Sequential maximum likelihood estimation for reflected Ornstein-Uhlenbeck processes

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Abstract

The paper studies the properties of a sequential maximum likelihood estimator of the drift parameter in a one dimensional reflected Ornstein-Uhlenbeck process. We observe the process until the observed Fisher information reaches a specified precision level. We derive the explicit formulas for the sequential estimator and its mean squared error. The estimator is shown to be unbiased and uniformly normally distributed. A simulation study is conducted to assess the performance of the estimator compared with the ordinary maximum likelihood estimator.

Keywords: Reflected Ornstein-Uhlenbeck processes, sequential maximum likelihood estimator, unbiasedness, mean squared error, efficiency.

1 Introduction

In this paper, we study a sequential parameter estimation problem for a one-dimensional reflected Ornstein-Uhlenbeck (ROU) process with infinitesimal drift $-\alpha x$ and infinitesimal variance σ^2, where $\alpha \in (-\infty, \infty)$ and $\sigma > 0$. The ROU process has received a lot of attention these days due to its

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diverse applications such as in physical, biological, and mathematical finance models (see, e.g., Ricciardi (1985); Attia (1991); Bo *et al.* (2010a,b)). Hence a parameter estimation problem in the ROU is an important problem. The ROU behaves as a standard OU process in the interior of its domain (b, ∞). However, when it reaches its boundary b, then the sample path returns to the interior in a manner exercising with minimal "pushing" force. Our main interest in this model stems from the fact that the ROU process arises as the key approximating process for queueing systems with reneging or balking customers (*balking* is defined as deciding not to join the queue at all, and *reneging* as joining a line but leaving without being served; see Ward and Glynn (2003a,b, 2005) and the references therein). In such cases, the drift parameter α carries the physical meaning of customers' reneging (or, balking) rate from the system.

Sequential maximum likelihood estimation in discrete time processes has been studied in Lai and Siegmund (1983) and the idea dates back to Anscombe (1952). In continuous time processes, sequential maximum likelihood estimation was first studied by Novikov (1972). The results of his work appear in Liptser and Shiryaev (2001). Since then there has been extension to semimartingales and SPDEs. To the best of our knowledge, the problem has not been studied for the ROU process and our aim is to bridge this gap. In a recent paper by Bo *et al.* (2011), the authors investigated an important statistical inference problem for the ROU process. Namely, they derived the maximum likelihood estimator (MLE) for an unknown drift parameter $\alpha \in (0, \infty)$ of the ROU process based on continuous observations, and established strong consistency and asymptotic normality of the MLE. Such results are valuable from statistical analysis viewpoint of applications, however, generally speaking, the MLE is a biased estimator and its mean squared error (MSE) depends on the parameter to be estimated (see Theorems 3.1–3.2 and Lemma 3.1 of Bo *et al.* (2011)). Moreover, the classical Cramer-Rao lower bound may not be attained for this MLE.

To overcome such limitations, in this paper, we propose to use the sequential estimation plan for the ROU process and verify that the proposed plan is significantly helpful both in asymptotic and non-asymptotic short time observation. More precisely, in contrast to the MLE, the proposed sequential maximum likelihood estimator (SMLE) is unbiased, exactly normally distributed (on the finite time observation), and its MSE has an explicit, simple expression that does not depend on the parameter to be estimated (cf. Theorem 2.1). The SMLE is uniformly normally distributed over the entire parameter space which is the real line. Such results would be of ample use in applications to several areas such as engineering, financial and biological modeling where unknown parameter estimation is based on relatively shorter time observation, which commonly arises in practical situations. Furthermore, an analog of the Cramer-Rao lower bound is proved and the SMLE is shown to be efficient among all unbiased estimation plans in the mean squared error sense (cf. Theorem 2.3).

The remainder of the paper is organized as follows. In Section 2, we precisely describe the ROU model in (2.1)–(2.2) and introduce a sequential estimation plan (2.4)–(2.5), which is adapted from the classical sequential estimation plan for regular diffusion processes. We present the main theoretical results in Theorems 2.1–2.3. Section 3 is devoted to the numerical studies on various comparison of performance between the ordinary MLE and the proposed sequential MLE. Section 4 concludes with some discussion on the further work.

2 Sequential maximum likelihood estimation

Let us introduce the reflected Ornstein-Uhlenbeck process. Let $\Lambda := (\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ be a filtered probability space with the filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the *usual* conditions. Define the ROU process $\{X_t : t \geq 0\}$ reflected at the boundary $b \in \mathbb{R}_+$ on Λ as follows. Let $\{X_t : t \geq 0\}$ be the strong solution (whose existence is guaranteed by an extension of the results of Lions and Sznitman (1984)) to the stochastic differential equation:

$$\left.\begin{aligned} dX_t &= -\alpha X_t dt + \sigma dW_t + dL_t, \\ X_t &\geq b \quad \text{for all } t \geq 0, \\ X_0 &= x, \end{aligned}\right\} \tag{2.1}$$

where $\alpha \in (-\infty, \infty)$, $\sigma \in (0, \infty)$, $x \in [b, \infty)$ and $W = (W_t)_{t\geq 0}$ is a one-dimensional standard Brownian motion on Λ. Here, $L = (L_t)_{t\geq 0}$ is the minimal non-decreasing and non-negative process, which makes the process $X_t \geq b$ for all $t \geq 0$. The process L increases only when X hits the boundary b, so that

$$\int_{[0,\infty)} I(X_t > b) dL_t = 0, \tag{2.2}$$

where $I(\cdot)$ is the indicator function. It is often the case that the reflecting barrier b is assumed to be zero in applications to queueing system, storage model, engineering, finance, etc. This is mainly due to the physical restriction of the state processes such as queue-length, inventory level, content process, stock prices and interest rates, which take non-negative values. We refer the reader to Harrison (1985) and Whitt (2002) for more details on reflected stochastic processes and their wide applications.

Our interest lies in the statistical inference for the ROU process (2.1). More specifically, we would like to estimate the unknown drift parameter $\alpha \in (-\infty, \infty)$ in (2.1) based on observation of the state process $\{X_t\}_{t\geq 0}$ up to a certain predetermined level of precision. Recently, Bo *et al.* (2011) studied the maximum likelihood estimator (MLE) for the ROU model above, and established several important properties. The MLE $\widehat{\alpha}_T$ of α, based on the process $\{X_t\}$ up to a previously determined fixed time T, is given by

$$\widehat{\alpha}_T := \frac{bL_T - \int_0^T X_t dX_t}{\int_0^T X_t^2 dt}. \tag{2.3}$$

The MLE $\widehat{\alpha}_T$ satisfies strong consistency and asymptotic normality as $T \to \infty$. However, this estimator is biased and its mean squared error (MSE) depends on the unknown parameter to be estimated. We note that exact estimates for the bias and the MSE of the estimator $\widehat{\alpha}_T$ are not available, and they are expressed in an implicit form in terms of a solution of a partial differential equation with Neumann boundary condition.

In this paper, we propose to use the sequential estimation plan $(\tau(H), \widehat{\alpha}_{\tau(H)})$ to estimate the parameter α of the ROU process $\{X_t\}$. Here, we assume that the process $\{X_t\}$ is observed until the observed Fisher information of the process exceeds a predetermined level of precision H, i.e., we observe $\{X_t\}$ over the random time interval $[0, \tau(H)]$ where the stopping time $\tau(H)$ is defined as

$$\tau(H) := \inf\left\{t \geq 0 : \int_0^t X_s^2 ds \geq H\right\}, \quad 0 < H < \infty, \tag{2.4}$$

and the $\mathcal{F}^X_{\tau(H)}$-measurable function $\widehat{\alpha}_{\tau(H)}$ (defined below at (2.5)) is a sequential estimator of the parameter α. Then the sequential estimation plan $(\tau(H), \widehat{\alpha}_{\tau(H)})$ has the following good properties (see Chapter 17.5 of Liptser and Shiryaev (2001) or Chapter 5.2 of Bishwal (2008)): (a) it is unbiased; (b) the plan is closed, i.e., the time of the observation $\tau(H)$ is finite with probability 1; (c) its MSE is a constant that does not depend on the parameter to be estimated; (d) not only it provides consistent estimation plan but also $\widehat{\alpha}_{\tau(H)}$ is exactly normally distributed, which makes it possible to construct an exact confidence interval for the parameter α. These results are stated in the following main theorem.

A proof of the next theorem follows along the similar lines of Section 17.5 of Liptser and Shiryaev (2001) by accommodating our basic model assumptions involving the reflection (i.e., state space) constraint. In what follows, the index α in $\mathbb{P}$ and $\mathbb{E}$ emphasizes the fact that the distribution of the state process is being considered for the prescribed value α.

Theorem 2.1. *The sequential estimation plan* $(\tau(H), \widehat{\alpha}_{\tau(H)})$, $0 < H < \infty$, *with the observation time* $\tau(H)$ *defined as* (2.4) *and the SMLE given by*

$$\widehat{\alpha}_{\tau(H)} := \frac{1}{H}\left[bL_{\tau(H)} - \int_0^{\tau(H)} X_s dX_s\right], \tag{2.5}$$

has the following properties:

(i) $\mathbb{P}_\alpha(\tau(H) < \infty) = 1$ *for all* $\alpha \in (-\infty, \infty)$,

(ii) $\mathbb{E}_\alpha(\widehat{\alpha}_{\tau(H)}) = \alpha$ *for all* $\alpha \in (-\infty, \infty)$,

(iii) $\mathbb{E}_\alpha(\widehat{\alpha}_{\tau(H)} - \alpha)^2 = \frac{\sigma^2}{H}$,

(iv) $\sqrt{H}(\widehat{\alpha}_{\tau(H)} - \alpha) \overset{\mathcal{D}}{=} N(0, \sigma^2)$.

Proof. We first show that the $\mathcal{F}^X_{\tau(H)}$-measurable random variable $\widehat{\alpha}_{\tau(H)}$ is indeed the SMLE as follows. Consider an arbitrary stopping time τ with respect to the filtration $\{\mathcal{F}^X_t\}_{t\geq 0}$ generated by the process X. According to Theorem 2.1 in Bo *et al.* (2011) (see also Theorem 1 in Ward and Glynn (2003b)), the probability measures $\mathbb{P}^\theta_{\tau,X}$ and $\mathbb{P}^\alpha_{\tau,X}$ corresponding to the processes

$$\left.\begin{aligned} dX^\theta_t &= -\theta X^\theta_t dt + \sigma dW_t + dL^\theta_t, \quad X^\theta_t \geq b \quad \text{for all } t \geq 0, \\ dX^\alpha_t &= -\alpha X^\alpha_t dt + \sigma dW_t + dL^\alpha_t, \quad X^\alpha_t \geq b \quad \text{for all } t \geq 0, \end{aligned}\right\}$$

respectively, are equivalent and their Radon-Nikodym derivative is given by

$$\frac{d\mathbb{P}^{\alpha}_{\tau,X}}{d\mathbb{P}^{\theta}_{\tau,X}}\Big|_{\mathcal{F}^X_{\tau,\theta}} = \exp\left\{-\frac{1}{\sigma}\int_0^{\tau}(\alpha-\theta)X_t^{\theta}dW_t - \frac{1}{2\sigma^2}\int_0^{\tau}(\alpha-\theta)^2(X_t^{\theta})^2dt\right\}, \tag{2.6}$$

where $\mathcal{F}^X_{\tau,\theta}$ is the natural filtration generated by $\{X_t^{\theta} : 0 \le t \le \tau\}$. Let X_t^0 be the reflected Brownian motion (RBM) satisfying $dX_t^0 = \sigma dW_t + dL_t$, $t \ge 0$ and $\mathbb{P}^0_{\tau}$ be the measure induced by the RBM X^0. Then the log likelihood function $\ell_{\tau}(\alpha)$ is given by

$$\ell_{\tau}(\alpha) := \sigma^2 \log\frac{d\mathbb{P}^{\alpha}_{\tau,X}}{d\mathbb{P}^{0}_{\tau,X}} = -\alpha\int_0^{\tau}X_tdX_t - \frac{\alpha^2}{2}\int_0^{\tau}X_t^2dt + \alpha\int_0^{\tau}X_tdL_t.$$

Then, by solving the equation

$$\dot{\ell}_{\tau}(\alpha) := \frac{d}{d\alpha}\left(\sigma^2\log\frac{d\mathbb{P}^{\alpha}_{\tau,X}}{d\mathbb{P}^{0}_{\tau,X}}\right) = 0,$$

we obtain the SMLE given by

$$\widehat{\alpha}_{\tau} = \frac{-\int_0^{\tau}X_sdX_s + \int_0^{\tau}X_sdL_s}{\int_0^{\tau}X_s^2ds} = \frac{-\int_0^{\tau}X_sdX_s + bL_{\tau}}{\int_0^{\tau}X_s^2ds}, \tag{2.7}$$

where the second equality follows from (2.1). Now, setting $\tau = \tau(H)$ in (2.7), we obtain for $\widehat{\alpha}_{\tau} = \widehat{\alpha}_{\tau(H)}$ the representation given by (2.5).

To verify (i), notice that for $T > 0$ we have

$$\mathbb{P}_{\alpha}(\tau(H) \ge T) = \mathbb{P}_{\alpha}\left(\int_0^T X_s^2ds < H\right).$$

Thus it suffices to show $\mathbb{P}_{\alpha}(\int_0^{\infty}X_s^2ds = \infty) = 1$, which is to say that $\int_0^T X_s^2ds \to \infty$ a.s. as $T \to \infty$. But this fact follows from a simple comparison result between the ROU process and the regular OU process with the same drift vector (see, e.g., p. 591 in Bo *et al.* (2011)). To keep the presentation self-contained, we provide the details. Let $Y = (Y_t)_{t\ge 0}$ be the regular OU process satisfying $dY_t = -\alpha Y_tdt + \sigma dW_t$. It can be shown that $\int_0^T Y_s^2ds \to \infty$ a.s. as $T \to \infty$ and $X_t - Y_t = e^{-\alpha t}(X_0 - Y_0) + e^{-\alpha t}\int_0^t e^{\alpha s}dL_s$. It follows that if $X_0 \ge Y_0$, then $X_t \ge Y_t$ a.s. for all $t \ge 0$, and therefore we have $\int_0^T X_s^2ds \to \infty$ a.s. as $T \to \infty$.

The proof of (ii) follows directly from noting that

$$\begin{aligned}
\widehat{\alpha}_{\tau(H)} &= \frac{1}{H}\left(bL_{\tau(H)} - \int_0^{\tau(H)} X_s dX_s\right) \\
&= \frac{1}{H}\left(bL_{\tau(H)} + \alpha \int_0^{\tau(H)} X_s^2 ds - \sigma \int_0^{\tau(H)} X_s dW_s - bL_{\tau(H)}\right) \\
&= \alpha - \frac{\sigma}{H}\int_0^{\tau(H)} X_s dW_s,
\end{aligned}$$

and the fact that the process $\{\int_0^{\tau(H)} X_s dW_s\}$ is a Wiener process with variance $H > 0$ (see, e.g., Lemma 17.4 in Liptser and Shiryaev (2001) or §4 in Gīhman and Skorohod (1972)). Hence

$$\mathbb{E}_\alpha(\widehat{\alpha}_{\tau(H)}) = \mathbb{E}_\alpha\left(\alpha - \frac{1}{H}\sigma \int_0^{\tau(H)} X_s dW_s\right) = \alpha.$$

Next, notice that (when α is the true parameter)

$$(\widehat{\alpha}_{\tau(H)} - \alpha)^2 = \frac{\sigma^2}{H^2}\left(\int_0^{\tau(H)} X_s dW_s\right)^2,$$

and the stochastic integral $\int_0^{\tau(H)} X_s dW_s$ is normally distributed with mean 0 and variance H. Hence, (iii) follows immediately since $\int_0^{\tau(H)} X_s^2 ds = H$ by the definition of $\tau(H)$ in (2.4).

Lastly, the result (iv) follows from the fact that

$$\begin{aligned}
\sqrt{H}(\widehat{\alpha}_{\tau(H)} - \alpha) &= -\frac{\sigma}{\sqrt{H}}\int_0^{\tau(H)} X_s dW_s \\
&\stackrel{\mathcal{D}}{=} \frac{\sigma}{\sqrt{H}} N(0, H) \stackrel{\mathcal{D}}{=} N(0, \sigma^2).
\end{aligned}$$

This completes the proof of theorem. □

Remark 2.1. *Although we focus on the non-asymptotic properties of SMLE, one can easily show the strong consistency of the SMLE, i.e.,* $\widehat{\alpha}_{\tau(H)} \to \alpha$ *a.s. as* $H \to \infty$*. To establish this property, it is enough to show that*

$$\frac{\sigma}{H}\int_0^{\tau(H)} X_s dW_s \to 0 \quad a.s. \quad as\ H \to \infty,$$

which follows immediately from noting that $\int_0^{\tau(H)} X_s dW_s \stackrel{\mathcal{D}}{=} N(0, H)$*.*

Remark 2.2. *Bo* et al. *(2011) deals with the ergodic case (i.e., $\alpha > 0$). We do not need ergodicity of our model. The behavior of SMLE holds for all $\alpha \in (-\infty, \infty)$, i.e., ergodic ($\alpha > 0$), non-ergodic ($\alpha < 0$) and non-stationary ($\alpha = 0$) cases. This uniform behavior of SMLE distinguishes it from the ordinary MLE.*

Remark 2.3. *Notice that the stopping time $\tau(H)$ substantially depends on the precision $H > 0$. Thus, a natural question arises how to choose H in practice. The result about MSE in Theorem 2.1 (iii) reveals the meaning of the constant H in the definition of the sequential estimation plan* (2.5)*. In practice, one can pre-specify the desired tolerance value for MSE and then take $H > 0$ which corresponds to this prescribed value.*

In view of the preceding remark, it is of interest to compute some bounds on the average observation time $\mathbb{E}_\alpha[\tau(H)]$ of the process $\{X_t\}$. For the sake of simplicity, we assume $X_0 = x = 0$ subsequently. In the following theorem, it is seen under mild conditions that the average observation time $\mathbb{E}_\alpha[\tau(H)]$ of our estimation plan has approximately a linear growth in H (or, in terms of the prescribed MSE associated with given H).

Theorem 2.2. *The moments of all orders of the stopping time exist, i.e., $\mathbb{E}_\alpha[\tau(H)]^n < \infty$ for all $n = 1, 2, \ldots$. Furthermore, the average observation time $\mathbb{E}_\alpha[\tau(H)]$ of the sequential estimation plan satisfies*

$$\mathbb{E}_\alpha[\tau(H)] \leq 2\left(\frac{|\alpha|}{\sigma^2}H + \frac{2}{\sigma}\sqrt{H}\right) + \sqrt{\frac{4\alpha^2}{\sigma^4}H^2 + \frac{16|\alpha|}{\sigma^3}H^{3/2} + \frac{18}{\sigma^2}H}. \tag{2.8}$$

In the case $b = 0$ and $\alpha > 0$, the following lower bound estimate holds for $\mathbb{E}_\alpha[\tau(H)]$:

$$\mathbb{E}_\alpha[\tau(H)] \geq \frac{2\alpha H}{\sigma^2}. \tag{2.9}$$

Proof. The proof is adapted from that of Theorem 17.7 in Liptser and Shiryaev (2001). By the Itô's formula, we have

$$\begin{aligned} X_t^2 &= 2\int_0^t X_s dX_s + \sigma^2 t \\ &= -2\alpha\int_0^t X_s^2 ds + 2\sigma\int_0^t X_s dW_s + 2bL_t + \sigma^2 t, \end{aligned} \tag{2.10}$$

where the second equality follows from (2.1) and (2.2). Hence, letting $t = \tau(H)$, we get

$$\begin{aligned}
\int_0^{\tau(H)} X_t^2 dt &= -2\alpha \int_0^{\tau(H)} \left(\int_0^t X_s^2 ds \right) dt + 2\sigma \int_0^{\tau(H)} \left(\int_0^t X_s dW_s \right) dt \\
&\quad + 2b \int_0^{\tau(H)} L_t dt + \frac{\sigma^2}{2} (\tau(H))^2 \\
&\geq -2\alpha \int_0^{\tau(H)} \left(\int_0^t X_s^2 ds \right) dt + 2\sigma \int_0^{\tau(H)} \left(\int_0^t X_s dW_s \right) dt + \frac{\sigma^2}{2} (\tau(H))^2 \quad (2.11) \\
&\geq -2\alpha H \tau(H) + 2\sigma \int_0^{\tau(H)} \left(\int_0^t X_s dW_s \right) dt + \frac{\sigma^2}{2} (\tau(H))^2, \quad (2.12)
\end{aligned}$$

where (2.11) follows from the fact that $(L_t)_{t \geq 0}$ is a non-negative and non-decreasing process and (2.12) is owing to the definition of $\tau(H)$ in (2.4). Therefore, by rearranging terms and noting that $\int_0^{\tau(H)} X_s^2 ds = H$, it follows that

$$\begin{aligned}
(\tau(H))^2 &\leq \frac{4\alpha}{\sigma^2} H \tau(H) - \frac{4}{\sigma} \int_0^{\tau(H)} \left(\int_0^t X_s dW_s \right) dt + \frac{2}{\sigma^2} H \\
&\leq \left(\frac{4|\alpha|}{\sigma^2} H + \frac{4}{\sigma} \beta \right) \tau(H) + \frac{2}{\sigma^2} H,
\end{aligned}$$

where $\beta := \sup_{0 \leq t \leq \tau(H)} | \int_0^t X_s dW_s|$. This implies that for each $\alpha \in (-\infty, \infty)$,

$$\tau(H) \leq 2 \left(\frac{|\alpha|}{\sigma^2} H + \frac{\beta}{\sigma} \right) + \sqrt{4 \left(\frac{|\alpha|}{\sigma^2} H + \frac{\beta}{\sigma} \right)^2 + \frac{2}{\sigma^2} H},$$

and thus by the Jensen's inequality we have

$$\mathbb{E}_\alpha[\tau(H)] \leq 2 \left(\frac{|\alpha|}{\sigma^2} H + \frac{1}{\sigma} (\mathbb{E}_\alpha[\beta^2])^{1/2} \right) + \sqrt{\frac{4\alpha^2}{\sigma^4} H^2 + \frac{4\mathbb{E}_\alpha[\beta^2]}{\sigma^2} + \frac{8|\alpha| H}{\sigma^3} (\mathbb{E}_\alpha[\beta^2])^{1/2} + \frac{2}{\sigma^2} H}. \quad (2.13)$$

Finally, from Doob's inequality for submartingales we obtain an upper bound

$$\mathbb{E}_\alpha[\beta^2] \leq 4\mathbb{E}_\alpha \left| \int_0^{\tau(H)} X_s dW_s \right|^2 = 4H,$$

from which the desired result (2.8) follows using (2.13). To show the existence of moments of all orders for $\tau(H)$, it is sufficient to notice that for $p = 2, 4, \ldots$, we have

$$\mathbb{E}_\alpha[\beta^p] \le \left(\frac{p}{p-1}\right)^p \mathbb{E}_\alpha \left| \int_0^{\tau(H)} X_s dW_s \right|^p = \left(\frac{p}{p-1}\right)^p (p-1)!! H^{p/2} < \infty.$$

To deduce (2.9), it suffices to note that the inequality

$$\begin{aligned} \mathbb{E}_\alpha[\sigma^2 \tau(H)] &\ge 2\alpha \mathbb{E}_\alpha \left(\int_0^{\tau(H)} X_s^2 ds \right) - 2\sigma \mathbb{E}_\alpha \left(\int_0^{\tau(H)} X_s dW_s \right) \\ &= 2\alpha H, \end{aligned}$$

which follows from (2.10) with substituting $t = \tau(H)$. □

Next, we show that the proposed sequential estimation plan $(\tau(H), \widehat{\alpha}_{\tau(H)})$ for estimating parameter α of the ROU process $\{X_t\}$ is efficient among all unbiased estimation plans in the mean squared error sense. To do this, we will assume that $b = 0$ and prove an analog of the Cramer-Rao lower bound for arbitrary unbiased estimation plans.

Theorem 2.3. *Assume $b = 0$. Let the sequential plan $(\tau, \widehat{\alpha}_\tau(X))$ be an arbitrary unbiased estimation plan for the ROU process $\{X_t\}$ with the parameter $\alpha \in (-\infty, \infty)$, namely,*

$$\mathbb{E}_\alpha(\widehat{\alpha}_\tau(X)) = \alpha \quad \textit{for all} \quad \alpha \in (-\infty, \infty). \tag{2.14}$$

Suppose also that $0 < \mathbb{E}_\alpha[\int_0^\tau X_s^2 ds] < \infty$. Then,

$$\mathrm{Var}_\alpha(\widehat{\alpha}_\tau) = \mathbb{E}_\alpha[\widehat{\alpha}_\tau - \alpha]^2 \ge \frac{\sigma^2}{\mathbb{E}_\alpha[\int_0^\tau X_s^2 ds]}. \tag{2.15}$$

Remark 2.4. *A sequential estimation plan $(\tau, \widehat{\alpha}_\tau)$ is said to be* efficient *in the MSE sense if for which (2.15) becomes an equality for all $\alpha \in (-\infty, \infty)$. Notice that since $\mathbb{E}_\alpha[\widehat{\alpha}_{\tau(H)} - \alpha]^2 = \frac{\sigma^2}{H}$ as established in Theorem 2.1 (iii) and $\mathbb{E}_\alpha[\int_0^{\tau(H)} X_s^2 ds] = H$ by the definition of $\tau(H)$, the sequential estimation plan $(\tau(H), \widehat{\alpha}_{\tau(H)})$ is efficient in the MSE sense.*

Proof. Without loss of generality, we assume that $\sigma = 1$. In view of the Radon-Nikodym derivative expression in (2.6) with $\theta = 0$, differentiating both sides of (2.14) with respect to α yields that

$$\mathbb{E}_\alpha \left[\widehat{\alpha}_\tau \left\{ -\int_0^\tau X_s dX_s - \alpha \int_0^\tau X_s^2 ds \right\} \right] = 1, \tag{2.16}$$

where $\tau := \tau^X$ (cf. the proof of Theorem 7.22 in Liptser and Shiryaev (2001)). Then, since

$$\begin{aligned} \mathbb{E}_\alpha \left[\int_0^\tau X_s dX_s + \alpha \int_0^\tau X_s^2 ds \right] &= \mathbb{E}_\alpha \left[-\alpha \int_0^\tau X_s^2 ds + \int_0^\tau X_s dW_s + \alpha \int_0^\tau X_s^2 ds \right] \\ &= 0, \end{aligned}$$

it follows that

$$\mathbb{E}_\alpha \left[(\widehat{\alpha}_\tau - \alpha) \left(-\int_0^\tau X_s dX_s - \alpha \int_0^\tau X_s^2 ds \right) \right] = 1.$$

Applying Cauchy-Schwarz inequality in (2.16), we obtain

$$\begin{aligned} 1 &\leq \mathbb{E}_\alpha[\widehat{\alpha}_\tau - \alpha]^2 \mathbb{E}_\alpha \left[\left(\int_0^\tau X_s dX_s + \alpha \int_0^\tau X_s^2 ds \right)^2 \right] \\ &= \mathbb{E}_\alpha[\widehat{\alpha}_\tau - \alpha]^2 \mathbb{E}_\alpha \left[\left(\int_0^\tau X_s dW_s \right)^2 \right] \\ &= \mathbb{E}_\alpha[\widehat{\alpha}_\tau - \alpha]^2 \mathbb{E}_\alpha \left[\int_0^\tau X_s^2 ds \right], \end{aligned} \tag{2.17}$$

where the first equality follows from the state equation (2.1). Since $0 < \mathbb{E}_\alpha[\int_0^\tau X_s^2 ds] < \infty$, we can divide both sides of (2.17) by this factor, and then the desired result follows. □

3 Numerical performance of the estimator

In this section, we examine and compare the performance of the ordinary MLE $\widehat{\alpha}_T$ and SMLE $\widehat{\alpha}_{\tau(H)}$. As seen in the previous section, the sequential estimator $\widehat{\alpha}_{\tau(H)}$ has several desirable properties such as unbiasedness, normal distribution, and optimality in the MSE sense. It would be interesting to know the relative efficiency of the estimators $\widehat{\alpha}_T$ and $\widehat{\alpha}_{\tau(H)}$, which is defined by

$$r(\alpha, T) := \frac{\mathbb{E}_\alpha(\widehat{\alpha}_T - \alpha)^2}{\mathbb{E}_\alpha(\widehat{\alpha}_{\tau(H)} - \alpha)^2}. \tag{3.1}$$

While $\mathbb{E}_\alpha(\widehat{\alpha}_{\tau(H)} - \alpha)^2$ is explicitly given by σ^2/H as in Theorem 2.1 (iii), the expression of MSE $\mathbb{E}_\alpha(\widehat{\alpha}_T - \alpha)^2$ cannot be obtained in such a simple form. In Bo *et al.* (2011), Lemma 3.1 provides a formula $\mathbb{E}_\alpha(\widehat{\alpha}_T - \alpha)^2 = \int_0^\infty \Psi_T(\alpha, a)da + \int_0^\infty a \frac{\partial^2}{\partial \alpha^2} \Psi_T(\alpha, a)da$, where an auxiliary function Ψ_T can be obtained by solving a partial differential equation with Neumann boundary condition (cf. Theorem 3.3 in Bo *et al.* (2011)), similar to the Feymann-Kac type formula. With the lack of an explicit, closed form formula for $\mathbb{E}_\alpha(\widehat{\alpha}_T - \alpha)^2$, it seems inevitable computing this via a Monte Carlo simulation procedure.

To compute $r(\alpha, T)$ numerically, we naturally impose a certain condition that the mean durations of the observations in both estimation procedures being compared are the same. More precisely, the choice of the level $H \in (0, \infty)$ should be such that

$$\mathbb{E}_\alpha[\tau(H)] = T, \tag{3.2}$$

for given α and T. Note that the distribution of $\tau(H)$ depends in general on the parameter α as well as $H \in (0, \infty)$. In the simulation study below, we actually fix the level $H \in (0, \infty)$ first, and choose $T > 0$ via a Monte Carlo simulation by approximating the left-hand side of (3.2). Thus, for this practical reasons, in the comparison study that follows, H is treated as a deriving variable rather than T. For a simulation of the ROU, we adopt the numerical scheme presented in Lépingle (1995), which is known to yield the same rate of convergence as the usual Euler-Maruyama scheme.

Denote the time between each discretization step by Δt. We generate $n = 1,000$ Monte Carlo simulations of the sample paths with $\Delta t = 0.01$ under two different settings:

(a) Set $\alpha = 0.5$, $\sigma = 1$;

(b) Set $\alpha = 1$, $\sigma = 2$.

Shown in Table 1 are the average squared errors along with the standard errors. The values of T that approximate $\mathbb{E}_\alpha[\tau(H)]$ over Monte Carlo simulations are reported as well. The two estimators eventually show similar performance as H becomes large, but for small values of H, the SMLE $\widehat{\alpha}_{\tau(H)}$ clearly outperforms the MLE $\widehat{\alpha}_T$. Notice also that the average observation time $\mathbb{E}_\alpha[\tau(H)]$, which is approximated from Monte Carlo simulation and set to be T in (3.2), shows a linear growth in H. This confirms the theoretical result shown in Theorem 2.2.

For Figures 1–5, the sample paths are generated under the setting (a). Figure 1 shows the MSE of the two estimators against H. The SMLE (dash) shows competitive performance compared to MLE (dash-dot). The exact theoretical MSE curve (i.e., σ^2/H) of SMLE is overlayed as a solid curve, which is almost overlapping with the dashed curve. The relative efficiency defined in (3.1) (except that H becomes a deriving variable) is seen in Figure 2. Eventually the curve goes down to 1, but when the paths are observed for a short period, say corresponding to $H = 10$, the SMLE is more than 1.6 times better than the MLE in terms of reducing the MSE. One of the reasons why the SMLE shows better performance than the MLE is due to its unbiasedness. The biases of the two methods are presented in Figure 3.

The distribution of each estimator (under $H = 20$) is illustrated as a histogram, depicted in Figures 4 and 5. For an easy comparison, estimators are centered and scaled appropriately so that both converge to the standard normal distribution. Specifically, the histograms of $\sqrt{\int_0^T X_s^2 ds}(\widehat{\alpha}_T - \alpha)/\sigma$ and $\sqrt{H}(\widehat{\alpha}_{\tau(H)} - \alpha)/\sigma$ are drawn in Figures 4 and 5, respectively. In each panel, the standard normal distribution density function is overlayed as a solid curve. The normal distribution is in fact the exact distribution for arbitrary $H > 0$ for the SMLE. The histogram in Figure 4 reasonably well approximates the standard Gaussian density, whereas the one in Figure 5 is skewed to the right.

		$\widehat{\alpha}_{\tau(H)}$	$\widehat{\alpha}_T$
$\alpha = 0.5, \sigma = 1$	$H = 10$ ($T \approx 11.96$)	0.1031 (0.1309)	0.1674 (0.4422)
	$H = 50$ ($T \approx 50.41$)	0.0200 (0.0295)	0.0242 (0.0432)
	$H = 100$ ($T \approx 97.68$)	0.0104 (0.0149)	0.0113 (0.0190)
$\alpha = 1, \sigma = 2$	$H = 10$ ($T \approx 5.92$)	0.4024 (0.5473)	0.5231 (1.0249)
	$H = 50$ ($T \approx 24.41$)	0.0823 (0.1215)	0.0942 (0.1804)
	$H = 100$ ($T \approx 47.63$)	0.0426 (0.0590)	0.0469 (0.0810)

Table 1: The mean and the standard error of $(\widehat{\alpha} - \alpha)^2$ for SMLE and MLE. The SMLE $\widehat{\alpha}_{\tau(H)}$ outperforms the MLE $\widehat{\alpha}_T$ for small values of H (or T).

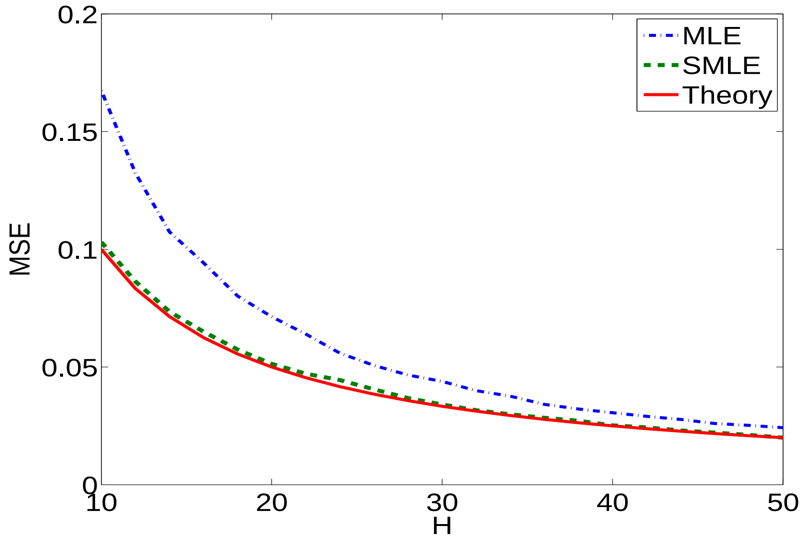

Figure 1: The MSE plot for $\widehat{\alpha}_{\tau(H)}$ (dash) and $\widehat{\alpha}_T$ (dash-dot). The theoretical MSE for $\widehat{\alpha}_{\tau(H)}$ is drawn as a solid line. The SMLE apparently outperforms the MLE, especially when an estimation is based on a process observed for a short period of time.

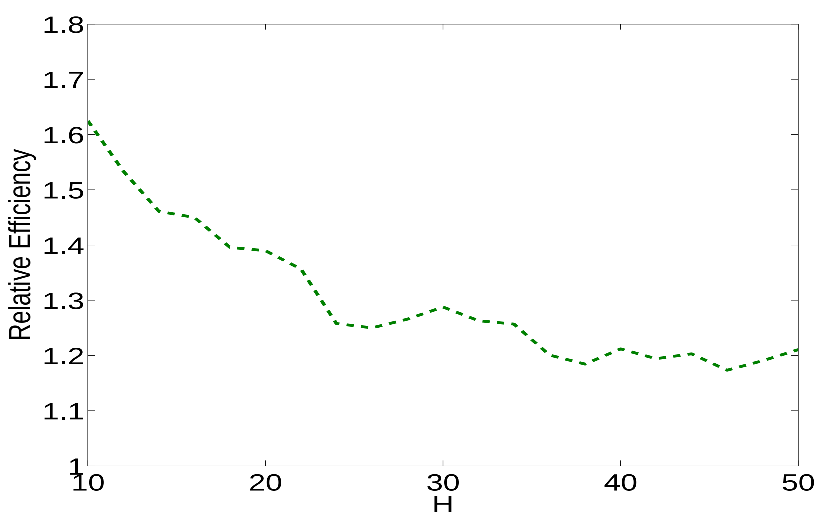

Figure 2: The relative efficiency of estimators defined as (3.1). The SMLE is uniformly more efficient than the MLE when the paths are observed for a short duration of time.

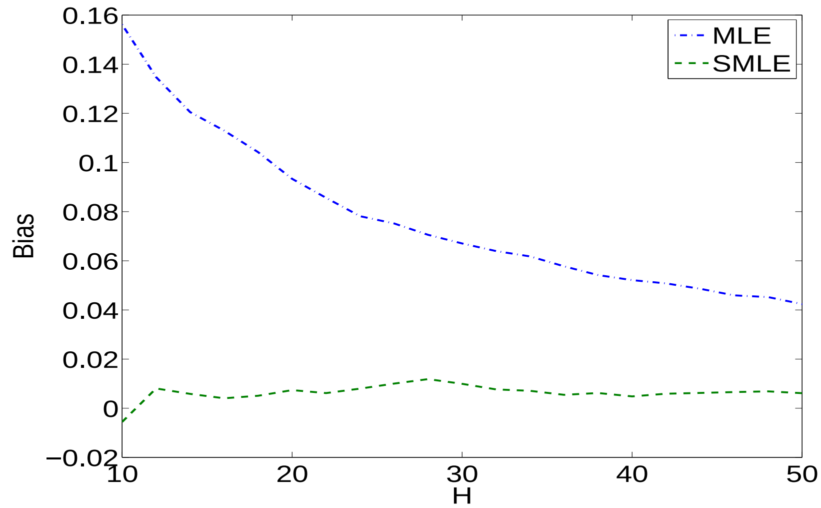

Figure 3: Bias plot for $\widehat{\alpha}_{\tau(H)}$ (dash) and $\widehat{\alpha}_T$ (dash-dot) against H. The main reason why SMLE works better than the MLE for shorter-time paths is due to its unbiasedness.

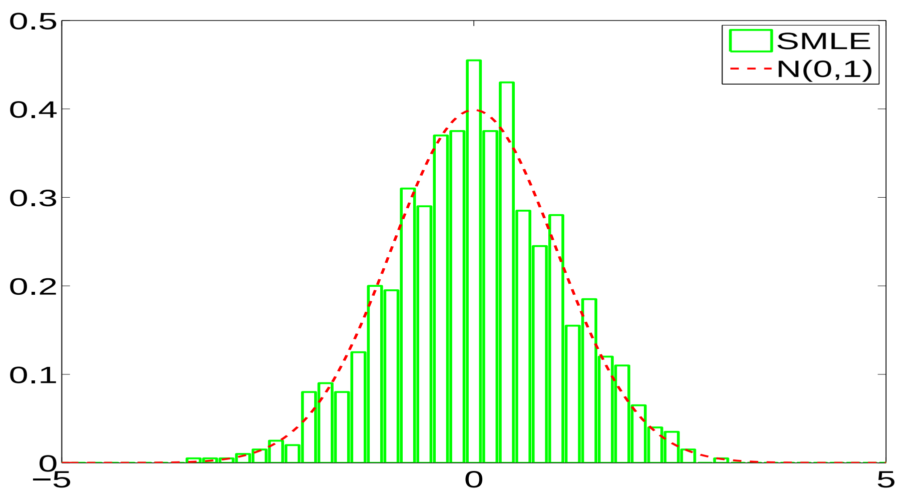

Figure 4: Histogram of $\sqrt{H}(\widehat{\alpha}_{\tau(H)} - \alpha)/\sigma$ with $H = 20$. It has the normal distribution for arbitrary $H > 0$. Thus, the sequential estimator works quite well even when the path is observed only for a a fairly short time.

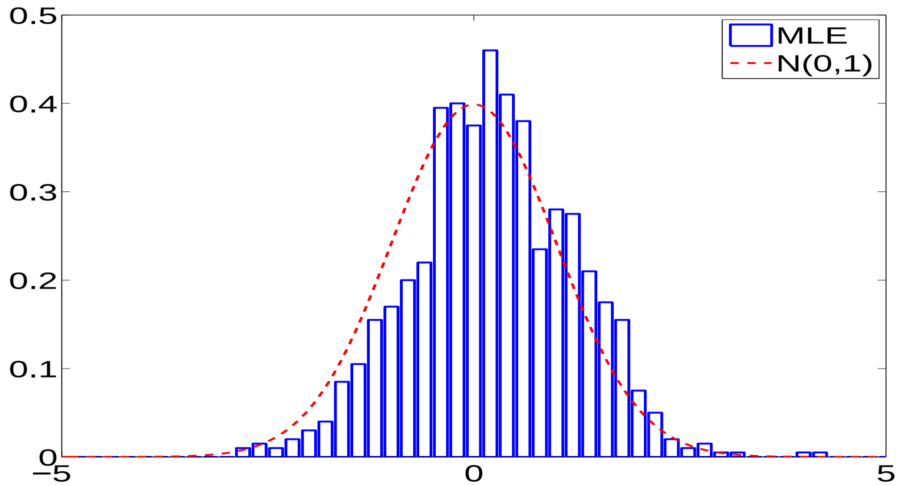

Figure 5: Histogram of $\sqrt{\int_0^T X_s^2 ds}(\widehat{\alpha}_T - \alpha)/\sigma$. This will converge to the standard Gaussian as $T \to \infty$. For a fair comparison with $\widehat{\alpha}_{\tau(H)}$ with $H = 20$, T is chosen around the value 21.74 so that (3.2) is satisfied. For small values of $T > 0$, the distribution is observed skewed to the right.

4 Conclusion

We studied sequential estimation problem for the parameter of a one-dimensional ROU process based on continuous observation. An explicit sequential estimator is provided and the theoretical properties of the estimator such as unbiasedness, exact uniform normality, expression of the mean squared error, and its efficiency results are derived. A simulation study on the comparison of performance with the MLE illustrates the proposed estimator's usefulness on the short time horizon data in the practical situations.

There are several important directions for future research. First, we would like to study estimation problems when only discrete observations are available, which is the scenario one often encounters in practice. In such cases, the parameter estimation based on the likelihood function will be considerably more involved, since the transition densities for reflected diffusions have quite complex representations (see, e.g., Linetsky (2005); Ricciardi and Sacerdote (1987)). Second, one may be interested in the sequential plan of hypothesis testing characterized by a certain stopping time (the end of the observation) and a decision rule. We note that the stopping time appearing in the sequential testing problem may be quite different from that used in the sequential estimation plan which makes the hypothesis testing problem for reflected diffusions quite nontrivial.

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