and $Z_2^0(\infty)$ satisfy

$$Z_1^0(\infty) = \sup_{0\leq s<\infty} \{W_1(s) - u_1 s\} \text{ and} \tag{4.5}$$

$$Z_1^0(\infty) + Z_2^0(\infty) = \sup_{0\leq r\leq s<\infty} \{(W_1(s) - u_1 s) + (\sigma W_2(r) - u_2 r)\}. \tag{4.6}$$

The random vector $\mathbf{Z}^0(\infty)$ has a proper distribution function. (More precisely, $Z_1^0(\infty) < \infty$ a.s. if $u_1 > 0$ and $Z_2^0(\infty) < \infty$ a.s. when $u_1 > 0$ and $u_1 + u_2 > 0$.)

(b) When $u_1 > 0$ and $u_2 > 0$, the tail distribution of $\mathbf{Z}^0(\infty)$ satisfies

$$\lim_{z\to\infty} z^{2H-2} \log I\!P[Z_1^0(\infty) \geq z] = -\theta^*(u_1) \text{ and} \tag{4.7}$$

$$\limsup_{z\to\infty} z^{2H-2} \log I\!P[Z_2^0(\infty) \geq z] \leq -\frac{1}{\sigma^2}\theta^*(u_2), \tag{4.8}$$

where

$$\theta^*(u) = \frac{u^{2H}}{2H^{2H}(1-H)^{2-2H}} > 0, \quad \text{for } u > 0.$$

Proof. (a) First we consider the case $u_1 > 0$ and $u_2 > 0$. Consider W_1 and W_2, which are correlated fBm's with a constant correlation coefficient $\rho \in [-1, 1]$. To construct such a process, we begin with two independent fBm's Y_1 and Y_2 and let for all $t \geq 0$

$$W_1(t) = Y_1(t), \quad W_2(t) = \rho Y_1(t) + \bar{\rho} Y_2(t), \tag{4.9}$$

where $\bar{\rho} = \sqrt{1-\rho^2}$. To prove part (a), our first step is to show that

$$(Z_1^0(T), Z_1^0(T) + Z_2^0(T)) \stackrel{D}{=} \left(\max_{0 \leq s \leq T} \{W_1(s) - u_1(s)\}, \max_{0 \leq r \leq s \leq T} \{W_1(s) - u_1 s + \sigma W_2(r) - u_2 r\} \right), \tag{4.10}$$

for each $T > 0$, where Z_1^0 and Z_2^0 satisfy (4.1) and (4.2), respectively. We keep $T > 0$ fixed and introduce $(B_1(s), B_2(s))_{0 \leq s \leq T}$ by

$$B_1(s) = W_1(T) - W_1(T-s), \quad B_2(s) = W_2(T) - W_2(T-s), \tag{4.11}$$

for each $0 \leq s \leq T$. Then, it is easy to verify that each $(B_i(s))_{0 \leq s \leq T}$ is a one-dimensional fBm for $i = 1, 2$ (see Ex. 5.1.1, page 286, [28]). Using (4.9) in (4.11), we also have

$$I\!E\,[B_1(t)B_2(s)] = I\!E\,[W_1(t)W_2(s)] = \rho \Upsilon_H(s,t),$$

where $\Upsilon_H(s,t)$ is as in (1.1). Since both $(B_1(s), B_2(s))_{0 \leq s \leq T}$ and $(W_1(s), W_2(s))_{0 \leq s \leq T}$ are Gaussian processes with the same mean and covariance function, they induce the same measure μ_W on $C[0,T]$. The point here is that even though $(B_1(s), B_2(s))$ depends on T in (4.11), the measure induced on $C[0,T]$ is the same as that of $(W_1(s), W_2(s))_{0 \leq s \leq T}$.

Using (4.1)–(4.4), we can write $Z_1^0(T)$ and $Z_2^0(T)$ in the form

$$\begin{aligned} Z_1^0(T) &= \max_{0 \leq r \leq T} \{B_1(T-r) - u_1(T-r)\} \text{ and} \\ Z_1^0(T) + Z_2^0(T) &= \max_{0 \leq r \leq s \leq T} \{B_1(T-r) - u_1(T-r) + \sigma B_2(T-s) - u_2(T-s)\} \\ &= \max_{0 \leq v \leq t \leq T} \{B_1(t) - u_1 t + \sigma B_2(v) - u_2 v\}. \end{aligned}$$

We have used the time substitutions $t = T - r$ and $v = T - s$ in the last equality. Since $(B_1, B_2) \stackrel{D}{=} (W_1, W_2)$ on $C[0,T]$ for each $T > 0$, the desired equality (4.10) follows. We let

$$M_1(T) = \max_{0 \leq s \leq T} \{W_1(s) - u_1 s\} \text{ and } M_2(T) = \max_{0 \leq r \leq s \leq T} \{W_1(s) - u_1 s + \sigma W_2(r) - u_2 r\}.$$

Since W_i has stationary and ergodic increments, we have $\lim_{T \to \infty} \frac{W_i(T)}{T} = 0$ a.s. for $i = 1, 2$ (see e.g. [28, 23] for additional properties and a more detailed description of fBm). Thus, $M_1(T) < \infty$ and $M_2(T) < \infty$ a.s. Clearly, $(M_1(T), M_2(T)) \to (M_1(\infty), M_2(\infty))$ as $T \to \infty$ a.s. and $M_i(\infty) < \infty$ for $i = 1, 2$. Hence, we can conclude that

$$(Z_1^0(T), Z_1^0(T) + Z_2^0(T)) \xrightarrow{D} (M_1(\infty), M_2(\infty)) \text{ as } T \to \infty,$$

and as a consequence, we have

$$(Z_1^0(T), Z_2^0(T)) \xrightarrow{D} (M_1(\infty), M_2(\infty) - M_1(\infty)) \text{ as } T \to \infty.$$

This completes the proof of part (a) for the case $u_1 > 0$ and $u_2 > 0$. For the case $u_1 > 0$ and $u_1 + u_2 > 0$, we pick an $\epsilon > 0$ so that $\min\{u_1, u_1 + u_2\} > \epsilon > 0$. We intend to show that the right hand side of (4.6) is finite. Using (4.6), we obtain

$$\begin{aligned}
&\sup_{0\le r\le s} \{(W_1(s) - u_1 s) + (\sigma W_2(r) - u_2 r)\}\\
&\le \sup_{0\le r\le s} \{(W_1(s) - \epsilon s) + (\sigma W_2(r) - (u_1 + u_2 - \epsilon)r)\}\\
&\le \sup_{s\ge 0} (W_1(s) - \epsilon s) + \sup_{s\ge 0} (\sigma W_2(s) - (u_1 + u_2 - \epsilon)s)\,.
\end{aligned}$$

From the above proof for the case $u_1 > 0$ and $u_2 > 0$, the right hand side of the last inequality is finite a.s. Hence, it follows that $Z_1^0(\infty) + Z_2^0(\infty) < \infty$ a.s. and this completes part (a).

(b) The result in (4.7) was shown in [7, 25, 9]. To prove (4.8), we begin with $Z_2^0(t)$ as in (4.2). Using (4.4), we notice that

$$L_2^0(t) \le L_1^0(t) + \max_{s\in[0,t]} (u_2 s - \sigma W_2(s))$$

and hence by (4.2), we have

$$\begin{aligned}
Z_2^0(t) &\le \sigma W_2(t) - u_2 t + \max_{s\in[0,t]} (u_2 s - \sigma W_2(s))\\
&= \max_{s\in[0,t]} \{\sigma (W_2(t) - W_2(s)) - u_2(t-s)\}\\
&= \max_{s\in[0,t]} \{\sigma B_2(s) - u_2 s\} \qquad \text{(using (4.11))}\\
&\stackrel{D}{=} \max_{s\in[0,t]} \{\sigma W_2(s) - u_2 s\}\,.
\end{aligned}$$

Thus,

$$I\!P[Z_2^0(T) \ge z] \le I\!P[Y(\infty) \ge z],$$

where $Y(\infty) = \sup_{0\le s}\{\sigma W_H(s) - u_2 s\} = \sigma \sup_{0\le s}\{W_H(s) - \frac{u_2}{\sigma} s\}$ and W_H is a one-dimensional fBm with the Hurst parameter H. Using (4.7), we can estimate $I\!P[Y(\infty) \ge z]$. Hence, the result (4.8) follows. ∎

When the drift rates u_1 and u_2 satisfy the condition $u_1 > 0 > u_2$ with $u_1 + u_2 > 0$ (i.e. $u_1 > u_1 + u_2 > 0$), we can replace (4.8) with a weaker upper bound as descried below.

Corollary 4.2. *Assume $u_1 > u_1 + u_2 > 0$. Then* (4.7) *holds and instead of* (4.8)*, the following estimate holds for $Z_2^0(\infty)$:*

$$\limsup_{z\to\infty} z^{2H-2} \log I\!P[Z_2^0(\infty) \ge z] \le -\frac{1}{2^{2(1-H)}}\theta^*\left(\frac{u_1 + u_2}{1 + \sigma^{\frac{1}{H}}}\right), \tag{4.12}$$

where $\theta^*(\cdot)$ is described in part (b) of Theorem 4.1.

Proof. The estimate (4.7) remains valid since $u_1 > 0$. It remains to estimate $I\!P[Z_2^0(\infty) \ge z]$. Since $Z_2^0(\infty) \le Z_1^0(\infty) + Z_2^0(\infty)$, we estimate $I\!P[Z_1^0(\infty) + Z_2^0(\infty) \ge z]$. We pick $0 < \varrho < u_1 + u_2$ and introduce $\varsigma = \frac{u_1+u_2-\varrho}{\sigma} > 0$. Then using (4.6), we obtain

$$Z_1^0(\infty) + Z_2^0(\infty) \le \sup_{t\ge 0}(W_1(t) - \varrho t) + \sigma \sup_{t\ge 0}(W_2(t) - \varsigma t).$$

Hence

$$I\!\!P[Z_1^0(\infty)+Z_2^0(\infty)\geq z]\leq I\!\!P\left[\sup_{t\geq 0}(W_1(t)-\varrho t)\geq \frac{z}{2}\right]+I\!\!P\left[\sup_{t\geq 0}(W_2(t)-\varsigma t)\geq \frac{z}{2\sigma}\right].$$

Since $\varrho>0$ and $\varsigma>0$, now we can use (4.7) and a straightforward calculation using the above estimate to obtain

$$\limsup_{z\to\infty} z^{2H-2}\log I\!\!P[Z_1^0(\infty)+Z_2^0(\infty)\geq z]\leq -\frac{1}{2^{2(1-H)}}\bar{\theta}(\varrho,\sigma), \tag{4.13}$$

where $\bar{\theta}(\varrho,\sigma)=\min\left\{\theta^*(\varrho),\frac{1}{\sigma^{2(1-H)}}\theta^*\left(\frac{u_1+u_2-\varrho}{\sigma}\right)\right\}$. Using the expression for $\theta^*(\cdot)$ in part (b) of Theorem 4.1, we observe that

$$\bar{\theta}(\varrho,\sigma)=\frac{1}{2H^{2H}(1-H)^{2(1-H)}}\min\left\{\varrho^{2H},\left(\frac{u_1+u_2-\varrho}{\sigma^{1/H}}\right)^{2H}\right\},$$

where $0<\varrho<u_1+u_2$. It is straightforward to compute the maximum value of $\bar{\theta}(\varrho,\sigma)$ when $0<\varrho<u_1+u_2$ and it is achieved at $\varrho=\varrho^*=\frac{u_1+u_2}{1+\sigma^{\frac{1}{H}}}$. Moreover, $\bar{\theta}(\varrho^*,\sigma)=\theta^*\left(\frac{u_1+u_2}{1+\sigma^{\frac{1}{H}}}\right)$. Since $I\!\!P[Z_2^0(\infty)\geq z]\leq I\!\!P[Z_1^0(\infty)+Z_2^0(\infty)\geq z]$, then as a consequence of the above estimates, we obtain (4.12). ■

Next, we intend to establish the existence of a stationary process on the same probability space on which $\mathbf{Z}^0$ is defined. The coupling arguments in the next section will establish the uniqueness in law for this stationary process (see Corollary 5.6). For a tandem network with two stations and only one random input process with stationary ergodic increments at the first station, the existence of a unique stationary process was established in [5]. In our situation, there are two input noise processes, which are correlated and have stationary ergodic increments. The following result complements the work of [5] and our stationary process has the more explicit description.

Theorem 4.3. *Let $u_1>0$ and $u_1+u_2>0$. Then there is a probability space $(\Omega,\mathcal{F},I\!\!P)$ supporting $\mathbf{Z}^0$ as described in (4.1)–(4.4) and a stationary process $\mathbf{Z}^*=(Z_1^*,Z_2^*)$, which satisfies the following equations with respect to the same fBm's W_1 and W_2 with the correlation coefficient $\rho\in[-1,1]$: For all $t\geq 0$,*

$$Z_1^*(t) = Z_1^*(0)+W_1(t)-u_1t+L_1^*(t), \text{ and} \tag{4.14}$$
$$Z_2^*(t) = Z_2^*(0)+\sigma W_2(t)-u_2t-L_1^*(t)+L_2^*(t), \tag{4.15}$$

Here $L_1^*(0)=L_2^*(0)=0$, $L_1^*(t)$ and $L_2^*(t)$ are non-decreasing, continuous processes adapted to the filtration of $\{\mathbf{W}_H=(W_1,W_2)^T\}$, which also satisfy

$$\int_0^\infty Z_i^*(t)dL_i^*(t)=0 \text{ for } i=1,2. \tag{4.16}$$

Let $g(x)=e^{x^\alpha}$ where $0<\alpha<2(1-H)$. Then $I\!\!E[g(\mathbf{Z}^*(t))]<\infty$ and consequently, $I\!\!E|\mathbf{Z}^*(t)|^N<\infty$ for every $N\geq 1$.

Proof. We begin with two independent two-sided fBm's, Y_1 and Y_2 defined on some probability space $(\Omega,\mathcal{F},I\!\!P)$ (cf. [23]). Thus Y_1, Y_2 are defined for all $-\infty<t<+\infty$. We let $W_1=Y_1$

and $W_2 = \rho Y_1 + \bar{\rho} Y_2$, where $\bar{\rho} = \sqrt{1-\rho^2}$. Then $\mathbf{W}_H(t) = (W_1(t), W_2(t))^T$ is defined for all $-\infty < t < +\infty$ and $\mathbf{W}_H(0) = \mathbf{0}$.

Next, we consider the two-dimensional "free process"

$$(W_1(t) - u_1 t, (W_1(t) - u_1 t) + (\sigma W_2(t) - u_2 t))^T$$

for all $-\infty < t < +\infty$. Introduce the two-dimensional process $(X(t), Y(t))^T$ using the reflection map as described below. We write

$$\begin{aligned} X(t) &= W_1(t) - u_1 t - \inf_{-\infty < s \le t} \{W_1(s) - u_1 s\} \text{ and} && (4.17)\\ Y(t) &= W_1(t) - u_1 t + \sigma W_2(t) - u_2 t - \inf_{-\infty < s \le r \le t} \{W_1(s) - u_1 s + \sigma W_2(r) - u_2 r\}, && (4.18) \end{aligned}$$

for all $-\infty < t < +\infty$. Using the fact $\lim_{|t| \to \infty} \frac{|W_i(t)|}{|t|} = 0$ a.s., it is easy to check that $X(t)$ and $Y(t)$ are finite for every $-\infty < t < +\infty$.

We intend to show that $(X(t), Y(t)) \stackrel{D}{=} (X(0), Y(0))$ for all $t \ge 0$. Let us fix $t > 0$ and notice that we can write

$$\begin{aligned} X(t) &= \sup_{-\infty < s \le t} [W_1(t) - W_1(s) - u_1(t-s)] \text{ and} && (4.19)\\ Y(t) &= \sup_{-\infty < s \le r \le t} [W_1(t) - W_1(s) - u_1(t-s) + \sigma(W_2(t) - W_2(r)) - u_2(t-r)]. && (4.20) \end{aligned}$$

For $i = 1, 2$, let $B_i(s) = W_i(t) - W_i(t-s)$ for all $s \ge 0$. Then it is straightforward to check that B_1 and B_2 are also one-dimensional fBm's with the same correlation coefficient ρ. It is important to notice that B_1 and B_2 depend on t by their definitions. Substituting B_1, B_2 in (4.19), (4.20) and then using the time substitution $\tilde{s} = t - s \ge 0$, $\tilde{r} = t - r \ge 0$, we obtain

$$\begin{aligned} X(t) &= \max_{0 \le \tilde{s} \le t} (B_1(\tilde{s}) - u_1 \tilde{s}) \text{ and}\\ Y(t) &= \max_{0 \le \tilde{r} \le \tilde{s} \le t} [(B_1(\tilde{s}) - u_1 \tilde{s}) + (\sigma B_2(\tilde{r}) - u_2 \tilde{r})]. \end{aligned}$$

We observe that $Y(t) \ge X(t)$ for all t, by letting $\tilde{r} = 0$. We recall

$$\{(B_1(s), B_2(s)) : s \ge 0\} \stackrel{D}{=} \{(W_1(s), W_2(s)) : s \ge 0\}$$

and therefore conclude that

$$\begin{pmatrix} X(t) \\ Y(t) \end{pmatrix} \stackrel{D}{=} \begin{pmatrix} \sup_{0 \le s < \infty} (W_1(s) - u_1 s) \\ \sup_{0 \le r \le s < \infty} [(W_1(s) - u_1 s) + (\sigma W_2(r) - u_2 r)] \end{pmatrix}. \quad (4.21)$$

Notice that the right hand side of (4.21) is independent of t and hence $(X(t), Y(t))$ is a stationary process. In particular, $(X(t), Y(t)) \stackrel{D}{=} (Z_1^0(\infty), Z_1^0(\infty) + Z_2^0(\infty))$ for all $t \ge 0$, where $Z_1^0(\infty)$ and $Z_2^0(\infty)$ are given in (4.5) and (4.6). Next, we define

$$\begin{pmatrix} Z_1^*(t) \\ Z_2^*(t) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} \begin{pmatrix} X(t) \\ Y(t) \end{pmatrix}, \text{ for all } t \ge 0.$$

Then, clearly $\mathbf{Z}^* = (Z_1^*, Z_2^*)^T$ is also a stationary process. Since $Y(t) \geq X(t) \geq 0$ for all t, we have $Z_1^*(t) \geq 0$ and $Z_2^*(t) \geq 0$ for all $t \geq 0$. We let

$$\begin{aligned}
\widetilde{L}_1^*(t) &= \sup_{-\infty < s \leq t} \left(u_1 s - W_1(s)\right), \\
\widetilde{L}_2^*(t) &= \sup_{-\infty < s \leq r \leq t} \left(u_1 s - W_1(s) + u_2 r - \sigma W_2(r)\right).
\end{aligned}$$

Notice that $Z_1^*(0) = \widetilde{L}_1^*(0)$ and $Z_2^*(0) = \widetilde{L}_2^*(0) - \widetilde{L}_1^*(0)$. Introduce the processes $L_1^*(\cdot)$ and $L_2^*(\cdot)$ given by

$$\begin{aligned}
L_1^*(t) &= \max\left\{0, \max_{s\in[0,t]} \left(u_1 s - W_1(s) - Z_1^*(0)\right)\right\}, \\
L_2^*(t) &= \max\left\{0, \max_{s\in[0,t]} \left(u_2 s - \sigma W_2(s) + L_1^*(s) - Z_2^*(0)\right)\right\},
\end{aligned}$$

for all $t \geq 0$. Then clearly,

$$L_1^*(t) = \widetilde{L}_1^*(t) - Z_1^*(0) \quad \text{and} \quad L_2^*(t) = \widetilde{L}_2^*(t) - Z_1^*(0) - Z_2^*(0)$$

hold for all $t \geq 0$. Now, it is a straightforward matter to check that the above defined processes $(Z_i^*(t), L_i^*(t))$ for $i = 1, 2$ satisfy (4.14)–(4.16).

Let $g(x) = e^{x^\alpha}$ where $0 < \alpha < 2(1-H)$. To show $\mathbb{E}[g(\mathbf{Z}^*(t))] < \infty$, we observe that $|\mathbf{Z}^*(t)| \leq Z_1^*(t) + Z_2^*(t) = Y(t)$ for all $t \geq 0$. Using (4.6) and (4.21), $Y(t) \stackrel{D}{=} Z_1^0(\infty) + Z_2^0(\infty)$ for all $t \geq 0$. When $u_1 > 0$ and $u_2 > 0$, we can employ the tail distribution bounds (4.7) and (4.8) in Theorem 4.1 to conclude that $\mathbb{E}[g(\mathbf{Z}^*(t))] < \infty$. If $u_1 > u_1 + u_2 > 0$, we can use (4.7) and the tail estimate in Corollary 4.2 to obtain $\mathbb{E}[g(\mathbf{Z}^*(t))] < \infty$. Hence, as a consequence, $\mathbb{E}[Z_1^0(\infty) + Z_2^0(\infty)]^N < \infty$ for each $N \geq 1$. This completes the proof. ■

Remark 4.4. *Consider the two-sided filtration* $(\mathcal{F}_t : -\infty < t < \infty)$ *defined by* $\mathcal{F}_t = \sigma(\{\mathbf{W}_H(s) : -\infty < s \leq t\})$ *for each* $-\infty < t < \infty$ *and allow each* $\mathcal{F}_t$ *to have all the null sets. Here* $\mathbf{W}_H$ *is the two-sided, two dimensional fBm introduced in the above proof. Then it is evident that the stationary process* $\mathbf{Z}^*$ *is adapted to the filtration* $(\mathcal{F}_t)_{t\geq 0}$.

5 A coupling time result

Consider the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ described in Theorem 4.3. Let $(\mathcal{F}_t)_{t\geq 0}$ be the filtration described in Remark 4.4. Then $\mathbf{W}_H = (W_1, W_2)^T$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and $\mathbf{Z}^*(0)$ is $\mathcal{F}_0$-measurable, where $\mathbf{Z}^*$ is the stationary process defined on $(\Omega, \mathcal{F}, \mathbb{P})$. Henceforth, all our processes are defined on $(\Omega, \mathcal{F}, \mathbb{P})$ and adapted to the filtration $(\mathcal{F}_t)_{t\geq 0}$. The one-dimensional fBm's W_1 and W_2 are correlated with a constant correlation coefficient ρ. Next, recall that our model $\mathbf{Z} = (Z_1, Z_2)^T$ is described by

$$Z_1(t) = Z_1(0) + W_1(t) - u_1 t + L_1(t), \text{ and} \tag{5.1}$$

$$Z_2(t) = Z_2(0) + \sigma W_2(t) - u_2 t - L_1(t) + L_2(t), \text{ for } t \geq 0. \tag{5.2}$$

Here, $\sigma > 0$ is a constant and $Z_1(0)$, $Z_2(0)$ are non-negative, $\mathcal{F}_0$-random variables, which satisfy the condition

$$\mathbb{E}[Z_1(0) + Z_2(0)] < \infty. \tag{5.3}$$

Recall the non-decreasing processes $L_1(\cdot)$ and $L_2(\cdot)$ are given by

$$L_1(t) = \max\left\{0, \max_{s\in[0,t]} (u_1 s - W_1(s) - Z_1(0))\right\}, \tag{5.4}$$

$$L_2(t) = \max\left\{0, \max_{s\in[0,t]} (u_2 s - \sigma W_2(s) + L_1(s) - Z_2(0))\right\}, \tag{5.5}$$

for all $t \geq 0$. Also, we introduce the processes $\widetilde{L}_1(\cdot)$ and $\widetilde{L}_2(\cdot)$ given by

$$\widetilde{L}_1(t) = Z_1(0) + L_1(t) = \max\left\{Z_1(0), \max_{s\in[0,t]} (u_1 s - W_1(s))\right\}, \tag{5.6}$$

$$\widetilde{L}_2(t) = Z_1(0) + Z_2(0) + L_2(t) = \max\left\{Z_1(0) + Z_2(0), \max_{s\in[0,t]} \left(u_2 s - \sigma W_2(s) + \widetilde{L}_1(s)\right)\right\}. \tag{5.7}$$

Then (5.1) and (5.2) can be written as

$$\begin{aligned} Z_1(t) &= W_1(t) - u_1 t + \widetilde{L}_1(t), \\ Z_2(t) &= \sigma W_2(t) - u_2 t - \widetilde{L}_1(t) + \widetilde{L}_2(t) \quad \text{for all } t \geq 0. \end{aligned}$$

It is evident that $\widetilde{L}_1(0) = Z_1(0)$, $\widetilde{L}_2(0) = Z_1(0) + Z_2(0)$, $\widetilde{L}_j(\cdot)$ is non-decreasing with continuous paths and $\int_0^\infty Z_j(t) d\widetilde{L}_j(t) = 0$ for $j = 1, 2$.

Our aim in this section is to show the existence of a stopping time $\tau \geq 0$ such that $\mathbf{Z}(t) = \mathbf{Z}^0(t)$ for all $t \geq \tau$ and $I\!\!E[\tau] < \infty$. Here, $\{\mathbf{Z}^0(t)\}_{t\geq 0}$ is the process described in (4.1) and (4.2). Furthermore, we show that if $I\!\!E[Z_1(0) + Z_2(0)]^N < \infty$ for some $N \geq 1$ then $I\!\!E[\tau^N] < \infty$. From these results, it also follows that the stationary process $\mathbf{Z}^*$ in (4.14) and (4.15) is unique in law. Our first lemma is a variant of Proposition 4.1 in [10], and the difference here is that we allow $Z_1(0)$ to be a random variable.

Lemma 5.1. *Assume the condition* (5.3). *Let the processes* L_1^0 *and* $\widetilde{L}_1$ *be as in* (4.3) *and* (5.6), *respectively. Then there is a stopping time* τ_1 *such that* $\widetilde{L}_1(t) = L_1^0(t)$ *for all* $t \geq \tau_1$ *and* $I\!\!E[\tau_1] < \infty$. *In addition, if we assume* $I\!\!E[Z_1(0)]^N < \infty$ *for some* $N \geq 1$ *then* $I\!\!E[\tau_1^N] < \infty$.

Proof. We begin with introducing the stopping time τ_1 and show that $I\!\!E[\tau_1^N] < \infty$ if $I\!\!E[Z_1(0)]^N < \infty$ for some $N \geq 1$. This establishes both parts of the lemma. Let

$$\tau_1 = \inf\{t \geq 0 : L_1^0(t) \geq Z_1(0)\}, \tag{5.8}$$

where the infimum over an empty set is defined to be ∞. Notice that

$$\begin{aligned} I\!\!E[\tau_1^N] &= N \int_0^\infty t^{N-1} I\!\!P[\tau_1 > t] dt \\ &= N \int_0^\infty t^{N-1} I\!\!P[L_1^0(t) < Z_1(0)] dt, \end{aligned} \tag{5.9}$$

and

$$\begin{aligned} I\!\!P\left[L_1^0(t) < Z_1(0)\right] &= I\!\!P\left[\max_{s\in[0,t]} (u_1 s - W_1(s)) < Z_1(0)\right] \\ &\leq I\!\!P\left[W_1(t) + Z_1(0) > u_1 t\right] \\ &\leq I\!\!P\left[W_1(t) > \frac{u_1}{2} t\right] + I\!\!P\left[Z_1(0) > \frac{u_1}{2} t\right]. \end{aligned} \tag{5.10}$$

For $t > 0$, $\mathcal{Z} \equiv \frac{W_1(t)}{t^H}$ is a standard normal random variable. For $y > 0$, it is known that

$$I\!P[\mathcal{Z} > y] \leq \frac{1}{\sqrt{2\pi}} \frac{1}{y} e^{-y^2/2}$$

and hence, for $t > 0$ we have

$$I\!P\left[W_1(t) > \frac{u_1}{2} t\right] = I\!P\left[\mathcal{Z} > \frac{u_1}{2} t^{1-H}\right] \leq \frac{1}{\sqrt{2\pi}} \frac{2}{u_1 t^{1-H}} e^{-\frac{u_1^2}{8} t^{2(1-H)}}. \tag{5.11}$$

Using (5.9)–(5.11), we have

$$\begin{aligned} E[\tau_1^N] &\leq 1 + N \int_1^\infty t^{N-1} I\!P[L_1^0(t) < Z_1(0)] dt \\ &\leq 1 + N \int_1^\infty t^{N-1} \left(I\!P\left[W_1(t) > \frac{u_1}{2} t\right] + I\!P\left[\frac{2}{u_1} Z_1(0) > t\right] \right) dt \\ &\leq 1 + \frac{2}{u_1} \frac{N}{\sqrt{2\pi}} \int_1^\infty t^{N+H-2} e^{-\frac{u_1^2}{8} t^{2(1-H)}} dt + \left(\frac{2}{u_1}\right)^N I\!E[Z_1(0)]^N. \end{aligned}$$

The above integral is finite since $H < 1$ and hence $E[\tau_1^N] < \infty$ by the assumption $I\!E[Z_1(0)]^N < \infty$. This completes the proof. ■

In the next lemma, we consider any stopping time τ with respect to the filtration $(\mathcal{F}_t)_{t \geq 0}$.

Lemma 5.2. *Let $u_i > 0$ for $i = 1, 2$ and τ be any stopping time such that $I\!E[\tau^N] < \infty$ for some $N \geq 1$. Then*

$$I\!E\left[\max_{s \in [0,\tau]} |u_1 s - W_1(s)|^N\right] + I\!E\left[\max_{s \in [0,\tau]} |u_2 s - \sigma W_2(s)|^N\right] \leq C_N \left[1 + I\!E[\tau^N]\right], \tag{5.12}$$

where $C_N > 0$ is a constant which depends only on u_1, u_2 and N.

Proof. For simplicity, we only show that $I\!E[\max_{s \in [0,\tau]} |u_2 s - \sigma W_2(s)|^N] \leq K_N[1 + I\!E[\tau^N]]$, where $K_N > 0$ is a constant which depends only on u_2 and N. An estimate for $I\!E[\max_{s \in [0,\tau]} |u_1 s - W_1(s)|^N]$ can be obtained along the same lines of the following proof. We begin with the fact

$$\max_{s \in [0,\tau]} |u_2 s - \sigma W_2(s)| \leq u_2 \tau + \sigma \max_{s \in [0,\tau]} |W_2(s)|$$

almost surely. Hence for each $N \geq 1$,

$$I\!E\left[\max_{s \in [0,\tau_1]} |u_2 s - \sigma W_2(s)|^N\right] \leq 2^{N-1} \left[u_2^N I\!E[\tau^N] + \sigma^N I\!E\left[\max_{s \in [0,\tau]} |W_2(s)|^N\right]\right]. \tag{5.13}$$

For $0 < H < 1$, it is known from Corollary 3.1 of [34] (see also, Theorem 1.2 of [27], Ex. 5.1.5 in [28] and the recent article [16] for the analysis of a related martingale) that

$$I\!E\left[\max_{s \in [0,\tau]} |W_2(s)|^N\right] \leq C_{N,H} I\!E[\tau^{NH}], \tag{5.14}$$

where $C_{N,H} > 0$ is a constant which depends only on N and H. Since $0 < NH < N$, $I\!E[\tau^{NH}] \leq 1 + I\!E[\tau^N]$. Now combining this estimate with (5.13) and (5.14), the desired result follows. ■

Lemma 5.3. *Assume that $I\!E[Z_1(0) + Z_2(0)]^N < \infty$ for some $N \geq 1$ and A is an $\mathcal{F}_0$-random variable such that $I\!E[A^N] < \infty$ for some $N \geq 1$. Let τ_1 be the stopping time defined in (5.8) of Lemma 5.1. Define $M(t) = \max_{s\in[0,t]} (u_2 s - \sigma W_2(s))$ for all $t \geq 0$. Then there exists a stopping time $\tau_2 > \tau_1$ such that*

$$(u_2\tau_2 - \sigma W_2(\tau_2)) \geq A + M(\tau_1) \text{ a.s. and} \tag{5.15}$$
$$I\!E[\tau_2^N] < \infty. \tag{5.16}$$

Proof. By Lemma 5.2, $M(\tau_1)$ is finite a.s. We let

$$\tau_2 = \inf\{t \geq 0 : u_2 t - \sigma W_2(t) \geq A + M(\tau_1)\}, \tag{5.17}$$

where we set infimum over an empty set as ∞. Since $\lim_{t\to\infty}(u_2 t - \sigma W_2(t)) = +\infty$ a.s., τ_2 is also finite a.s. By the definition of $M(\tau_1)$, it clearly follows that $\tau_2 > \tau_1$ a.s. Next, we show that τ_2 is indeed a stopping time. Since $\tau_2 > \tau_1$ a.s. and τ_1 is a stopping time, we have for fixed $s > 0$ that $\{\tau_2 > s\} \cap \{\tau_1 \geq s\} = \{\tau_1 \geq s\} \in \mathcal{F}_s$. On the other hand,

$$\begin{aligned}\{\tau_2 > s\} \cap \{\tau_1 < s\} &= \{M(s) < A + M(\tau_1), \tau_1 < s\} \\ &= \{M(s) < A + M(\tau_1)\mathbf{1}_{\{\tau_1 < s\}}, \tau_1 < s\}.\end{aligned}$$

Observe that $M(\cdot)$ is a non-negative, continuous, non-decreasing process adapted to $(\mathcal{F}_t)$. Therefore, we have (by standard discrete approximation) that $M(\tau_1)\mathbf{1}_{\{\tau_1<s\}}$ is an $\mathcal{F}_s$-random variable and recall A is an $\mathcal{F}_0$-random variable. Hence $\{\tau_2 > s\} \cap \{\tau_1 < s\} \in \mathcal{F}_s$ and consequently the result in (5.15) follows.

To establish (5.16), note that

$$\begin{aligned}I\!E[\tau_2^N] &= N\int_0^\infty t^{N-1} I\!P[\tau_2 > t]dt \\ &\leq 1 + N\int_1^\infty t^{N-1} I\!P[\tau_2 > t]dt,\end{aligned} \tag{5.18}$$

and from (5.17)

$$\begin{aligned}I\!P[\tau_2 > t] &= I\!P[A + M(\tau_1) + \sigma W_2(t) \geq u_2 t] \\ &\leq I\!P\left[A \geq \frac{u_2}{3}t\right] + I\!P\left[M(\tau_1) \geq \frac{u_2}{3}t\right] + I\!P\left[W_2(t) \geq \frac{u_2}{3\sigma}t\right].\end{aligned} \tag{5.19}$$

Since $I\!E[A^N] < \infty$, we have $N\int_0^\infty t^{N-1} I\!P\left[A \geq \frac{u_2}{3}t\right] dt < \infty$. The assumption $I\!E[Z_1(0)+Z_2(0)]^N < \infty$ implies that $I\!E[\tau_1^N] < \infty$ by Lemma 5.1. Next, we can employ Lemma 5.2 with the stopping time τ_1 to conclude that $I\!E[M(\tau_1)]^N < \infty$. Therefore,

$$N\int_0^\infty t^{N-1} I\!P\left[M(\tau_1) \geq \frac{u_2}{3}t\right] dt < \infty. \tag{5.20}$$

Finally,

$$I\!P\left[W_2(t) \geq \frac{u_2}{3\sigma}t\right] \leq \frac{1}{\sqrt{2\pi}}\frac{3\sigma}{u_2 t^{1-H}} e^{-\frac{u_2^2}{18\sigma^2}t^{2(1-H)}}$$

holds for $t > 0$ as in (5.11) and hence we can conclude that

$$N\int_1^\infty t^{N-1} I\!P\left[W_2(t) \geq \frac{u_2}{3\sigma}t\right] dt < \infty. \tag{5.21}$$

Combining (5.18)–(5.21) it follows that $I\!E[\tau_2^N] < \infty$. This completes the proof of lemma. ■

Next, we employ the above three lemmas to prove the following proposition.

Proposition 5.4. *Assume that $I\!E[Z_1(0) + Z_2(0)]^N < \infty$ for some $N \geq 1$, and choose $A = 1 + Z_1(0) + Z_2(0) > 0$ in Lemma 5.3. Let the stopping times τ_1 and τ_2 be as defined in (5.8) and in (5.17), respectively. Then for all $t \geq \tau_2$*

(a) $\max_{s\in[\tau_1,t]} \left(u_2 s - \sigma W_2(s) + L_1^0(s)\right) \geq \max\left\{L_2^0(\tau_1), \widetilde{L}_2(\tau_1)\right\}$ *a.s. and*

(b) $\widetilde{L}_1(t) = L_1^0(t)$, $\widetilde{L}_2(t) = L_2^0(t)$ *and* $I\!E[{\tau_2}^N] < \infty$.

Proof. By the definition of τ_2 as in (5.17), we have

$$\begin{aligned} u_2\tau_2 - \sigma W_2(\tau_2) &\geq A + \max_{s\in[0,\tau_1]} (u_2 s - \sigma W_2(s)) \\ &\geq Z_1(0) + Z_2(0) \quad \text{a.s.} \end{aligned}$$

Since $\tau_2 \geq \tau_1$, when $t \geq \tau_2$ we obtain

$$\begin{aligned} \max_{s\in[\tau_1,t]} \left(u_2 s - \sigma W_2(s) + L_1^0(s)\right) &\geq u_2\tau_2 - \sigma W_2(\tau_2) + L_1^0(\tau_2) \\ &\geq A + \max_{s\in[0,\tau_1]} \left(u_2 s - \sigma W_2(s) + L_1^0(s)\right) \\ &\geq Z_2(0) + \max_{s\in[0,\tau_1]} \left(u_2 s - \sigma W_2(s) + \widetilde{L}_1(s)\right). \end{aligned}$$

Also, we get

$$\max_{s\in[\tau_1,t]} \left(u_2 s - \sigma W_2(s) + L_1^0(s)\right) \geq u_2\tau_2 - \sigma W_2(\tau_2) \geq Z_1(0) + Z_2(0) \quad \text{a.s.}$$

Hence, it follows that

$$\max_{s\in[\tau_1,t]} \left(u_2 s - \sigma W_2(s) + L_1^0(s)\right) \geq \max\{L_2^0(\tau_1), \widetilde{L}_2(\tau_1)\} \quad \text{a.s.}$$

for all $t \geq \tau_2$. Therefore, part (a) follows.

For part (b), by Lemma 5.1 we already know that $\widetilde{L}_1(t) = L_1^0(t)$ for $t \geq \tau_1$. Using this fact, whenever $t \geq \tau_2$, we can write

$$\begin{aligned} \widetilde{L}_2(t) &= \max\left\{Z_1(0) + Z_2(0), \max_{s\in[0,\tau_1]} \left(u_2 s - \sigma W_2(s) + \widetilde{L}_1(s)\right), \max_{s\in[\tau_1,t]} \left(u_2 s - \sigma W_2(s) + \widetilde{L}_1(s)\right)\right\} \\ &= \max\left\{\widetilde{L}_2(\tau_1), \max_{s\in[\tau_1,t]} \left(u_2 s - \sigma W_2(s) + \widetilde{L}_1(s)\right)\right\} \\ &= \max_{s\in[\tau_1,t]} \left(u_2 s - \sigma W_2(s) + \widetilde{L}_1(s)\right). \end{aligned}$$

Therefore, $\widetilde{L}_2(t) \leq L_2^0(t)$ whenever $t \geq \tau_2$. On the other hand,

$$\begin{aligned} L_2^0(t) &= \max\left\{L_2^0(\tau_1), \max_{s\in[\tau_1,t]} \left(u_2 s - \sigma W_2(s) + L_1^0(s)\right)\right\} \\ &= \max_{s\in[\tau_1,t]} \left(u_2 s - \sigma W_2(s) + L_1^0(s)\right) \quad \text{for all} \quad t \geq \tau_2. \end{aligned}$$

This yields that $\widetilde{L}_2(t) = L_2^0(t)$ for all $t \geq \tau_2$. We have already established $I\!E[{\tau_2}^N] < \infty$ in Lemma 5.3. This completes the proof. ∎

We now state and prove the main coupling time result in this section.

Theorem 5.5. *Let* $\mathbf{Z}$ *be a process which satisfies* (5.1) *and* (5.2) *with* $\mathbb{E}[Z_1(0)+Z_2(0)]^N < \infty$ *for some* $N \geq 1$. *Then the following statements hold.*

(a) There exists a stopping time τ *such that* $\mathbf{Z}(t) = \mathbf{Z}^0(t)$ *for all* $t \geq \tau$ *and* $\mathbb{E}[\tau^N] < \infty$, *where* $\mathbf{Z}^0$ *is the process which satisfies* (4.1) *and* (4.2).

(b) There exists a stopping time $\widehat{\tau}$ *such that* $\mathbf{Z}(t) = \mathbf{Z}^*(t) = \mathbf{Z}^0(t)$ *for all* $t \geq \widehat{\tau}$ *and* $\mathbb{E}[\widehat{\tau}^N] < \infty$, *where* $\mathbf{Z}^*$ *is the stationary process described in Theorem 4.3.*

Proof. Part (a) clearly follows from Proposition 5.4 and the representation of the process $\mathbf{Z}$ in (5.1)–(5.7). To establish part (b), notice that

$$Z_1^*(0) + Z_2^*(0) \overset{D}{=} Z_1^0(\infty) + Z_2^0(\infty)$$

where $Z_1^0(\infty)$ and $Z_2^0(\infty)$ are described in (4.5) and (4.6). Using the tail estimates in (4.7) and (4.8), it clearly follows that

$$\mathbb{E}[Z_1^*(0) + Z_2^*(0)]^n < \infty \text{ for every } n \geq 1.$$

Consequently, we can apply part (a) of the theorem and thus there is a stopping time $\tau^* > 0$ such that $\mathbf{Z}^*(t) = \mathbf{Z}^0(t)$ for all $t \geq \tau^*$. Let $\mathbf{Z}$ be any other process, which satisfies (5.1) and (5.2) with $\mathbb{E}[Z_1(0)+Z_2(0)]^N < \infty$ for some $N \geq 1$. Then there is a stopping time τ satisfying part (a) of the theorem. We can take $\widehat{\tau} = \tau + \tau^*$ and then

$$\mathbf{Z}(t) = \mathbf{Z}^*(t) = \mathbf{Z}^0(t) \quad \text{for all} \quad t \geq \widehat{\tau}.$$

Since $\mathbb{E}[\tau^N] < \infty$ and $\mathbb{E}[(\tau^*)^N] < \infty$, it follows that $\mathbb{E}[\widehat{\tau}^N] < \infty$. This completes the proof. ■

The following corollary is an immediate consequence of the above theorem.

Corollary 5.6. *Let* $\widetilde{\mathbf{Z}}^*$ *be any other stationary process which satisfies* (4.14)–(4.16) *with the moment condition* $\mathbb{E}[\widetilde{Z}_1^*(0) + \widetilde{Z}_2^*(0)] < \infty$. *Then there is a stopping time* $\widetilde{\tau}$ *so that* $\widetilde{\mathbf{Z}}^*(t) = \mathbf{Z}^*(t)$ *for all* $t \geq \widetilde{\tau}$ *and* $\mathbb{E}[\widetilde{\tau}] < \infty$. *Hence, the stationary process* $\mathbf{Z}^*$ *is unique in law.*

6 Cost minimization

In this section, we analyze the cost structure described in (2.7) and address the associated cost minimization problem. Our state process $\mathbf{Z} = (Z_1, Z_2)^T$ is adapted to $(\mathcal{F}_t)_{t\geq 0}$ and it satisfies (5.1) and (5.2). We assume that the initial data $\mathbf{Z}(0)$ satisfies the moment condition (5.3). For the cost minimization problem with cost functional $I(\mathbf{u}, \mathbf{Z}(0))$ in (2.7), the running cost function C satisfies the assumptions (H1)–(H3) described in Section 2. Henceforth, we say that a state process $\mathbf{Z}$ is an *admissible state process* if the initial data $\mathbf{Z}(0)$ satisfies the moment condition

$$\mathbb{E}|Z_1(0) + Z_2(0)|^{m+1} < \infty, \text{ where } m \geq 1 \text{ is as in (H3).} \tag{6.1}$$

To address the cost minimization problem, first we show that the cost functional $I(\mathbf{u}, \mathbf{Z}(0))$ described in (2.7) is independent of the initial data $\mathbf{Z}(0)$, and we obtain a representation for it using the stationary distribution $\mathbf{Z}^*$ of Theorem 4.3. This representation will be used to address the cost minimization problem. We begin with the following lemma.

Lemma 6.1. *Let the process* $\mathbf{Z}$ *satisfy* (5.1)*,* (5.2) *and* (5.3)*. Then*

$$\begin{aligned} \lim_{T\to\infty} \frac{1}{T} I\!E[L_1(T)] &= u_1, & (6.2)\\ \lim_{T\to\infty} \frac{1}{T} I\!E[L_2(T)] &= u_1 + u_2. & (6.3) \end{aligned}$$

Proof. Since $I\!E[Z_1(0)] < \infty$ from the condition (5.3), the conclusion (6.2) can be obtained by following the proof of Lemma 3.1 in [10]. For (6.3), we begin with the definition of the standard one-dimensional reflection mapping (Skorokhod map) $\Gamma : C([0,\infty), I\!R) \to C([0,\infty), I\!R)$, which is defined as

$$\Gamma(f)(t) = f(t) + \max\left\{0, \max_{s\in[0,t]}(-f(s))\right\}$$

for $f \in C([0,\infty), I\!R)$ and $t \geq 0$. Then we have $Z_2(t) = \Gamma\left(Z_2(0) + \sigma W_2 - u_2 e - L_1\right)(t)$, where $e(t) \equiv t$ for all $t \geq 0$. Since $u_2 t + L_1(t)$ is non-negative and non-decreasing in t, we have

$$Z_2(0) + \sigma W_2(t) - u_2 t - L_1(t) \leq Z_2(0) + \sigma W_2(t).$$

Therefore, from the basic properties of the Skorokhod map (see for instance [30]), we have

$$\begin{aligned} 0 \leq Z_2(t) &\leq \Gamma\left(Z_2(0) + \sigma W_2\right)(t) \\ &\leq 2\left(Z_2(0) + \max_{s\in[0,t]} \sigma|W_2(s)|\right). & (6.4) \end{aligned}$$

Hence

$$\begin{aligned} 0 \leq \frac{I\!E[Z_2(T)]}{T} &\leq \frac{2}{T}\left(I\!E[Z_2(0)] + I\!E\left[\max_{s\in[0,T]} \sigma|W_2(s)|\right]\right) \\ &\leq \frac{2}{T}\left(I\!E[Z_2(0)] + K_1 T^H\right) \to 0, \end{aligned}$$

as $T \to \infty$. Here $K_1 \in (0,\infty)$ is a generic constant independent of T (see [28], page 296). Since

$$L_2(T) = Z_2(T) - Z_2(0) - \sigma W_2(T) + u_2 T + L_1(T), \quad I\!E[W_2(T)] = 0,$$

and using (6.2), we obtain $\lim_{T\to\infty} \frac{1}{T} I\!E[L_2(T)] = u_1 + u_2$. ■

Remark 6.2. *Indeed the estimate in* (6.4) *also implies that* $\lim_{T\to\infty} \frac{Z_2(T)}{T} = 0$ *a.s. Similar estimate in [10] can be used to show* $\lim_{T\to\infty} \frac{Z_1(T)}{T} = 0$ *a.s. Consequently,* $\lim_{T\to\infty} \frac{L_1(T)}{T} = u_1$ *a.s. Using this with* (6.4) *also leads to* $\lim_{T\to\infty} \frac{L_2(T)}{T} = u_1 + u_2$ *a.s.*

Proposition 6.3. *Let* $\mathbf{Z}$ *be an admissible state process which satisfies* (5.1)*,* (5.2) *and* (6.1)*. Then*

$$\lim_{T\to\infty} \frac{1}{T} I\!E \int_0^T C\left(Z_1(t), Z_2(t)\right) dt = I\!E\left[C(Z_1^0(\infty), Z_2^0(\infty))\right]. \tag{6.5}$$

Proof. Since $I\!E|Z_1(0) + Z_2(0)|^{m+1} < \infty$, we can use Theorem 5.5 to conclude that there exists a stopping time $\widehat{\tau}$ such that $\mathbf{Z}(t) = \mathbf{Z}^*(t)$ for all $t \geq \widehat{\tau}$ and $I\!E[\widehat{\tau}^{m+1}] < \infty$. Furthermore, $\mathbf{Z}^*$ is a stationary process and $(Z_1^*(t), Z_2^*(t)) \stackrel{D}{=} (Z_1^0(\infty), Z_2^0(\infty))$ for all $t \geq 0$. Therefore, to establish (6.5), it suffices to show that

$$\lim_{T\to\infty} \frac{1}{T} I\!E \int_0^{\widehat{\tau}} C\left(Z_1(t), Z_2(t)\right) dt = 0. \tag{6.6}$$

This will follow if $I\!E \int_0^{\widehat{\tau}} C\left(Z_1(t), Z_2(t)\right) dt < \infty$, and hence we establish this fact in the argument below.

Using (H3), there is $m \geq 1$ such that $0 \leq C(x,y) \leq K(1 + |x+y|^m)$ for all $x \geq 0$ and $y \geq 0$, where $K > 0$ is a constant. Therefore,

$$I\!E \int_0^{\widehat{\tau}} C\left(Z_1(t), Z_2(t)\right) dt \leq K I\!E \left[\left(1 + \max_{t\in[0,\widehat{\tau}]} |Z_1(t) + Z_2(t)|^m \right) \widehat{\tau} \right].$$

We have by Theorem 5.5 that $I\!E[\widehat{\tau}^{m+1}] < \infty$ and thus the right hand side of the above inequality is finite, if we establish

$$I\!E \left[\left(\max_{t\in[0,\widehat{\tau}]} |Z_1(t) + Z_2(t)|^m \right) \widehat{\tau} \right] < \infty. \tag{6.7}$$

But using Hölder's inequality with $p = \frac{m+1}{m}$ and $q = m+1$, we have

$$I\!E \left[\left(\max_{t\in[0,\widehat{\tau}]} |Z_1(t) + Z_2(t)|^m \right) \widehat{\tau} \right] \leq \left[I\!E \left(\max_{t\in[0,\widehat{\tau}]} |Z_1(t) + Z_2(t)|^{m+1} \right) \right]^{m/(m+1)} \cdot \left(I\!E[\widehat{\tau}^{m+1}] \right)^{1/(m+1)}.$$

Since $I\!E[\widehat{\tau}^{m+1}] < \infty$, it remains to establish $I\!E(\max_{t\in[0,\widehat{\tau}]} |Z_1(t) + Z_2(t)|^{m+1}) < \infty$ to guarantee (6.7). Using (5.1)–(5.5), we obtain

$$\begin{aligned} I\!E \left(\max_{t\in[0,\widehat{\tau}]} |Z_1(t) + Z_2(t)|^{m+1} \right) \quad \leq \quad & K_2 \left(I\!E\, |Z_1(0) + Z_2(0)|^{m+1} + I\!E \left[\max_{t\in[0,\widehat{\tau}]} |u_2 s - \sigma W_2(s)|^{m+1} \right] \right. \\ & \left. + \, I\!E \left[\max_{t\in[0,\widehat{\tau}]} |u_1 s - W_1(s)|^{m+1} \right] \right), \end{aligned} \tag{6.8}$$

where $K_2 > 0$ is a generic constant which may depend on m. Following the proof of Lemma 5.2 and using the fact $I\!E[\widehat{\tau}^{m+1}] < \infty$, we have

$$I\!E \left[\max_{t\in[0,\widehat{\tau}]} |u_2 s - \sigma W_2(s)|^{m+1} \right] < \infty \quad \text{as well as} \quad I\!E \left[\max_{t\in[0,\widehat{\tau}]} |u_1 s - W_1(s)|^{m+1} \right] < \infty.$$

Therefore, we can conclude that the left hand side of inequality in (6.8) is finite and this establishes (6.7). Hence, (6.6) follows. To complete the proof, we should check that $I\!E[C(Z_1^0(\infty), Z_2^0(\infty))]$ is finite. But, this directly follows from the tail distribution asymptotics of $\mathbf{Z}^0(\infty)$ described in (4.7) and (4.8). This completes the proof. ■

Let us introduce

$$F(u_1, u_2) \equiv I\!E[C(Z_1^0(\infty), Z_2^0(\infty))]$$

for all $u_1 > 0$ and $u_2 \geq 0$. We intend to establish the continuity of F on the domain

$$\mathcal{D} \equiv \{(u_1, u_2) : u_1 > 0 \text{ and } u_2 \geq 0\}.$$

To help the arguments in the next proposition, we introduce the following notation. Let the random variables $G(u)$ and $H(u,v)$ be defined by

$$G(u) = \sup_{0\leq s} \left(W_1(s) - us \right), \quad H(u,v) = \sup_{0\leq r\leq s} \left[(W_1(s) - us) + (\sigma W_2(r) - vr) \right] \tag{6.9}$$

for all $u > 0$ and $v \geq 0$. By Theorem 4.1, $Z_1^0(\infty) = G(u_1)$ and $Z_1^0(\infty) + Z_2^0(\infty) = H(u_1, u_2)$. Also, notice that $H(u, v)$ is finite if $u > 0$ and $u + v \geq 0$ as noted in Theorem 4.1(a).

We also introduce the function $\widehat{C}$ on the set $\{(x, y) : y \geq x \geq 0\}$ by

$$\widehat{C}(x, y) = C(x, y - x). \tag{6.10}$$

Hence, it follows that

$$F(u, v) = \mathbb{E}[\widehat{C}(G(u), H(u, v))] \tag{6.11}$$

for all $u > 0$ and $v \geq 0$. For each $u > 0$ and $v \geq 0$, $F(u, v)$ is finite. If $v > 0$, this follows from the facts that

$$H(u, v) \leq G(u) + \sup_{0 \leq r} (\sigma W_2(r) - vr) < \infty$$

and the polynomial growth condition of C in (H3), and the tail estimates in (4.7), (4.8). If $v = 0$, a very similar argument using the estimate in the proof of Theorem 4.1(a) guarantees the finiteness of $F(u, v)$.

Proposition 6.4. *Under the assumptions of Proposition 6.3, the following statements hold.*

(a) The function $F(u, v)$ is continuous on the domain $\mathcal{D} = \{(u, v) : u > 0$ and $v \geq 0\}$.

(b) The function $F(u, v)$ is decreasing in the variable v and $\lim_{(u,v) \to (0,b)} F(u, v) = \infty$ for each $b > 0$.

Proof. Let $(a, b) \in \mathcal{D}$. We pick $\delta > 0$ such that $0 < 3\delta < a$. To show the continuity of F at (a, b), we pick a sequence (a_n, b_n) which converges to (a, b) as $n \to \infty$. Without loss of generality, we assume $a_n > 3\delta$ for all n. Our first step is to show that $G(a_n) \to G(a)$ a.s. and $H(a_n, b_n) \to H(a, b)$ a.s. as n tends to infinity. Since $\lim_{t \to \infty} \frac{W^{(i)}(t)}{t} = 0$ a.s. for $i = 1, 2$, there exists a $T_1(\omega) > 0$ such that $\max\{W_1(t) - \delta t, \sigma W_2(t) - \delta t\} < 0$ for all $t \geq T_1(\omega)$. We let

$$T_0(\omega) = \max\left\{T_1(\omega), \frac{1}{\delta} \max_{t \in [0, T_1(\omega)]} (\sigma W_2(t) - \delta t)\right\}$$

and $T_0(\omega) > 0$ is finite. Then clearly it follows that

$$G(a_n) = \max_{0 \leq s \leq T_0(\omega)} (W_1(s) - a_n s) \quad \text{and} \quad G(a) = \max_{0 \leq s \leq T_0(\omega)} (W_1(s) - as).$$

From this, it is evident that $|G(a_n) - G(a)| \leq T_0(\omega)|a_n - a|$ and thus $G(a_n) \to G(a)$ as n tends to infinity.

Next, we consider

$$H(a_n, b_n) = \sup_{0 \leq r \leq s} [(W_1(s) - a_n s) + (\sigma W_2(r) - b_n r)].$$

Since $a_n > 3\delta > 0$ and $b_n \geq 0$, we obtain the following estimates. For any $s > T_0(\omega)$ and $r \leq s$,

$$\begin{aligned}
(W_1(s) - a_n s) + (\sigma W_2(r) - b_n r) &\leq W_1(s) - 3\delta s + \sigma W_2(r) \\
&\leq (W_1(s) - \delta s) - \delta s + (\sigma W_2(r) - \delta r) \\
&\leq -\delta s + \max_{r \in [0, T_1(\omega)]} (\sigma W_2(r) - \delta r) \\
&< 0.
\end{aligned}$$

Therefore, we can write

$$H(a_n, b_n) = \max_{0 \le r \le s \le T_0(\omega)} [(W_1(s) - a_n s) + (\sigma W_2(r) - b_n r)]$$

and similarly,

$$H(a, b) = \max_{0 \le r \le s \le T_0(\omega)} [(W_1(s) - as) + (\sigma W_2(r) - br)] .$$

Hence we obtain

$$|H(a_n, b_n) - H(a, b)| \le T_0(\omega)(|a_n - a| + |b_n - b|).$$

Since $T_0(\omega) > 0$ is finite, we have $H(a_n, b_n) \to H(a, b)$ as $n \to \infty$.

Our second step is to obtain an integrable upper bound for $C(G(a_n), H(a_n, b_n) - G(a_n))$. Since $C(x, y)$ is non-decreasing in each variable, we have

$$0 \le C(G(a_n), H(a_n, b_n) - G(a_n)) \le C(G(a_n), H(a_n, b_n)).$$

Since $a_n > 3\delta > 0$ and $b_n \ge 0$, we also have

$$0 \le G(a_n) \le G(\delta) \quad \text{and} \quad 0 \le H(a_n, b_n) \le G(\delta) + \sup_{0 \le r}(\sigma W_2(r) - \delta r).$$

Therefore,

$$0 \le C(G(a_n), H(a_n, b_n) - G(a_n)) \le C(G(\delta), G(\delta) + \widetilde{G}(\delta)), \tag{6.12}$$

where $\widetilde{G}(\delta) = \sup_{0 \le r}(\sigma W_2(r) - \delta r)$. Using the tail estimates (4.7), (4.8) and the polynomial growth condition of C in (H3), it follows that

$$I\!\!E[C(G(\delta), G(\delta) + \widetilde{G}(\delta))] < \infty. \tag{6.13}$$

In our third step, we apply dominated convergence theorem to establish the continuity of F at (a, b). Since $C(x, y)$ is continuous, using our first step above, we have

$$C(G(a_n), H(a_n, b_n) - G(a_n)) \to C(G(a), H(a, b) - G(a))$$

a.s. as $n \to \infty$. Next, using (6.12), (6.13) and the aforementioned a.s. convergence together with the dominated convergence theorem, we conclude that

$$F(a_n, b_n) = I\!\!E[C(G(a_n), H(a_n, b_n) - G(a_n))] \to F(a, b) = I\!\!E[C(G(a), H(a, b) - G(a))]$$

as $n \to \infty$. This completes the proof of part (a).

For part (b), observe that if $b_1 < b_2$ then $H(a, b_2) < H(a, b_1)$. Since $C(x, y)$ is increasing in the variable y, we obtain $F(a, b_2) \le F(a, b_1)$, whenever $b_2 > b_1$. Finally, we intend to compute the limit $\lim_{(u,v) \to (0,b)} F(u, v)$ for $b > 0$. Let $b > 0$ and $(a_n, b_n) \to (0, b)$ as $n \to \infty$. We can simply assume that a_n is decreasing to zero as n tends to infinity. Hence, $G(a_n)$ is increasing to infinity. Next, using the assumption (H2) for the cost function C, we can conclude that

$$\lim_{(a_n, b_n) \to (0,b)} F(a_n, b_n) = \infty.$$

This completes the proof. ■

Remark 6.5. *Using* (6.10) *and* (6.11)*, it can be shown that* $F(u, v)$ *is decreasing in both variables* u *and* v *under the assumption* $\frac{\partial C(x,y)}{\partial x} \ge \frac{\partial C(x,y)}{\partial y}$ *for all* x *and* y.

Our next theorem is the main result of this section.

Theorem 6.6. *Let* $\mathbf{Z}$ *be an admissible process which satisfies* (5.1), (5.2) *with control* $\mathbf{u} = (u_1, u_2)$ *in* $\mathcal{D}$. *Then, the following results hold.*

(a) The cost functional $I(\mathbf{u}, \mathbf{Z}(0))$ *described in* (2.7) *is independent of* $\mathbf{Z}(0)$ *and has the representation*

$$I(u_1, u_2) \equiv I(\mathbf{u}, \mathbf{Z}(0)) = h(u_1, u_2) + p_1 u_1 + p_2(u_1 + u_2) + F(u_1, u_2), \tag{6.14}$$

where $F(u_1, u_2)$ *is as defined in* (6.11).

(b) There exists an optimal control $\mathbf{u}^* = (u_1^*, u_2^*)$ *in* $\mathcal{D}$ *such that* $I(u_1^*, u_2^*) = \inf_{\mathbf{u} \in \mathcal{D}} I(u_1, u_2)$. *Moreover, the process* $\mathbf{Z}^*$ *defined as in Theorem 4.3 with control* $\mathbf{u}^* = (u_1^*, u_2^*)$ *is an optimal stationary process.*

Proof. Part (a) directly follows from the definition of $I(\mathbf{u}, \mathbf{Z}(0))$ in (2.7) and from the results obtained in Lemma 6.1, Propositions 6.3 and 6.4. For part (b), with the representation (6.14) for $I(u_1, u_2)$ in hand, we have $I(u_1, u_2)$ is finite and continuous on the domain $\mathcal{D} \equiv \{(u_1, u_2) : u_1 > 0, u_2 \geq 0\}$. Also, from Proposition 6.4 and the representation (6.14), we have

$$\lim_{u_1+u_2 \to \infty} I(u_1, u_2) = +\infty, \quad \lim_{(u_1,u_2)\to(a,0)} I(u_1, u_2) = I(a, 0), \quad \lim_{(u_1,u_2)\to(0,b)} I(u_1, u_2) = +\infty$$

for any $a > 0$ and $b > 0$. In the following argument, we consider the stationary state process $\mathbf{Z}^*_{\mathbf{u}}$ associated with control $\mathbf{u} \in \mathcal{D}$ as described in Theorem 4.3. Then it automatically satisfies the assumed moment condition for the initial data since the tail estimates (4.7) and (4.8) imply the finiteness of all the moments of $|\mathbf{Z}^*_{\mathbf{u}}(0)|$. Consequently, $\mathbf{Z}^*_{\mathbf{u}}$ is an admissible state process. Any state process with non-random initial data also satisfies the moment condition for initial data and hence admissible.

Next, consider a control (a_0, b_0) in $\mathcal{D}$ with $a_0 > 0$ and $b_0 > 0$. We keep (a_0, b_0) fixed. Let $M \equiv I(a_0, b_0)$, which is finite and define the set $\mathcal{D}_0 \subset \mathcal{D}$ by

$$\mathcal{D}_0 \equiv \{(u_1, u_2) \in \mathcal{D} : I(u_1, u_2) \leq M\}.$$

Then $\inf_{\mathcal{D}} I(u_1, u_2) = \inf_{\mathcal{D}_0} I(u_1, u_2)$. With the above described limits and properties of $I(u_1, u_2)$, it clearly follows that $\mathcal{D}_0$ is a bounded set. Now let $\{(a_n, b_n) : n \geq 1\}$ be a sequence in $\mathcal{D}_0$ such that $I(a_n, b_n) \to \inf_{\mathcal{D}_0} I(u_1, u_2)$ as $n \to \infty$. Thus $\{(a_n, b_n) : n \geq 1\}$ has a convergent subsequence. Therefore, we simply assume $(a_n, b_n) \to (u_1^*, u_2^*)$ as n tends to infinity. Hence, $u_1^* \geq 0$ and $u_2^* \geq 0$. Clearly, $u_1^* > 0$ since $\lim_{(u_1,u_2)\to(0,b)} I(u_1, u_2) = +\infty$. Therefore, there exists (u_1^*, u_2^*) in $\mathcal{D}$ such that $I(u_1^*, u_2^*) = \inf_{\mathcal{D}} I(u_1, u_2)$. This completes the proof. ■

7 Concluding remarks

Our methods can be readily extended to the case of a tandem queueing network with n stations, where $n \geq 2$. Let $W_1, \ldots, W_n$ be possibly correlated one-dimensional fBm's with constant correlation coefficients. Then we can represent the n-dimensional state process $\{\mathbf{Z}(t) = (Z_1(t), \ldots, Z_n(t))^T\}_{t \geq 0}$, where $\mathbf{Z}(t) \in I\!R^n_+$ by

$$Z_i(t) = Z_i(0) + \sigma_i W_i - u_i t - L_{i-1}(t) + L_i(t),$$

for $i = 1, \ldots, n$. Here $\sigma_i > 0$ and $u_i > 0$ are constants (with $\sigma_1 \equiv 1$) and the constant $u_i > 0$ represents the controllable drift rate at the i-th station. Also, $L_0(t) \equiv 0$ for all $t \geq 0$ and the local time process L_i (corresponding to Z_i) is a continuous, non-decreasing process which increases only when $Z_i(t) = 0$. That is, $\int_0^\infty Z_i(t) dL_i(t) = 0$ a.s. and $Z_i(t) \geq 0$ for all $t \geq 0$. In this situation, one can obtain a stationary state process $\mathbf{Z}^*$ and conclude the existence of an optimal control vector $\mathbf{u}^* = (u_1^*, \ldots, u_n^*)^T$ following our methods in the previous sections. Moreover, the distribution of $\mathbf{Z}^*(t)$ can be explicitly described as follows. Let

$$\xi_i = \sup_{0 \leq s_1 \leq \cdots \leq s_i} \left(\sum_{j=1}^{i} \sigma_j W_j(s_j) - u_j s_j\right) \quad \text{for} \quad i = 1, \ldots, n.$$

Then for all $t \geq 0$,

$$\mathbf{Z}^*(t) \stackrel{D}{=} R(\xi_1, \ldots, \xi_n)^T,$$

where $R = \begin{pmatrix} 1 & 0 & \cdots & 0 & 0 \\ -1 & 1 & \cdots & 0 & 0 \\ 0 & -1 & 1 & \cdots & 0 \\ \vdots & \cdots & -1 & 1 & 0 \\ 0 & \cdots & 0 & -1 & 1 \end{pmatrix}$.

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