Spatial Lasso with Application to GIS Model Selection

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Outline

- Regression for sparse spatial sample
  - layers and neighborhoods for GIS data
  - linear model formulation with many parameters
- Lasso
  - shrinkage and model selection
  - numerical experiment I
- Spatial Lasso
  - modifications for spatial smoothness
  - numerical experiment II
- Application: Prediction of soil moisture index

Responses and Covariates

- Data $\{Y(\boldsymbol{s}_i) : i = 1, \ldots, n\}$
  - observed at spatial locations $\boldsymbol{s}_i \in D \subset \mathbb{Z}^2$
  - D is a regular grid
- Covariate layers $\{x_k(\boldsymbol{s}) : \boldsymbol{s} \in D\}$; $k = 1, \ldots, p$
- **Goal:** find an appropriate model for $Y(\boldsymbol{s})$ as a linear function of $\{x_k(\boldsymbol{s}) : \boldsymbol{s} \in D\}$; $k = 1, \ldots, p$

Model Formulation

- Each layer has an associated neighborhood: $\mathcal{N}_k \subset \mathbb{Z}^2$ is a neighborhood set of $\mathbf{0}$

$$
\begin{aligned}
Y(\boldsymbol{s}_i) = & \sum_{j=1}^{J} a_j \phi_j(\boldsymbol{s}_i) \\
& + \sum_{k=1}^{p} \sum_{\boldsymbol{u} \in \mathcal{N}_k} b_k(\boldsymbol{s}_i, \boldsymbol{u}) x_k(\boldsymbol{s}_i + \boldsymbol{u}) + \varepsilon(\boldsymbol{s}_i)
\end{aligned}
$$

  - $\phi_1(\cdot), \ldots, \phi_J(\cdot)$ are known functions
  - $\{\varepsilon(\boldsymbol{s}_1), \ldots, \varepsilon(\boldsymbol{s}_n)\}$ are iid $\mathrm{N}(0, \sigma^2)$

Spatial Homogeneity

- For each layer k and pixel $\boldsymbol{u}$, expect some spatial homogeneity in $b_k(\cdot, \boldsymbol{u})$:

$$b_k(\cdot, \boldsymbol{u}) = \sum_{l=1}^{L_k} c_{k,l}(\boldsymbol{u}) \psi_{k,l}(\cdot); \quad k = 1, \ldots, p, \ \boldsymbol{u} \in \mathcal{N}_k \ ,$$

  where $\psi_{k,l}(\cdot)$'s are known functions possibly depending on some covariates.

  - can adjust for spatial features like flow directions
  - take $L_k \equiv 1$, $\psi_{k,1} \equiv 1$ in this discussion

$$b_k(\cdot, \boldsymbol{u}) = c_{k,1}(\boldsymbol{u})$$

Linear Model and Least Squares Estimation

- Rewrite as linear model:

$$\boldsymbol{Y} \equiv (Y(\boldsymbol{s}_1), \ldots, Y(\boldsymbol{s}_n))' = \boldsymbol{X}\boldsymbol{\beta} + \boldsymbol{\epsilon} \,,$$

- Ordinary least squares estimates?
  - large number of parameters: $J + \sum_{k=1}^{p} |\mathcal{N}_k|$
  - OLS has low bias, large variance: **shrinkage**
  - OLS difficult to interpret: **model selection**

Lasso

- **Lasso** (Tibshirani, 1996): least absolute shrinkage and selection operator

  – standardize $\boldsymbol{X}$ as $\boldsymbol{X}^*$

- Minimize RSS subject to L_1 constraint:

$$(\boldsymbol{Y} - \boldsymbol{X}^*\boldsymbol{\beta}^*)'(\boldsymbol{Y} - \boldsymbol{X}^*\boldsymbol{\beta}^*),$$

  subject to $\sum |\beta_j^*| \leq t$, a tuning parameter

- **Key feature:** some estimated β_j^* can be exactly zero

Lasso, Continued

- Equivalently, minimize

$$(\boldsymbol{Y}-\boldsymbol{X}^{*}\boldsymbol{\beta}^{*})'(\boldsymbol{Y}-\boldsymbol{X}^{*}\boldsymbol{\beta}^{*})+\lambda\sum_{j=1}^{m}|\beta_j^{*}|,$$

  - if priors are independent Laplace, $\{\hat{\beta}_j^{*}\}$ are posterior modes
- **Least angle regression** (LARS, Efron et al., 2004) provides fast algorithm for Lasso
  - same computational order as OLS applied to full set of covariates
  - `lars` package in R (`http://cran.r-project.org`)

Numerical Experiment I: Mean Function

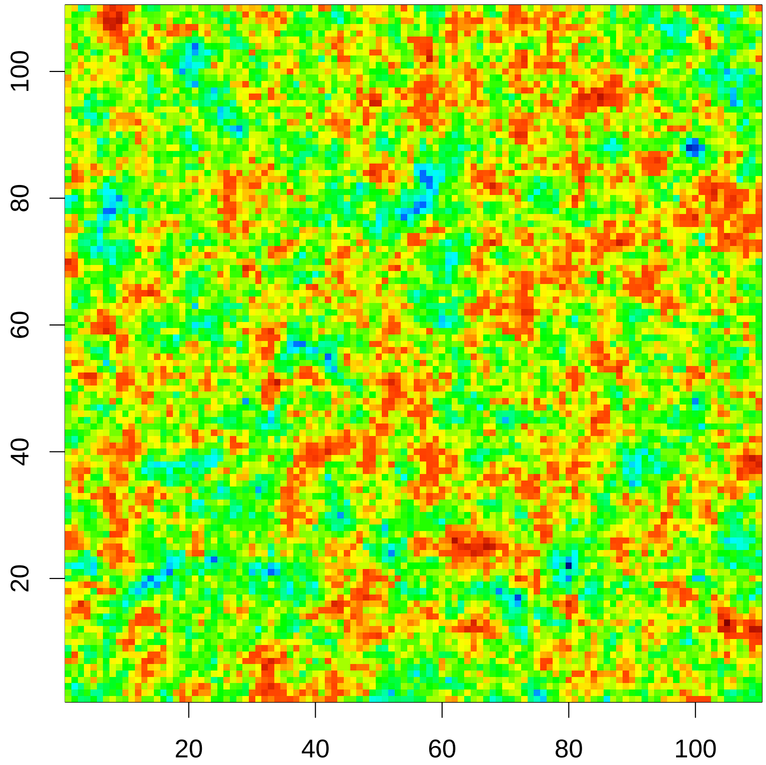
- Consider six covariate layers
  - simulated as Gaussian random fields
  - exponential covariance, strong dependence
- Construct true mean function:

$$\mu(\boldsymbol{s}) = 1 + \frac{1}{3}\left\{x_1(\boldsymbol{s}) + \sum_{j=-1}^{1} x_1(\boldsymbol{s} + (j,1)) + \sum_{j=-2}^{2} x_1(\boldsymbol{s} + (j,2))\right\} + \frac{1}{3}\sum_{j=-1}^{1}\sum_{k=-1}^{1} x_2(\boldsymbol{s} + (j,k))$$

- True neighborhoods:
  - x_1: inverted 5-3-1 pyramid
  - x_2: centered 3×3 block
  - x_3, x_4, x_5, x_6: empty

Simulated Covariate Layer

- Strong spatial dependence in $x_1(\cdot)$



Estimation and Prediction

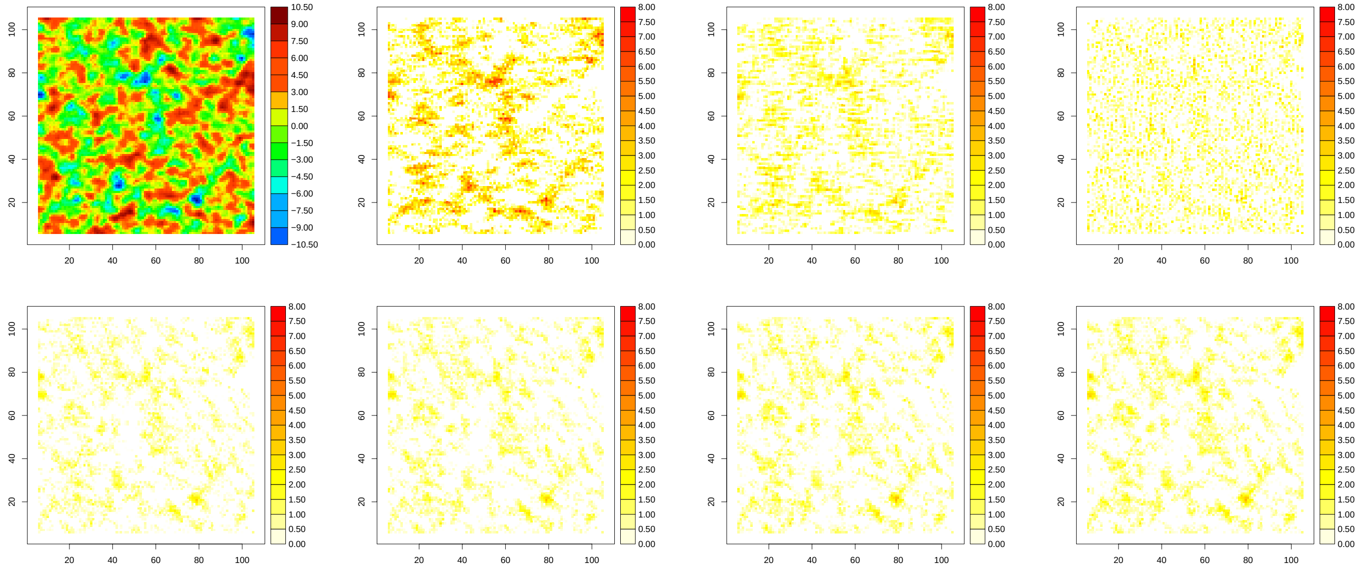
- Sample 100 sites from 100×100 spatial domain
  - observed response = mean function plus noise
- Estimate and predict using OLS or Lasso
  - OLS models with correct layers:

$$
\begin{aligned}
Y(\boldsymbol{s}_i) &= \beta_0 + \sum_{l=1}^{2} \beta_l x_l(\boldsymbol{s}_i) + \varepsilon(\boldsymbol{s}_i); \quad i = 1, \ldots, n, \\
Y(\boldsymbol{s}_i) &= \beta_0 + \sum_{l=1}^{2} \sum_{j=-1}^{1} \sum_{k=-1}^{1} \beta_{l,j,k} x_l(\boldsymbol{s}_i + (j,k)) + \varepsilon(\boldsymbol{s}_i); \quad i = 1, \ldots, n, \\
Y(\boldsymbol{s}_i) &= \beta_0 + \sum_{l=1}^{2} \sum_{j=-2}^{2} \sum_{k=-2}^{2} \beta_{l,j,k} x_l(\boldsymbol{s}_i + (j,k)) + \varepsilon(\boldsymbol{s}_i); \quad i = 1, \ldots, n,
\end{aligned}
$$

  - Lasso neighborhoods, $\mathcal{N}^{(2q+1)}$: $(2q+1) \times (2q+1)$ blocks

True Mean Function and Absolute Prediction Errors

- Qualitatively similar for weak dependence



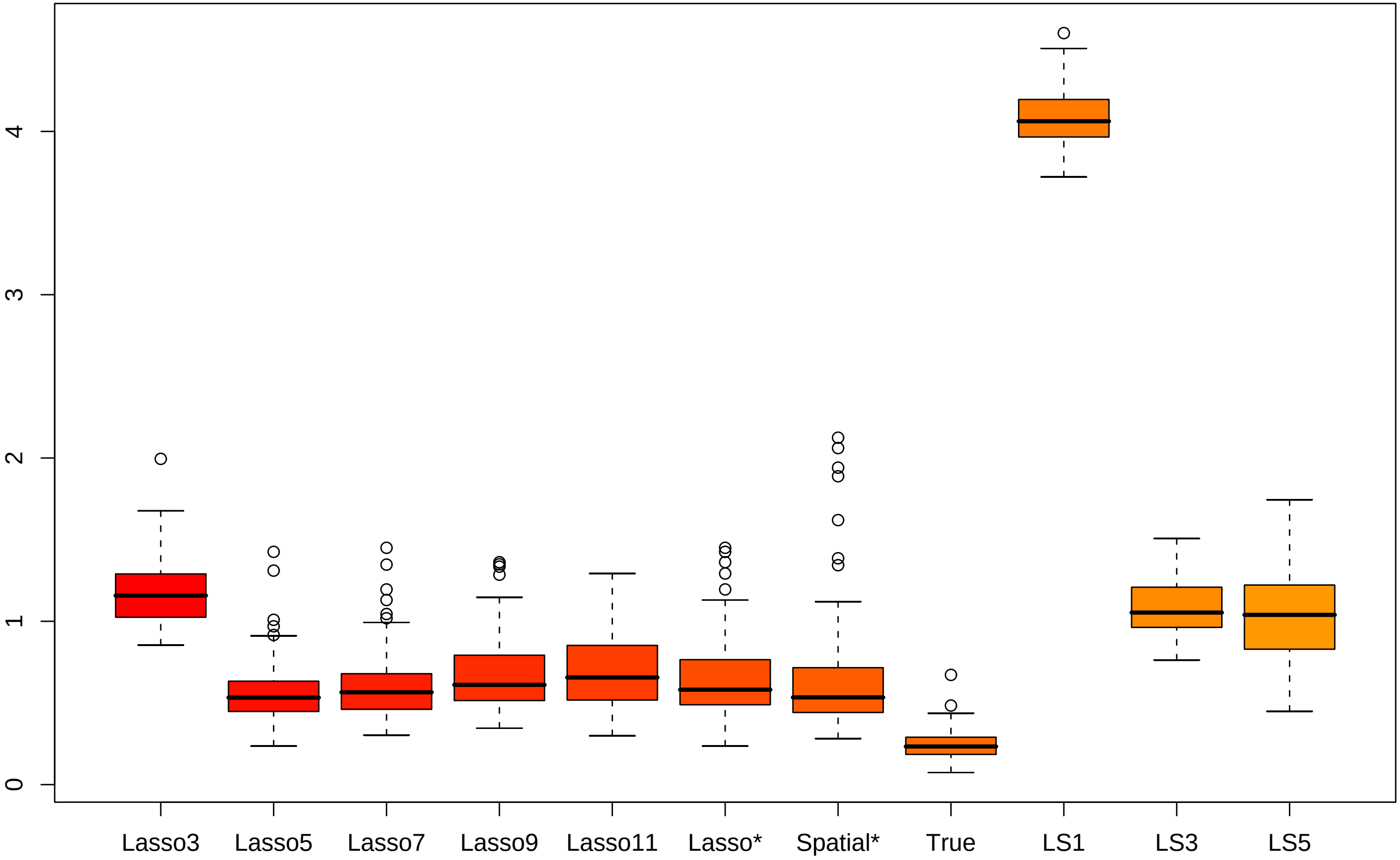
Average Squared Error

- ASE over 100×100 spatial domain:

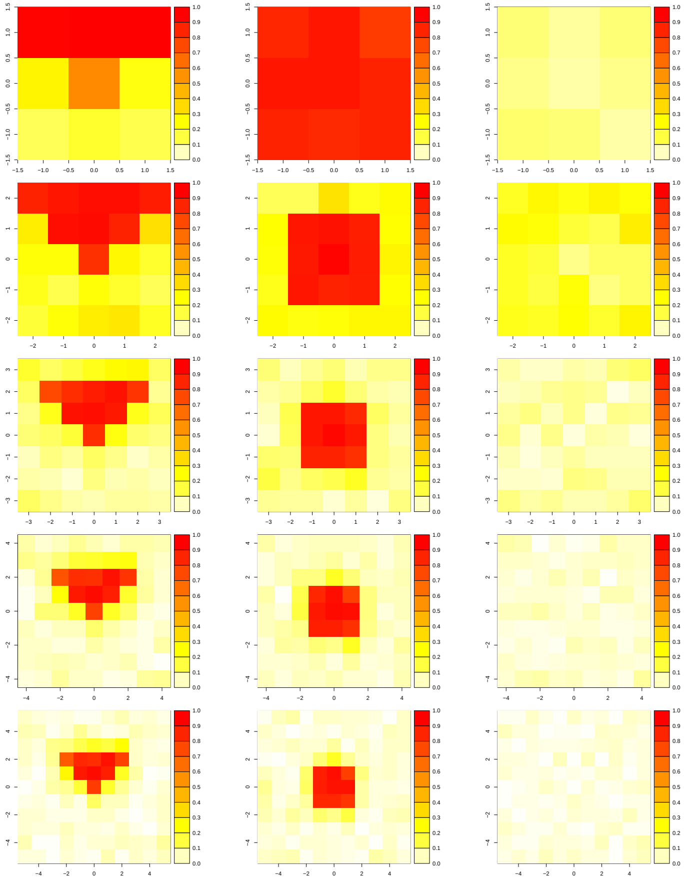
$$ASE = \frac{1}{(100)^2} \sum_{\boldsymbol{s} \in D} (\hat{\mu}(\boldsymbol{s}) - \mu(\boldsymbol{s}))^2,$$

- Conduct 100 simulation replicates
  - get 100 ASE's for LS1, LS3, LS5, Lasso3, . . . , Lasso11
  - produce boxplots of ASE's

Average Squared Error Under Strong Dependence



Estimated Neighborhoods Using Lasso (Strong)



Spatial Smoothness

- Above model assumes **spatial homogeneity** of regression coefficients *across* neighborhoods
- No assumption of **spatial smoothness** of regression coefficients *within* neighborhoods
  - in many applications, reasonable to assume smoothness of $\{c_{k,l}(\boldsymbol{s}) : \boldsymbol{s} \in \mathcal{N}_k\}$
- Ordinary Lasso does not account for spatial smoothness

Spatial Lasso

- Allow for smoothness of coefficients
- Assume spatial dependence prior for $\boldsymbol{\beta}$:
  - $\boldsymbol{\Gamma}$ is prior correlation matrix of $\boldsymbol{\beta}$
  - $\boldsymbol{\beta}^{**} \equiv \boldsymbol{\Gamma}^{-1/2}\boldsymbol{\beta^*} \sim$ independent Laplace
- **Spatial Lasso** obtained by minimizing

$$(\boldsymbol{Y} - \boldsymbol{X}^{**}\boldsymbol{\beta}^{**})'(\boldsymbol{Y} - \boldsymbol{X}^{**}\boldsymbol{\beta}^{**}) + \lambda \sum_{j=1}^{m} |\beta_j^{**}|,$$

where $\boldsymbol{X}^{**} \equiv \boldsymbol{X}^{*}\boldsymbol{\Gamma}^{1/2}$

  - computation via modification of LARS: **equi-projection regression**

Numerical Experiment II

- Mean function: for $x(\boldsymbol{s})$ iid $\mathsf{N}(0,1)$

$$\mu(\boldsymbol{s}) = 1 + 5\sum_{j=-2}^{2}\sum_{k=-2}^{2} w_{j,k}\, x(\boldsymbol{s} + (j,k)); \quad \boldsymbol{s} \in D,$$

  and Gaussian weight function

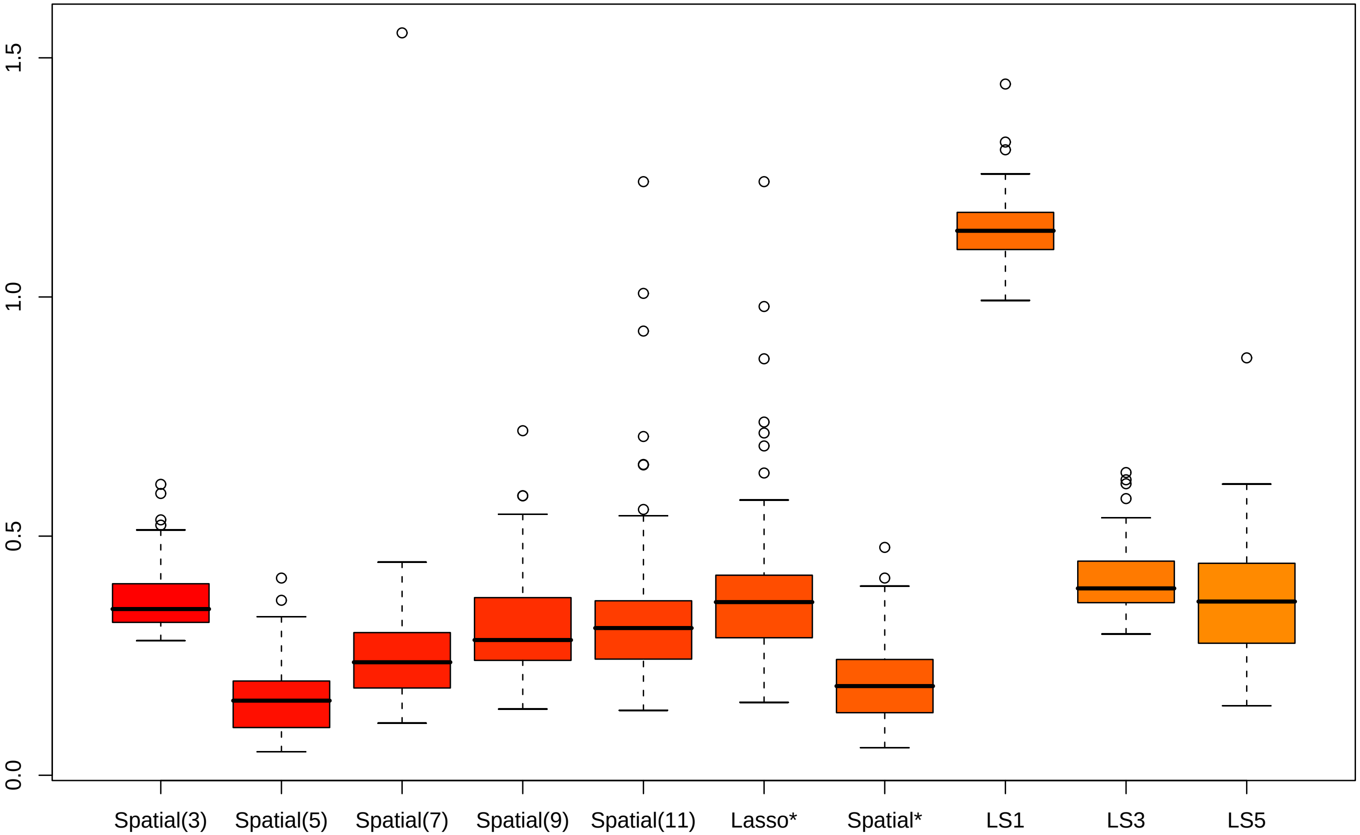
$$w_{j,k} \equiv \frac{\exp\left(-(j^2+k^2)/4\right)}{\sum_{j=-2}^{2}\sum_{k=-2}^{2}\exp\left(-(j^2+k^2)/4\right)}$$

- Sample $n = 100$ sites and generate $\{Y(\boldsymbol{s}_1), \ldots, Y(\boldsymbol{s}_n)\}$ by adding noise

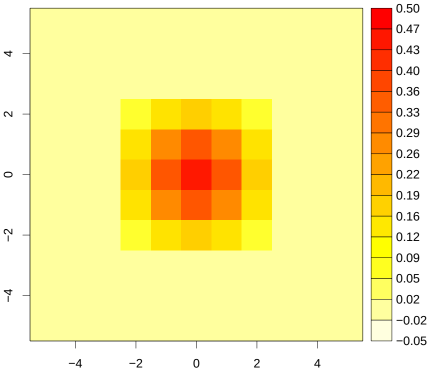
Spatial Lasso for Numerical Experiment II

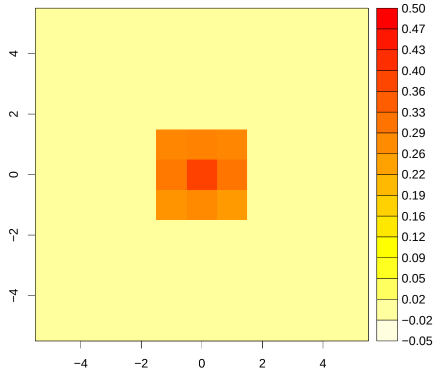
- Compare OLS, Lasso, Spatial Lasso
- Apply Spatial Lasso with exponential covariance structure
- Choose smoothness parameter and neighborhood size via grid search:
  - neighborhood sets $\{\mathcal{N}^{(2q+1)} : q = 1, \ldots, 5\}$
  - smoothness parameters $\gamma = 0, 1, \ldots, 5$
  - choose combination with best ten-fold cross-validation

Average Squared Error for Experiment II



Estimated Coefficients Using Spatial Lasso





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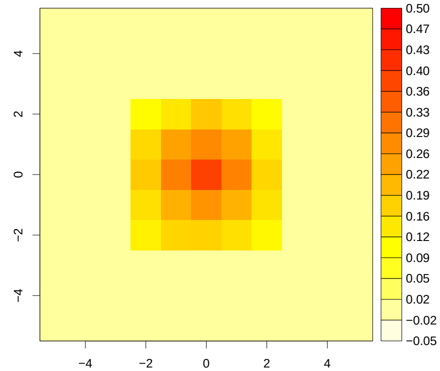

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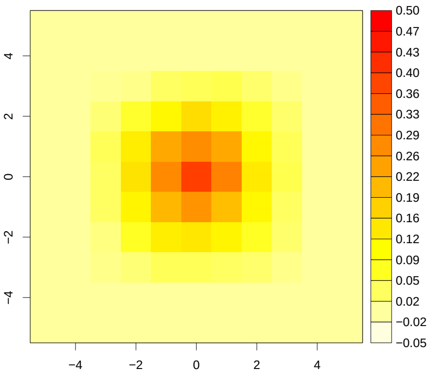

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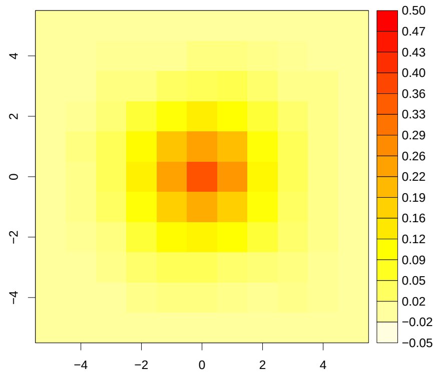

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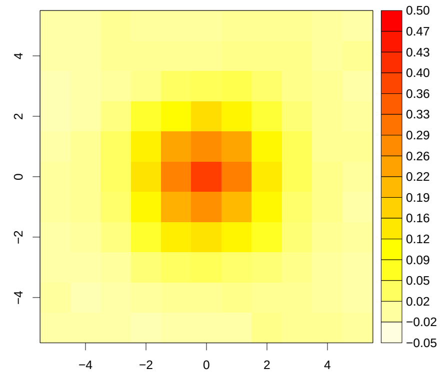
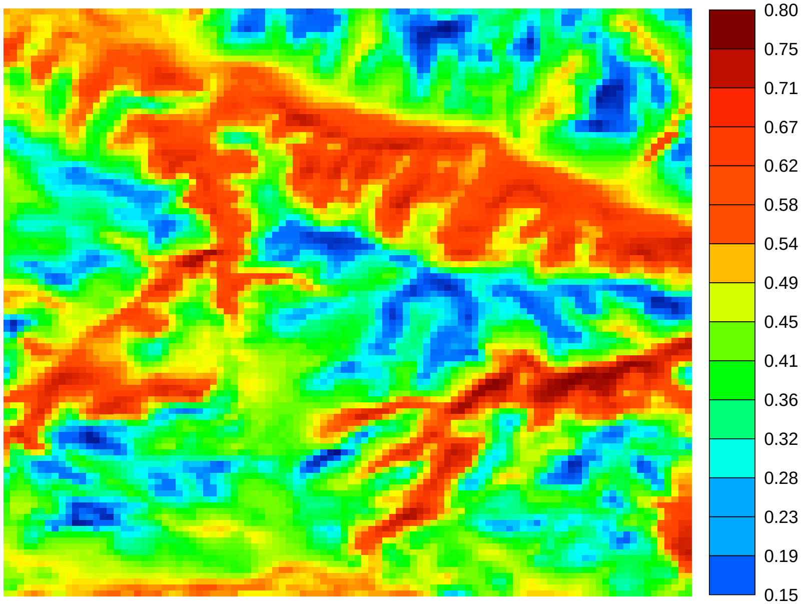


(f)

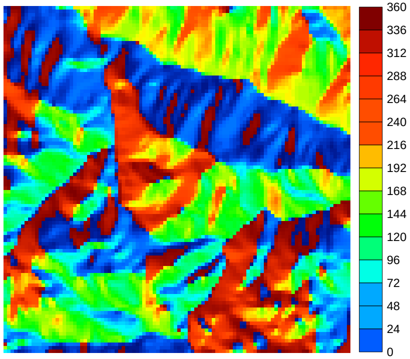
Application: Prediction of Soil Moisture Index

- Select $n = 100$ sites via simple random sampling without replacement

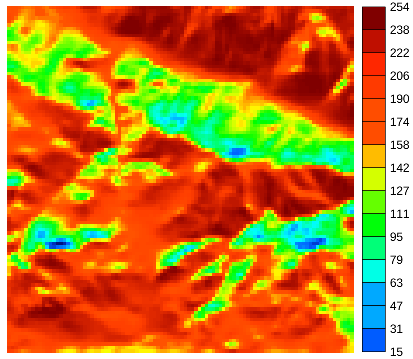


Covariates for Soil Moisture Prediction

- Aspect, hill shade, elevation, slope, precipitation



(a)



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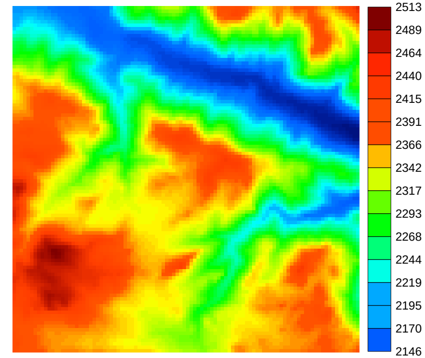

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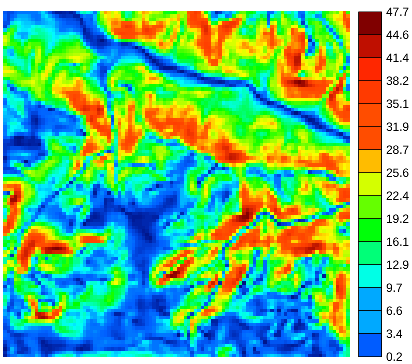

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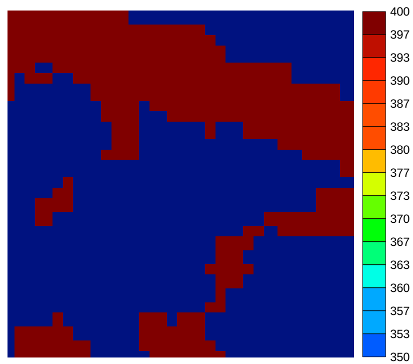


(e)

Covariates for Soil Moisture Prediction

- Response: soil moisture index
- Basic covariate layers:
  - aspect, hill shade, elevation, slope, precipitation
  - 100×100 regular grid
- Expanded covariate layers (14 total):
  - average elevation, slope, precipitation on 3×3 blocks
  - average elevation, slope, precipitation on 9×9 blocks
  - aspect*elevation, aspect*slope, aspect*precipitation

Neighborhoods for Soil Moisture Prediction

- Neighborhoods:
  - use single pixel for elevation, 3×3 elevation, 9×9 elevation
  - same for slope and precipitation
  - (up to 9×9 neighborhood, but highly constrained parameters)
  - for remaining variables, use

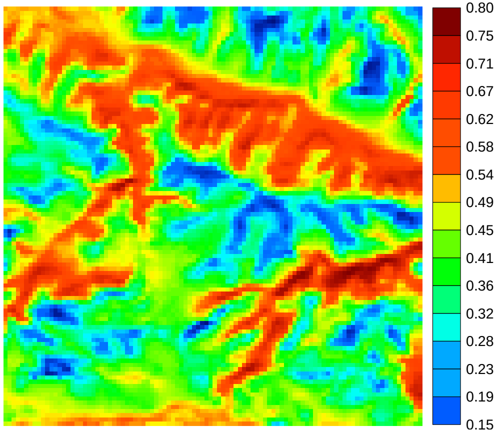
$$\mathcal{N}^{(2q+1)} = \{0, \pm 1, \pm 2, \ldots, \pm(2q+1)\}^2$$

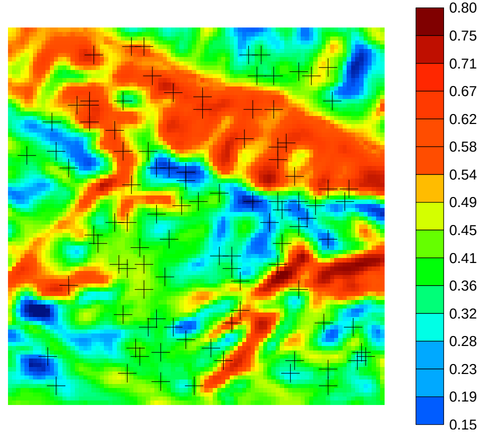
with $q = 0, 1, \ldots, 5$

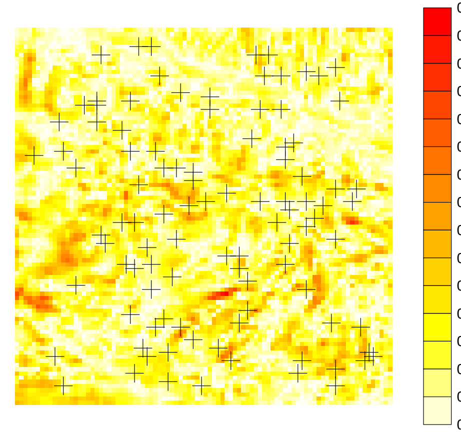
Spatial Lasso for Soil Moisture Prediction

- Apply spatial Lasso with exponential covariance structure
  - grid search on $\gamma = 0, 1, \ldots, 5$, crossed with neighborhoods above
  - 36 possible combinations
- Smallest ten-fold cross-validation value at $\gamma = 2$ and $q = 5$

True and Predicted Soil Moisture Index







Summary

- Lasso approach to GIS model selection and estimation
  - simultaneous layer selection, neighborhood selection, and estimation
  - dominates OLS for unknown layers/neighborhoods
  - extensible to neighborhood transformation
- Spatial Lasso
  - accounts for spatial smoothness within neighborhoods
  - dominates Lasso if smoothness is present
- Promising results in prediction of soil moisture index