Linear Time Series with "Nonlinear" Behavior

F. Jay Breidt

Department of Statistics
Iowa State University

`http://www.public.iastate.edu/~jbreidt`

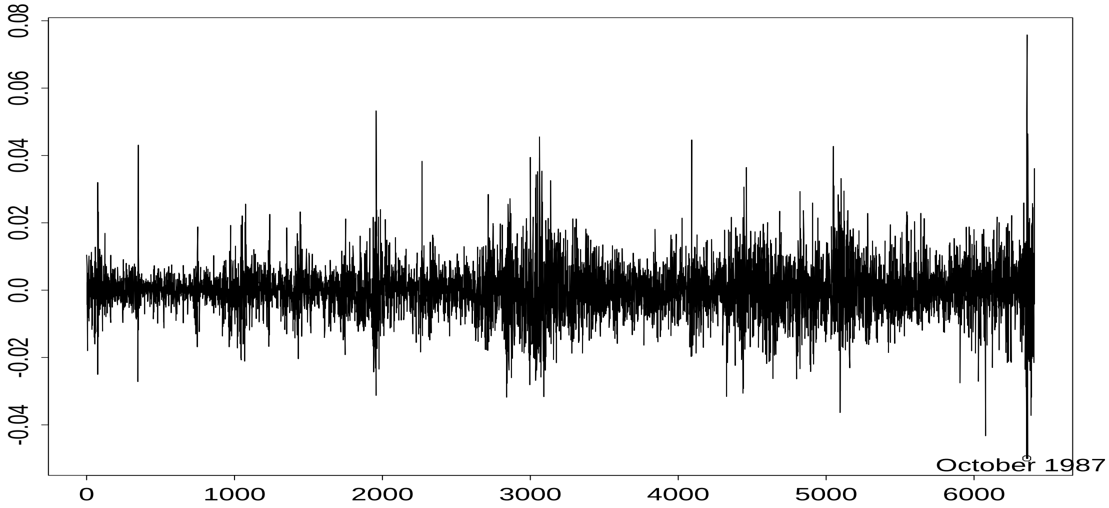
August 13, 2000
Joint work with Richard Davis, Colorado State University,
and Alex Trindade, University of Florida

Outline

- Introduction
  - motivating example
  - all-pass models and their properties
- Estimation
  - likelihood approximation
  - least absolute deviations (LAD)
  - asymptotic results
  - order selection
  - simulation
- Noncausal autoregressive processes
  - a two-step estimation procedure
  - Microsoft trading volume
- Summary

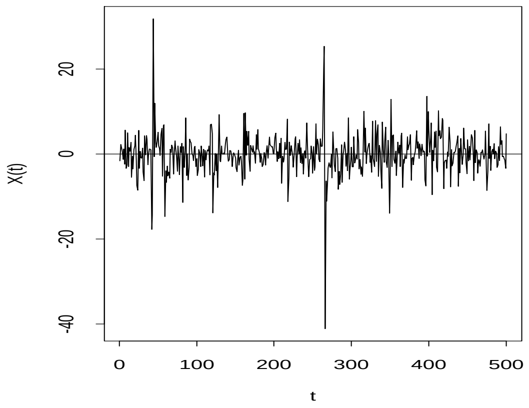
Financial Time Series

- Returns on financial assets often exhibit:
  - lack of serial correlation
  - heavy-tailed marginal distributions
  - bursts of outliers / volatility clustering
  - dependence outside 2nd-order moment structure
- Nonlinear models: $X_t = \sigma_t Z_t$
  - ARCH and its variants (Engle 1982; Bollerslev, Chou, and Kroner 1992)
  - Stochastic volatility (Clark 1973; Taylor 1986)

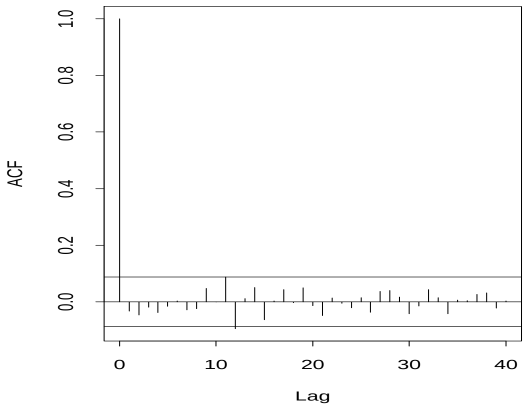


A Simulated Example

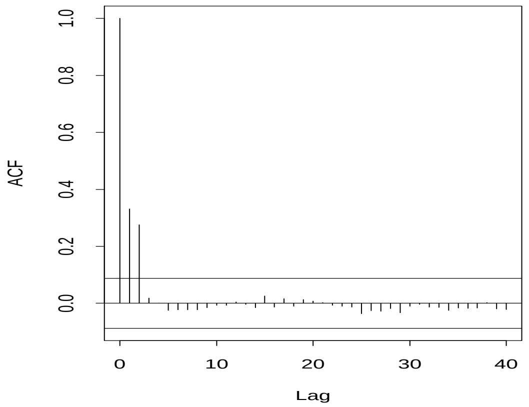
- White noise with "volatility clustering"



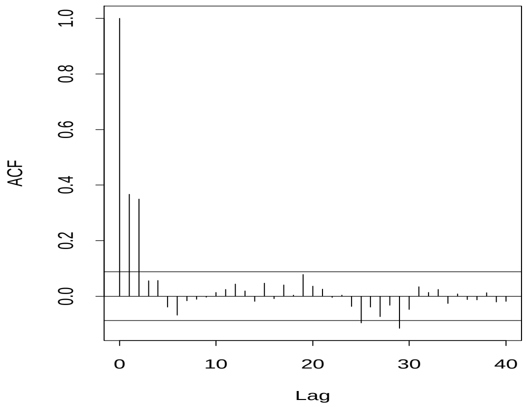
(a) Data From Allpass Model



(b) ACF of Allpass Data



(c) ACF of Squares



(d) ACF of Absolute Values

A Special ARMA Model

- Set up for later order selection:

$$\begin{aligned} r &= \text{unknown } \textit{r}\text{eal model order} \\ &\le s = \text{known } \textit{s}\text{ufficiently large model order} \\ p &= \textit{p}\text{roposed model order} \le s \end{aligned}$$

- Causal AR(s) polynomial:

$$\phi_0(z) = 1 - \phi_{01} z - \cdots - \phi_{0s} z^s$$

  where $\phi_{0r} \neq 0$ and $\phi_{0j} = 0$ for $r < j \le s$

- Causal all-pass of order r is $\{X_t\}$ satisfying

$$\phi_0(B) X_t = \frac{B^s \phi_0(B^{-1})}{-\phi_{0r}} Z_t, \ \{Z_t\} \text{ iid} f_\sigma$$

- AP(r) is causal, noninvertible ARMA(r, r)

Properties of All-Pass Models

- Spectral density of AP(r) is

$$\frac{|e^{-is\omega}|^2|\phi_0(e^{i\omega})|^2}{\phi_{0r}^2|\phi_0(e^{-i\omega})|^2}\frac{\sigma^2}{2\pi}=\frac{\sigma^2}{\phi_{0r}^2 2\pi}$$

- $\{X_t\}$ ∼AP(r) is
  - zero mean
  - serially uncorrelated (flat spectrum)
  - dependent if $\{Z_t\}$ is non-Gaussian
  - heavy-tailed if $\{Z_t\}$ is heavy-tailed
- Linear time series with “non-linear” behavior
  - illustration of general result (Bickel and Bühlmann, 1996)

Estimation for All-Pass Models

- Second-order moment techniques do not work
  - least squares
  - Gaussian likelihood
- Higher-order cumulant methods
  - Giannakis and Swami (1990)
  - Chi and Kung (1995)
- Non-Gaussian log-likelihood approximation:

$$\begin{aligned}\mathcal{L}(\boldsymbol{\phi},\sigma) &= \sum_{t=1}^{n-s} \ln f_\sigma\left(\phi_p z_t(\boldsymbol{\phi})\right) + (n-s)\ln|\phi_p| \\ &= (n-s)\ln\sigma^{-1}|\phi_p| + \sum_{t=1}^{n-s}\ln f(\sigma^{-1}|\phi_p| z_t(\boldsymbol{\phi}))\end{aligned}$$

  - least absolute deviations

Asymptotic Results

- Let $\gamma(h)$ = ACVF of AR $\phi_0(\cdot)$ and

$$\boldsymbol{\Gamma}_s = [\gamma(j-k)]_{j,k=1}^{s}$$

- For LAD estimators of AP(s),

$$n^{1/2}(\hat{\boldsymbol{\phi}}_{LAD} - \boldsymbol{\phi}_0) \xrightarrow{\mathcal{L}} \mathrm{N}\left(\mathbf{0}, \frac{\mathrm{Var}\left(|Z_1|\right)}{2\sigma^4 f_\sigma^2(0)}\sigma^2\boldsymbol{\Gamma}_s^{-1}\right)$$

- For LS estimators of AR(s),

$$n^{1/2}(\hat{\boldsymbol{\phi}}_{LS} - \boldsymbol{\phi}_0) \xrightarrow{\mathcal{L}} \mathrm{N}\left(\mathbf{0}, \sigma^2\boldsymbol{\Gamma}_s^{-1}\right)$$

Order Selection

- True model is AP(r) and fitted model is AP(s), $s > r$:

$$n^{1/2}\hat{\phi}_{s,LAD} \xrightarrow{\mathcal{L}} N\left(0, \frac{\mathrm{Var}\,(|Z_1|)}{2\sigma^4 f_\sigma^2(0)}\right)$$

- Model selection procedure:

1. Fit AP(s), s large and obtain residuals $\{z_t(\hat{\boldsymbol{\phi}})\}$

$$\begin{aligned}\hat{\theta}^2 &= \frac{\mathrm{var}\{|z_t(\hat{\boldsymbol{\phi}})|\}}{2\left[\mathrm{var}\{z_t(\hat{\boldsymbol{\phi}})\}\right]^2\left\{\hat{f}_{z_t(\hat{\phi})}(0)\right\}^2} \\ &\xrightarrow{P} \frac{\mathrm{Var}\,(|Z_1|)}{2\sigma^4 f_\sigma^2(0)}\end{aligned}$$

2. Fit AP(p) $p = 1, 2, \ldots, s$ via LAD and obtain $\hat{\phi}_{pp}$
3. Choose the model order $\hat{r}$:

$$\hat{r} = \min\{0 \le p \le s : |\hat{\phi}_{jj}| < 1.96\hat{\theta}n^{-1/2} \text{ for } j > p\}$$

AIC: $2p$ or not $2p$?

- Approximately unbiased estimator of the Kullback-Leibler index of fitted to true model:

$$\mathrm{AIC}(p) := -2\mathcal{L}_X(\hat{\boldsymbol{\phi}}, \hat{\kappa}) + \frac{\mathrm{Var}|Z_1|}{\mathrm{E}|Z_1|\sigma^2 f_\sigma(0)} p$$

- Penalty term for Laplace case:

$$\frac{\mathrm{Var}|Z_1|}{\mathrm{E}|Z_1|\sigma^2 f_\sigma(0)} p = \frac{\sigma^2/2}{(\sigma/\sqrt{2})\sigma^2(1/\sqrt{2}\sigma)} p = p$$

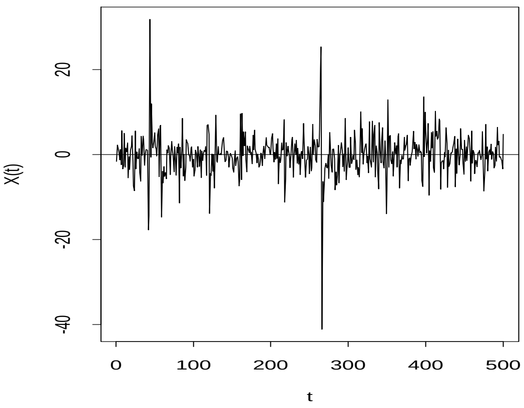
- Estimated penalty term:

$$\frac{\mathrm{var}\{|z_t(\hat{\boldsymbol{\phi}})|\}}{\mathrm{ave}\{|z_t(\hat{\boldsymbol{\phi}})|\}\mathrm{var}\{z_t(\hat{\boldsymbol{\phi}})\}\hat{f}_{z_t(\hat{\phi})}(0)} p \xrightarrow{P} \frac{\mathrm{Var}|Z_1|}{\mathrm{E}|Z_1|\sigma^2 f_\sigma(0)} p$$

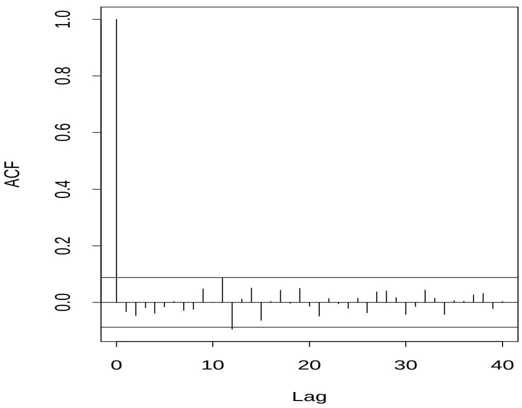
Example: Simulated AP(2)

- $\phi_1 = 0.3$, $\phi_2 = 0.4$, $n = 500$, noise is iid t_3

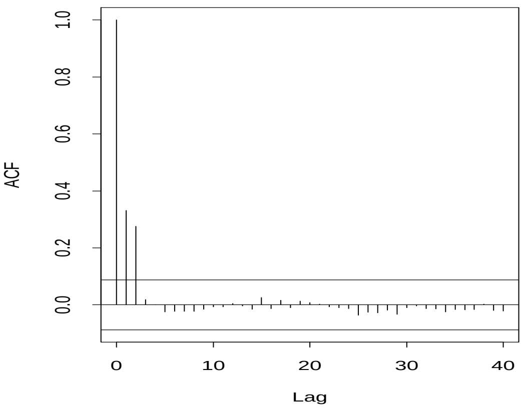
(a) Data From Allpass Model



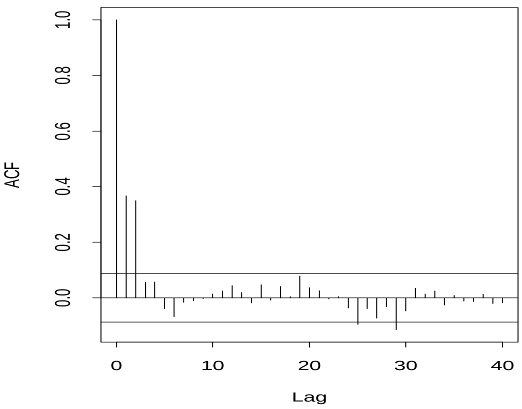
(b) ACF of Allpass Data



(c) ACF of Squares

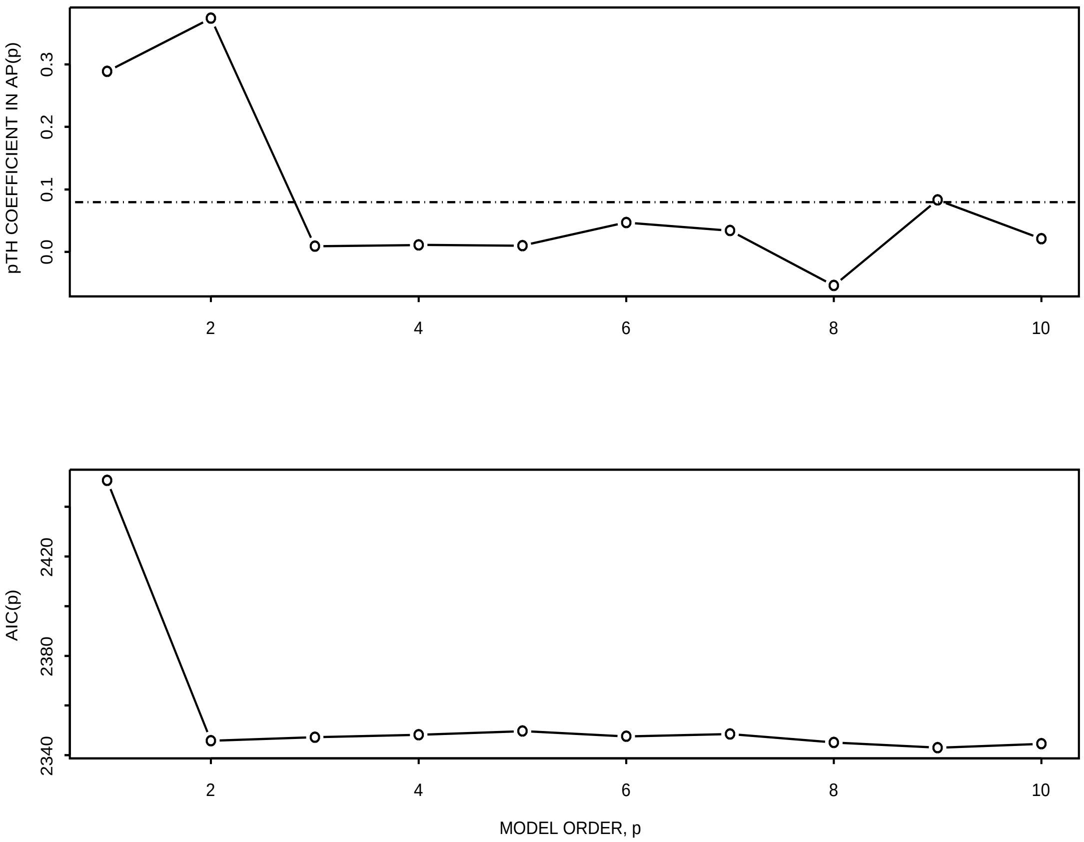


(d) ACF of Absolute Values



Example, Continued: Simulated AP(2)

- Truth: $\phi_1 = 0.3$, $\phi_2 = 0.4$, $n = 500$, noise is iid t_3
- Order selection:



- Estimates: $\hat{\phi}_1 = 0.297\ (0.0381)$; $\hat{\phi}_2 = 0.374\ (0.0381)$

Simulation Results: AP(1)

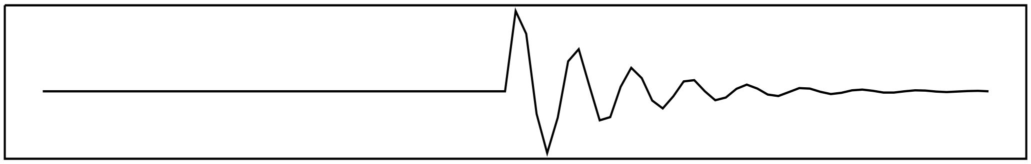
- Noise distribution is t_3; 1000 replications

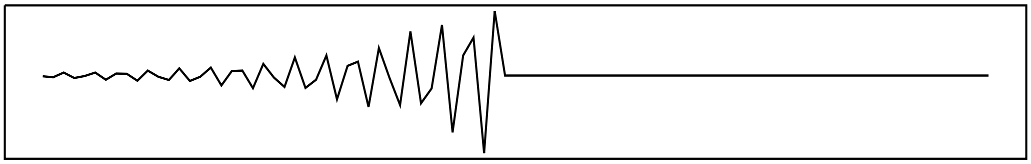
	Asymptotic		Empirical			
n	mean	std.dev.	mean	std.dev.	% coverage	rel. effic.
500	$\phi_{01} = 0.5$	0.0332	0.4979	0.0397	94.2	11.4
5000	$\phi_{01} = 0.5$	0.0105	0.4998	0.0109	95.4	9.3

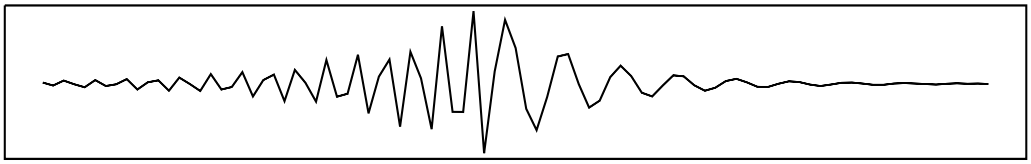
- Efficiency relative to maximum absolute residual kurtosis
- Similar results for AP(2)

Noncausal Autoregressive Processes

- Definitions: AR with polynomial $\phi(\cdot)$ is
  - *stationary* if no roots on unit circle
  - *causal* if no roots inside unit circle
  - *purely noncausal* if all roots inside unit circle
  - *mixed noncausal* if some roots inside unit circle
- Deconvolution problems: seismic, astronomical, speech, image
- Impulse response functions:

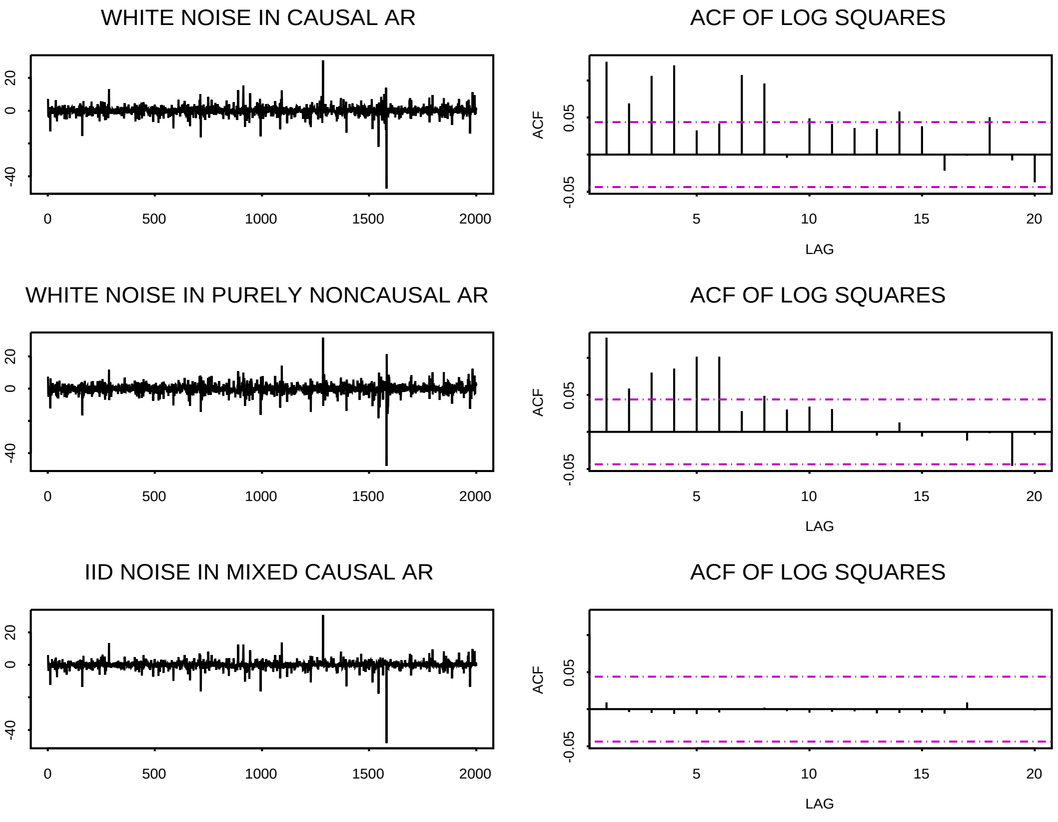

LOW FREQUENCY, CAUSAL


HIGH FREQUENCY, NONCAUSAL


HIGH AND LOW FREQUENCY, MIXED CAUSALITY

Second-Order Equivalent Representations

- $2^{\#\text{ roots}}$ AR's with differing causality
  - different white noise sequences
  - only one is iid in non-Gaussian case



Estimation for Noncausal Autoregressions

- Mixed AR(s) = AR(q)AR(r):

$$\phi(B)X_t = \phi_c(B)\phi_{nc}(B)X_t = Z_t, \quad \{Z_t\} \text{ iid}$$

- Maximum likelihood: Breidt, Davis, Lii, and Rosenblatt (1991)
  - maximize criterion function over all 2^s possible root configurations

- Alternative: note second-order equivalent causal representation:

$$\begin{aligned} U_t &= \phi_c(B)\phi_{nc}^{(c)}(B)X_t \\ &= \phi_c(B)\phi_{nc}^{(c)}(B)\frac{Z_t}{\phi_c(B)\phi_{nc}(B)} \\ &= \frac{\phi_{nc}^{(c)}(B)}{-\phi_{nc,r}B^r\phi_{nc}^{(c)}(B^{-1})}Z_t \end{aligned}$$

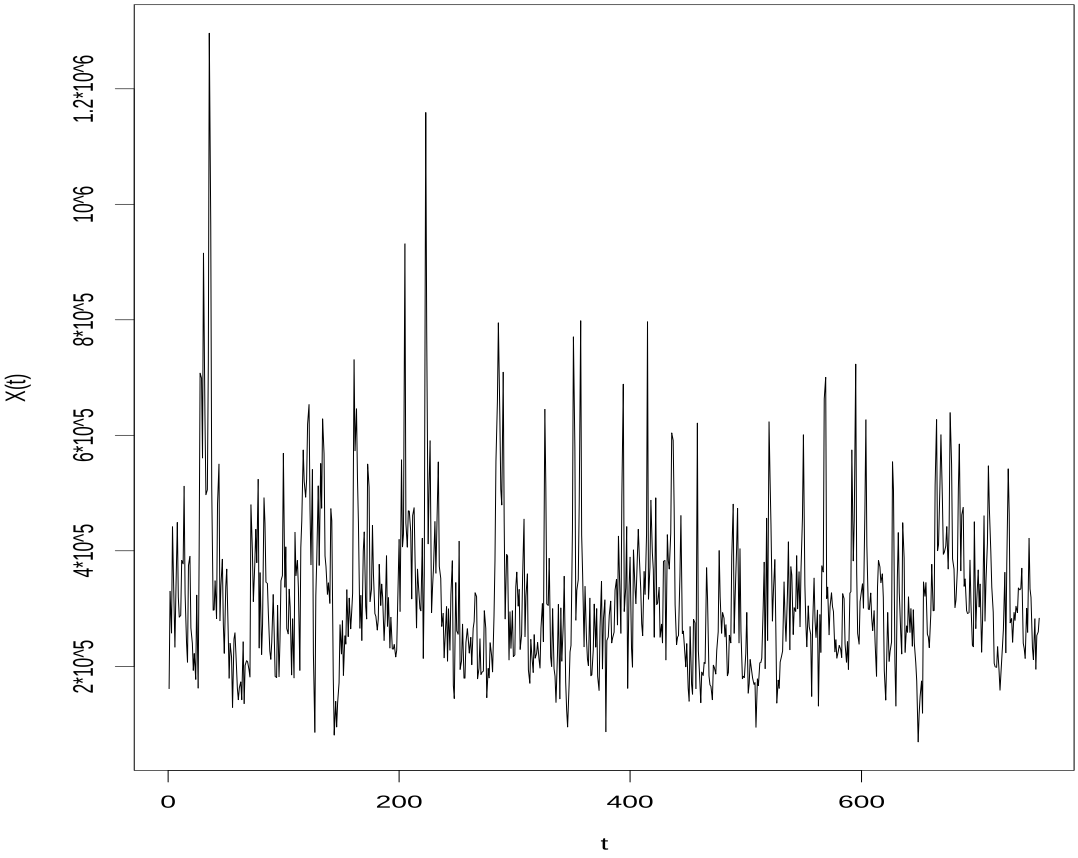
- $\{U_t\}$ is purely noncausal AP(r), hence WN

Two-Step Fitting Procedure

1. Fit causal AR($\hat{s}$):
    - use standard order selection and estimation
    - obtain residuals $\{\hat{U}_t\}$
2. Fit purely noncausal All-Pass to residuals
    - select order $\hat{r}$
    - look for iid noise (not merely white)
    - obtain purely noncausal AR($\hat{r}$), $\phi_{nc}(\cdot)$
    - find inverse roots of AR($\hat{r}$)
    - cancel corresponding roots in causal AR($\hat{s}$) to obtain causal AR($\hat{q}$), $\phi_c(\cdot)$

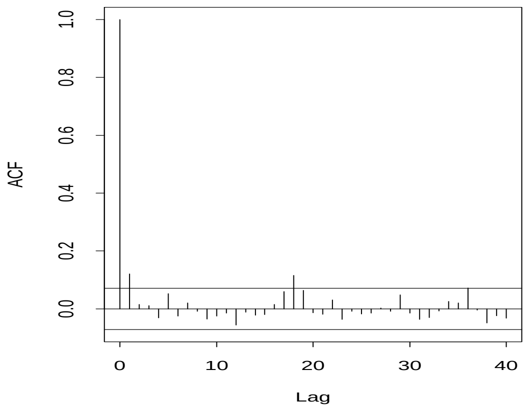
Example: Microsoft Trading Volume

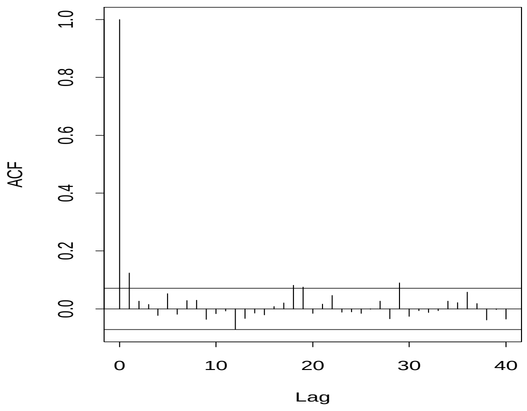
- Volumes of MSFT stock traded over 754 transaction days from 06/03/96 to 05/27/99

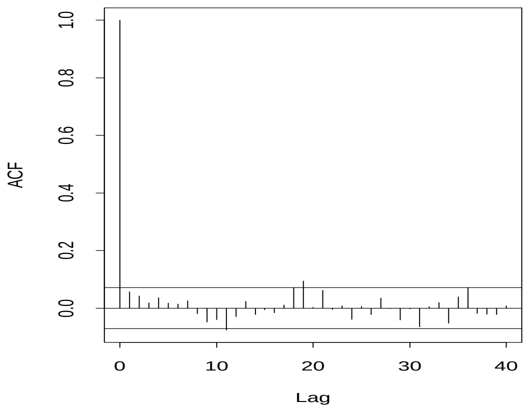


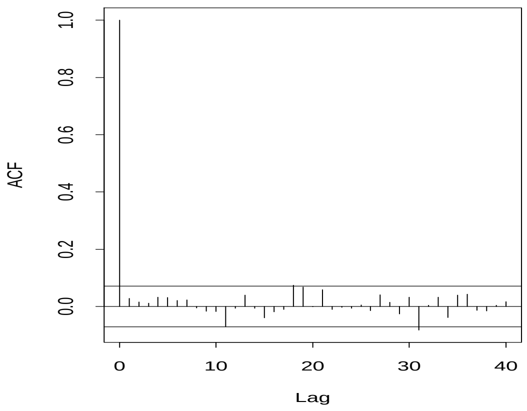
Microsoft Trading Volume

- Two-step fit of noncausal AR(1): $1 - 1.7522B$
  - causal AR(1); residuals not iid
  - purely noncausal AP(1); residuals iid
- Direct fit of noncausal AR(1): $1 - 1.7141B$
- For ATML and MCHP, causal AR models fit


(a) ACF of Squares of Ut


(b) ACF of Absolute Values of Ut


(c) ACF of Squares of Zt


(d) ACF of Absolute Values of Zt

Summary

- All-pass models and their properties
  - linear time series with “nonlinear” behavior
- Estimation
  - likelihood approximation
  - least absolute deviations
  - asymptotics and order selection
  - simulation study
- Noncausal autoregressive processes
  - two-step estimation procedure using all-pass
  - noncausal AR(1) for Microsoft trading volume
- Further work
  - MLE, other M-estimation procedures
  - heavy-tailed case
  - comparisons with cumulant-based methods