

Spatial Lasso with Application to GIS Model Selection

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Outline

- Regression for sparse spatial sample
 - layers and neighborhoods for GIS data
 - linear model formulation with many parameters
- Lasso
 - shrinkage and model selection
 - numerical experiment I
- Spatial Lasso
 - modifications for spatial smoothness
 - numerical experiment II
- Application: Prediction of soil moisture index

Responses and Covariates

- Data $\{Y(\mathbf{s}_i) : i = 1, \dots, n\}$
 - observed at spatial locations $\mathbf{s}_i \in D \subset \mathbb{Z}^2$
 - D is a regular grid
- Covariate layers $\{x_k(\mathbf{s}) : \mathbf{s} \in D\}; k = 1, \dots, p$
- **Goal:** find an appropriate model for $Y(\mathbf{s})$ as a linear function of $\{x_k(\mathbf{s}) : \mathbf{s} \in D\}; k = 1, \dots, p$

Model Formulation

- Each layer has an associated neighborhood: $\mathcal{N}_k \subset \mathbb{Z}^2$ is a neighborhood set of $\mathbf{0}$

$$Y(\mathbf{s}_i) = \sum_{j=1}^J a_j \phi_j(\mathbf{s}_i) + \sum_{k=1}^p \sum_{\mathbf{u} \in \mathcal{N}_k} b_k(\mathbf{s}_i, \mathbf{u}) x_k(\mathbf{s}_i + \mathbf{u}) + \varepsilon(\mathbf{s}_i)$$

- $\phi_1(\cdot), \dots, \phi_J(\cdot)$ are known functions
- $\{\varepsilon(\mathbf{s}_1), \dots, \varepsilon(\mathbf{s}_n)\}$ are iid $N(0, \sigma^2)$

Spatial Homogeneity

- For each layer k and pixel \mathbf{u} , expect some **spatial homogeneity** in $b_k(\cdot, \mathbf{u})$:

$$b_k(\cdot, \mathbf{u}) = \sum_{l=1}^{L_k} c_{k,l}(\mathbf{u}) \psi_{k,l}(\cdot); \quad k = 1, \dots, p, \mathbf{u} \in \mathcal{N}_k,$$

where $\psi_{k,l}(\cdot)$'s are known functions possibly depending on some covariates.

- can adjust for spatial features like flow directions
- take $L_k \equiv 1, \psi_{k,1} \equiv 1$ in this discussion

$$b_k(\cdot, \mathbf{u}) = c_{k,1}(\mathbf{u})$$

Linear Model and Least Squares Estimation

- Rewrite as linear model:

$$\mathbf{Y} \equiv (Y(\mathbf{s}_1), \dots, Y(\mathbf{s}_n))' = \mathbf{X}\boldsymbol{\beta} + \boldsymbol{\epsilon},$$

- Ordinary least squares estimates?

- large number of parameters: $J + \sum_{k=1}^p |\mathcal{N}_k|$
- OLS has low bias, large variance: **shrinkage**
- OLS difficult to interpret: **model selection**

Lasso

- **Lasso** (Tibshirani, 1996): least absolute shrinkage and selection operator
 - standardize \mathbf{X} as \mathbf{X}^*
- Minimize RSS subject to L_1 constraint:
$$(\mathbf{Y} - \mathbf{X}^* \boldsymbol{\beta}^*)' (\mathbf{Y} - \mathbf{X}^* \boldsymbol{\beta}^*),$$
subject to $\sum |\beta_j^*| \leq t$, a tuning parameter
- **Key feature:** some estimated β_j^* can be exactly zero

Lasso, Continued

- Equivalently, minimize

$$(\mathbf{Y} - \mathbf{X}^* \boldsymbol{\beta}^*)'(\mathbf{Y} - \mathbf{X}^* \boldsymbol{\beta}^*) + \lambda \sum_{j=1}^m |\beta_j^*|,$$

- if priors are independent Laplace, $\{\hat{\beta}_j^*\}$ are posterior modes
- **Least angle regression** (LARS, Efron et al., 2004) provides fast algorithm for Lasso
 - same computational order as OLS applied to full set of covariates
 - lars package in R (<http://cran.r-project.org>)

Numerical Experiment I: Mean Function

- Consider six covariate layers
 - simulated as Gaussian random fields
 - exponential covariance, strong dependence

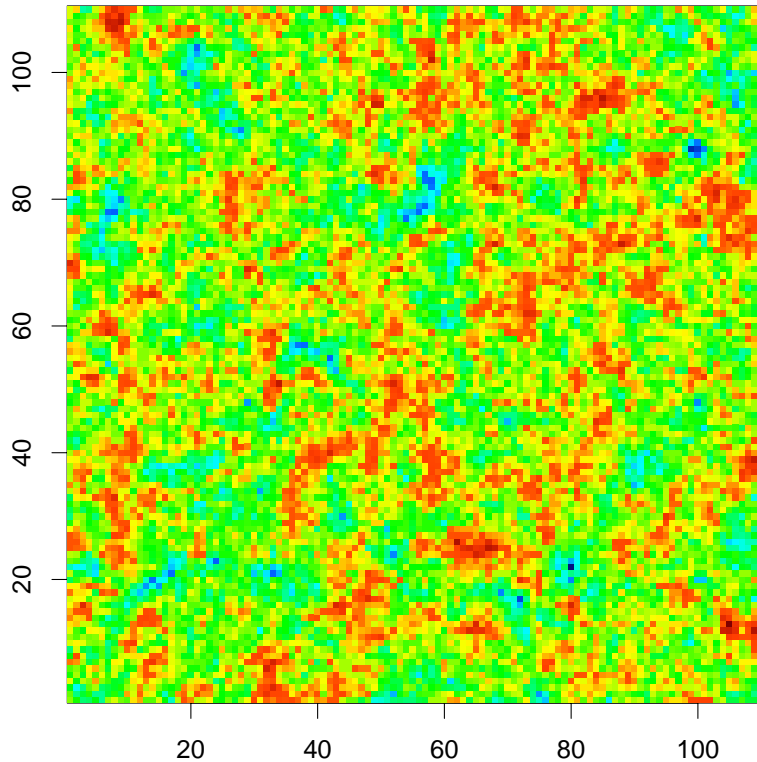
- Construct true mean function:

$$\mu(\mathbf{s}) = 1 + \frac{1}{3} \left\{ x_1(\mathbf{s}) + \sum_{j=-1}^1 x_1(\mathbf{s} + (j, 1)) + \sum_{j=-2}^2 x_1(\mathbf{s} + (j, 2)) \right\} + \frac{1}{3} \sum_{j=-1}^1 \sum_{k=-1}^1 x_2(\mathbf{s} + (j, k))$$

- True neighborhoods:
 - x_1 : inverted 5-3-1 pyramid
 - x_2 : centered 3×3 block
 - x_3, x_4, x_5, x_6 : empty

Simulated Covariate Layer

- Strong spatial dependence in $x_1(\cdot)$



Estimation and Prediction

- Sample 100 sites from 100×100 spatial domain
 - observed response = mean function plus noise
- Estimate and predict using OLS or Lasso
 - OLS models with correct layers:

$$Y(\mathbf{s}_i) = \beta_0 + \sum_{l=1}^2 \beta_l x_l(\mathbf{s}_i) + \varepsilon(\mathbf{s}_i); \quad i = 1, \dots, n,$$

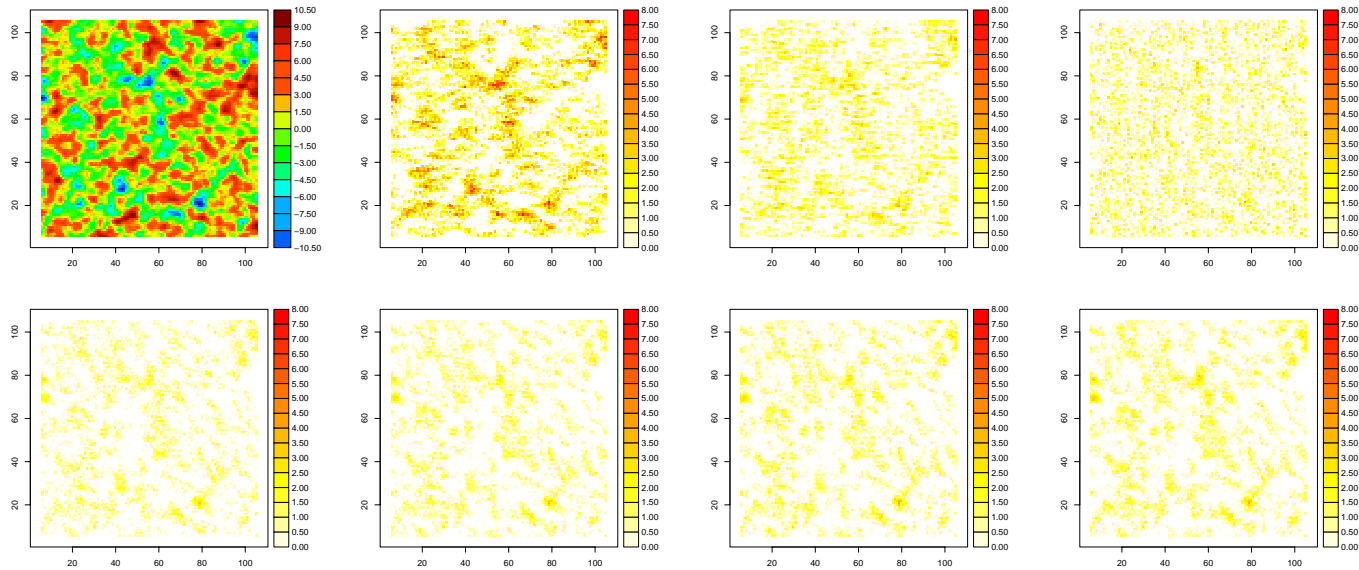
$$Y(\mathbf{s}_i) = \beta_0 + \sum_{l=1}^2 \sum_{j=-1}^1 \sum_{k=-1}^1 \beta_{l,j,k} x_l(\mathbf{s}_i + (j, k)) + \varepsilon(\mathbf{s}_i); \quad i = 1, \dots, n,$$

$$Y(\mathbf{s}_i) = \beta_0 + \sum_{l=1}^2 \sum_{j=-2}^2 \sum_{k=-2}^2 \beta_{l,j,k} x_l(\mathbf{s}_i + (j, k)) + \varepsilon(\mathbf{s}_i); \quad i = 1, \dots, n,$$

- Lasso neighborhoods, $\mathcal{N}^{(2q+1)}$: $(2q + 1) \times (2q + 1)$ blocks

True Mean Function and Absolute Prediction Errors

- Qualitatively similar for weak dependence



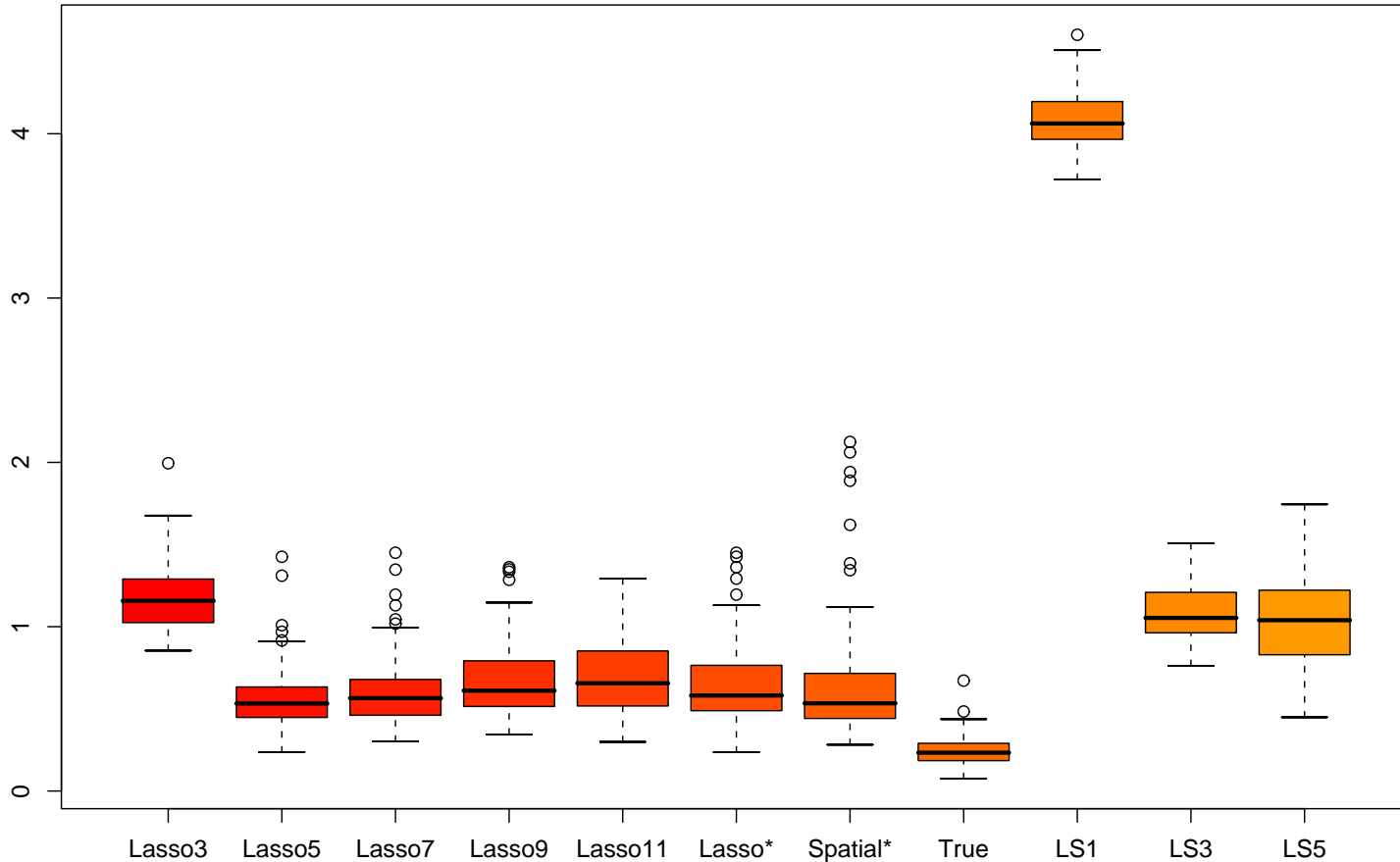
Average Squared Error

- ASE over 100×100 spatial domain:

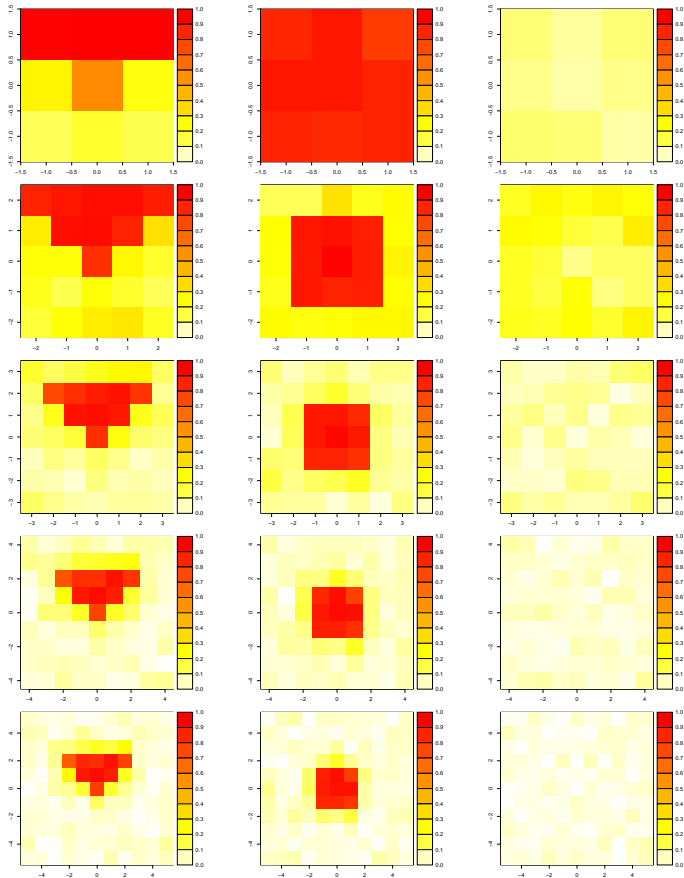
$$ASE = \frac{1}{(100)^2} \sum_{\mathbf{s} \in D} (\hat{\mu}(\mathbf{s}) - \mu(\mathbf{s}))^2,$$

- Conduct 100 simulation replicates
 - get 100 ASE's for LS1, LS3, LS5, Lasso3, ..., Lasso11
 - produce boxplots of ASE's

Average Squared Error Under Strong Dependence



Estimated Neighborhoods Using Lasso (Strong)



Spatial Smoothness

- Above model assumes **spatial homogeneity** of regression coefficients *across* neighborhoods
- No assumption of **spatial smoothness** of regression coefficients *within* neighborhoods
 - in many applications, reasonable to assume smoothness of $\{c_{k,l}(\mathbf{s}) : \mathbf{s} \in \mathcal{N}_k\}$
- Ordinary Lasso does not account for spatial smoothness

Spatial Lasso

- Allow for smoothness of coefficients
- Assume spatial dependence prior for β :
 - Γ is prior correlation matrix of β
 - $\beta^{**} \equiv \Gamma^{-1/2}\beta^* \sim$ independent Laplace
- **Spatial Lasso** obtained by minimizing

$$(\mathbf{Y} - \mathbf{X}^{**}\beta^{**})'(\mathbf{Y} - \mathbf{X}^{**}\beta^{**}) + \lambda \sum_{j=1}^m |\beta_j^{**}|,$$

where $\mathbf{X}^{**} \equiv \mathbf{X}^*\Gamma^{1/2}$

- computation via modification of LARS: **equi-projection regression**

Numerical Experiment II

- Mean function: for $x(\mathbf{s})$ iid $\mathbf{N}(0, 1)$

$$\mu(\mathbf{s}) = 1 + 5 \sum_{j=-2}^2 \sum_{k=-2}^2 w_{j,k} x(\mathbf{s} + (j, k)); \quad \mathbf{s} \in D,$$

and Gaussian weight function

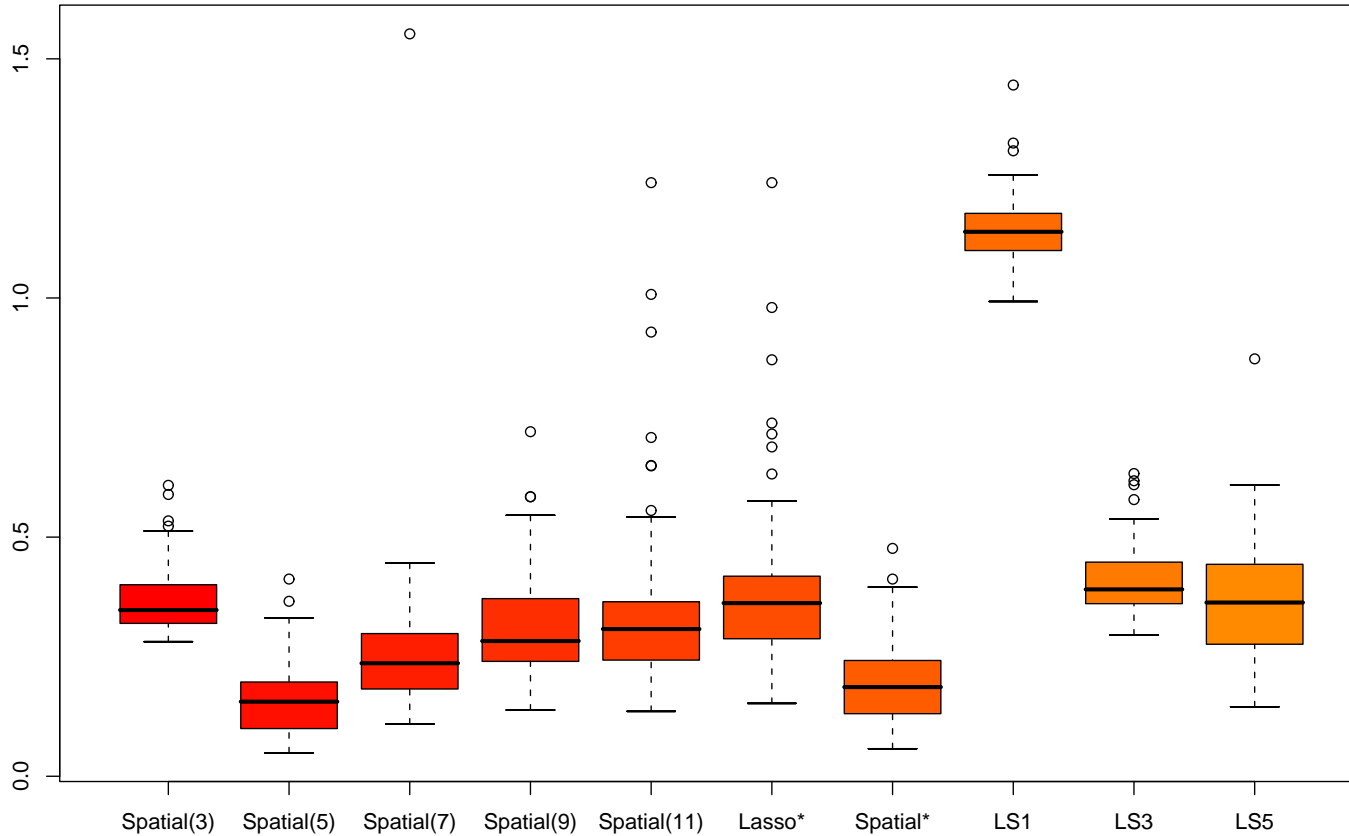
$$w_{j,k} \equiv \frac{\exp(- (j^2 + k^2)/4)}{\sum_{j=-2}^2 \sum_{k=-2}^2 \exp(- (j^2 + k^2)/4)}$$

- Sample $n = 100$ sites and generate $\{Y(\mathbf{s}_1), \dots, Y(\mathbf{s}_n)\}$ by adding noise

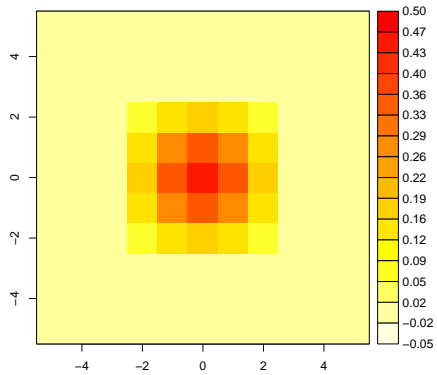
Spatial Lasso for Numerical Experiment II

- Compare OLS, Lasso, Spatial Lasso
- Apply Spatial Lasso with exponential covariance structure
- Choose smoothness parameter and neighborhood size via grid search:
 - neighborhood sets $\{\mathcal{N}^{(2q+1)} : q = 1, \dots, 5\}$
 - smoothness parameters $\gamma = 0, 1, \dots, 5$
 - choose combination with best ten-fold cross-validation

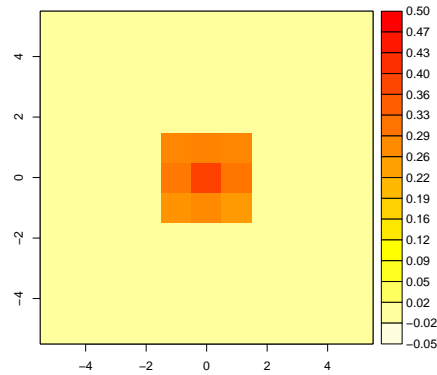
Average Squared Error for Experiment II



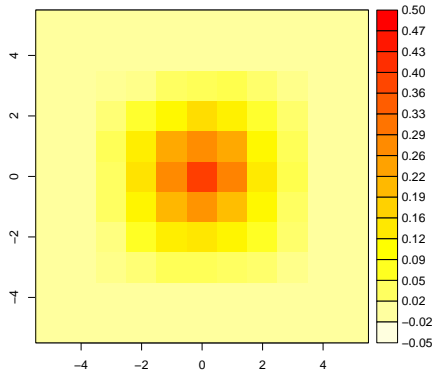
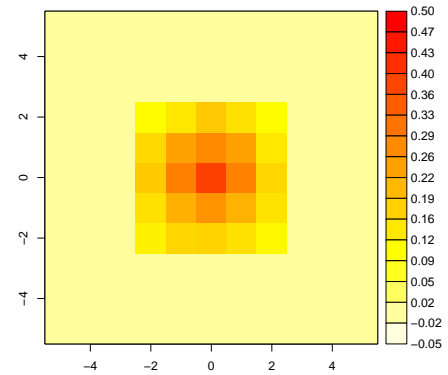
Estimated Coefficients Using Spatial Lasso



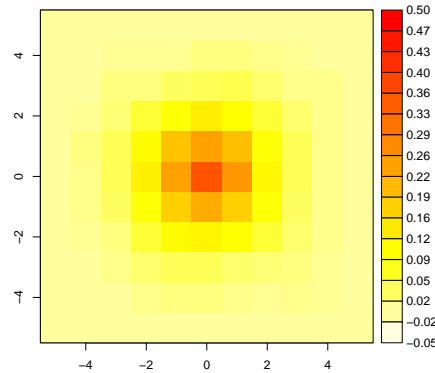
(b)



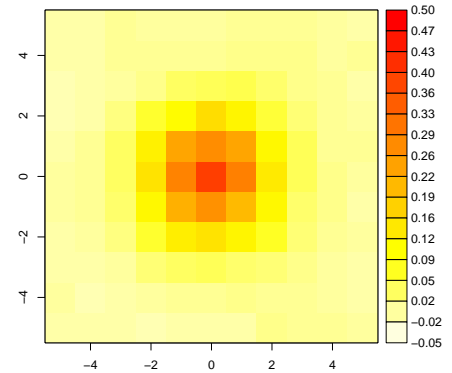
(c)



(d)



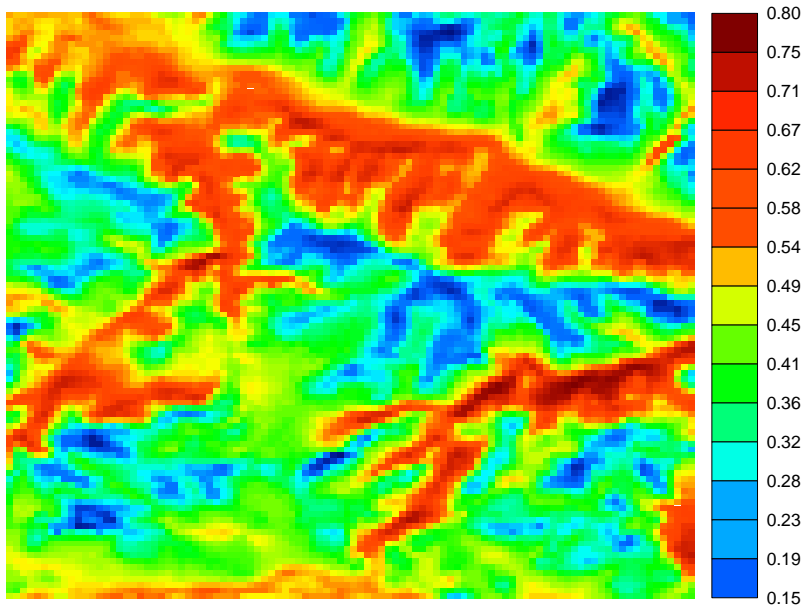
(e)



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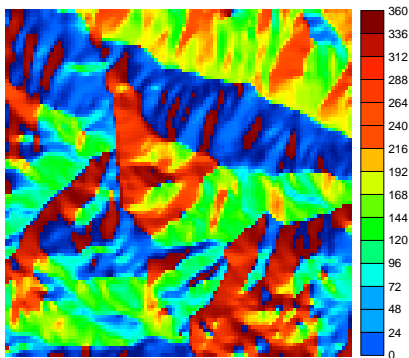
Application: Prediction of Soil Moisture Index

- Select $n = 100$ sites via simple random sampling without replacement

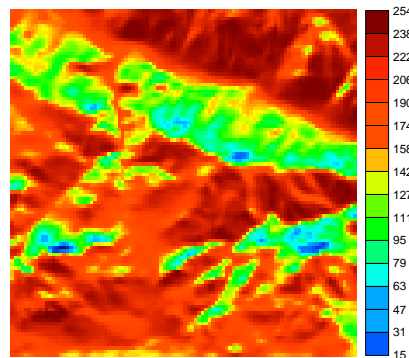


Covariates for Soil Moisture Prediction

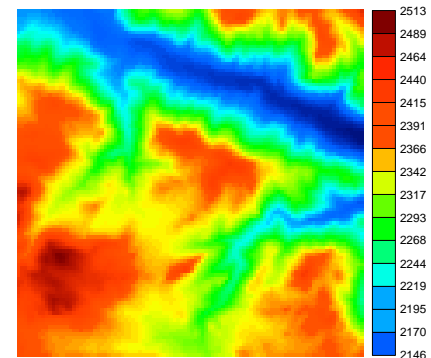
- Aspect, hill shade, elevation, slope, precipitation



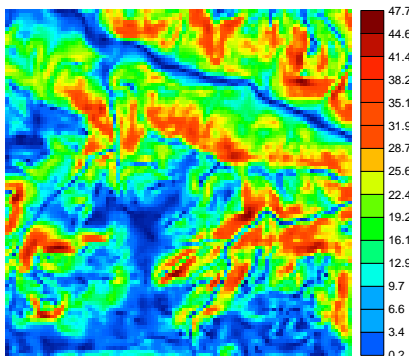
(a)



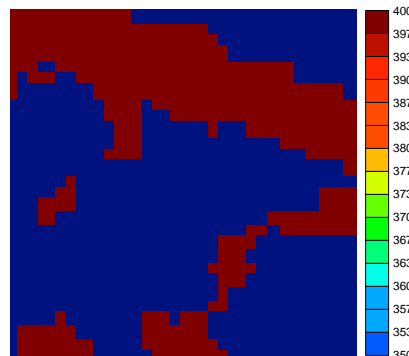
(b)



(c)



(d)



(e)

Covariates for Soil Moisture Prediction

- Response: soil moisture index
- Basic covariate layers:
 - aspect, hill shade, elevation, slope, precipitation
 - 100×100 regular grid
- Expanded covariate layers (14 total):
 - average elevation, slope, precipitation on 3×3 blocks
 - average elevation, slope, precipitation on 9×9 blocks
 - aspect**elevation*, aspect**slope*, aspect**precipitation*

Neighborhoods for Soil Moisture Prediction

- Neighborhoods:

- use single pixel for elevation, 3×3 elevation, 9×9 elevation
- same for slope and precipitation
- (up to 9×9 neighborhood, but highly constrained parameters)
- for remaining variables, use

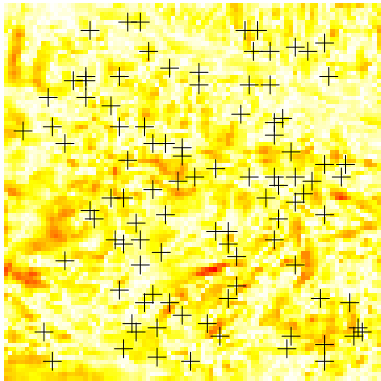
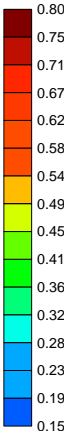
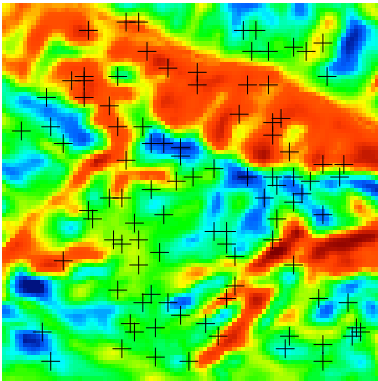
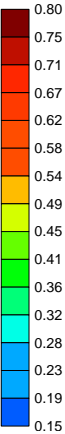
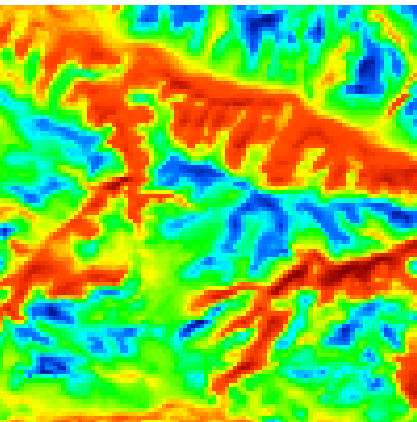
$$\mathcal{N}^{(2q+1)} = \{0, \pm 1, \pm 2, \dots, \pm(2q + 1)\}^2$$

with $q = 0, 1, \dots, 5$

Spatial Lasso for Soil Moisture Prediction

- Apply spatial Lasso with exponential covariance structure
 - grid search on $\gamma = 0, 1, \dots, 5$, crossed with neighborhoods above
 - 36 possible combinations
- Smallest ten-fold cross-validation value at $\gamma = 2$ and $q = 5$

True and Predicted Soil Moisture Index



Summary

- Lasso approach to GIS model selection and estimation
 - simultaneous layer selection, neighborhood selection, and estimation
 - dominates OLS for unknown layers/neighborhoods
 - extensible to neighborhood transformation
- Spatial Lasso
 - accounts for spatial smoothness within neighborhoods
 - dominates Lasso if smoothness is present
- Promising results in prediction of soil moisture index