

Heavy Tails and Financial Time Series Models

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Outline

Financial time series modeling

- General comments
- Characteristics of financial time series
- Examples (exchange rate, Amazon)
- Multiplicative models for log-returns (GARCH, SV)

Regular variation

- univariate case
- multivariate case

Applications of regular variation

- Stochastic recurrence equations (GARCH)
- Stochastic volatility
- Time-reversibility
- Point process convergence
- Extremes and extremal index
- Limit behavior of sample correlations

Wrap-up

Financial Time Series Modeling

One possible goal: Develop models that capture essential features of financial data.

Strategy: Formulate families of models that at least exhibit these key characteristics. (e.g., GARCH and SV)

Linkage with goal: Do fitted models actually capture the desired characteristics of the real data?

Answer wrt to GARCH and SV models: Yes and no. Answer may depend on the features.

Stărică's paper: “Is GARCH(1,1) Model as Good a Model as the Nobel Accolades Would Imply?”

This paper discusses inadequacy of GARCH(1,1) model as a “data generating process” for the data.

Financial Time Series Modeling (cont)

Goal of this talk: compare and contrast some of the features of GARCH and SV models.

- Regular-variation of finite dimensional distributions
- Time-reversibility
- Point process convergence
- Extreme value behavior
- Sample ACF

Characteristics of financial time series

Define $X_t = \ln(P_t) - \ln(P_{t-1})$ (log returns)

- heavy tailed

$$P(|X_1| > x) \sim RV(-\alpha), \quad 0 < \alpha < 4.$$

- uncorrelated

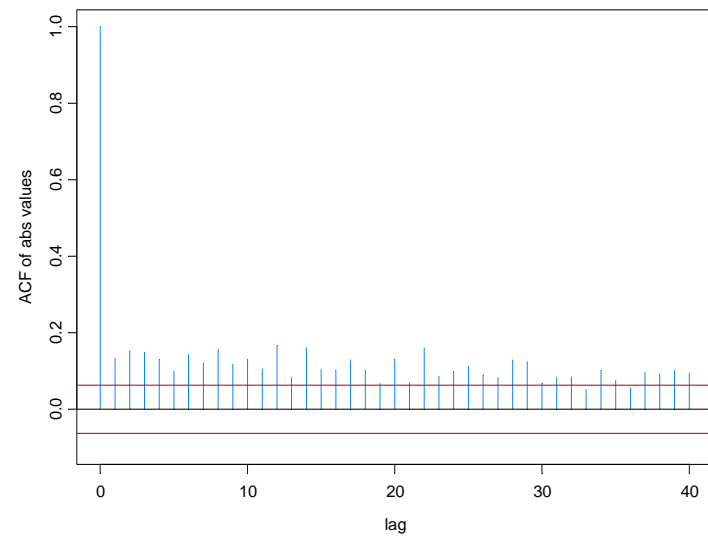
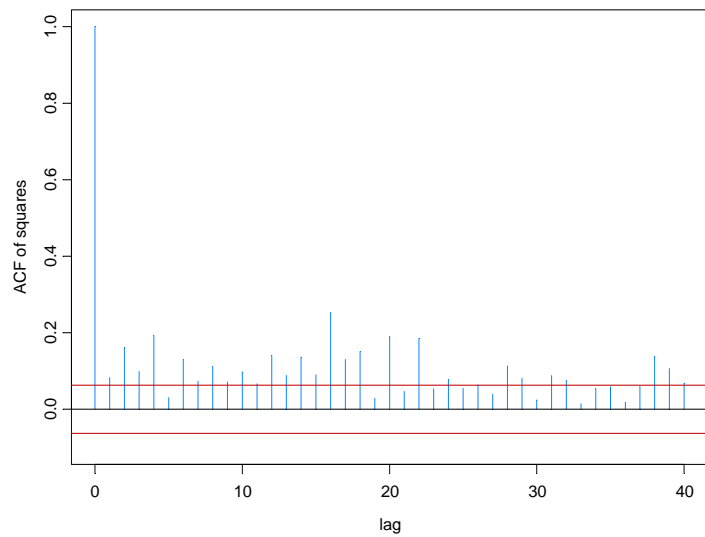
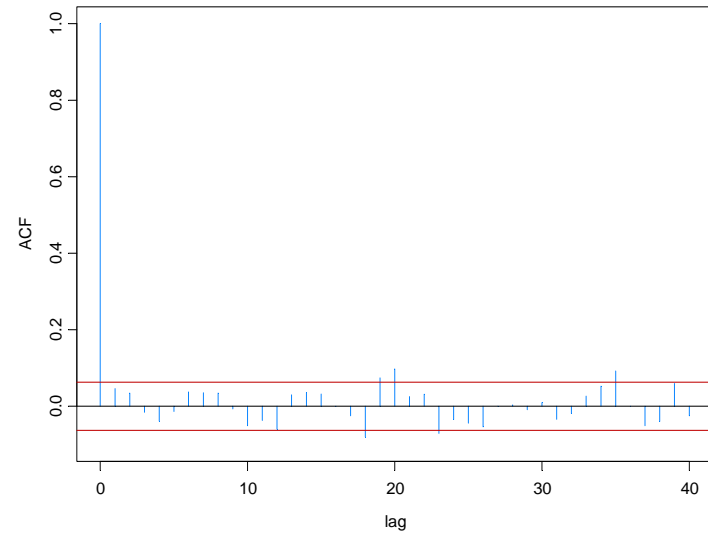
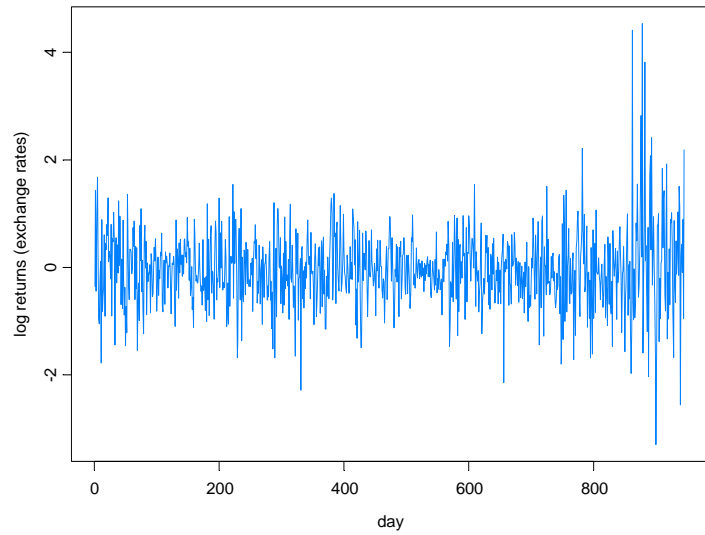
$$\hat{\rho}_x(h) \text{ near } 0 \text{ for all lags } h > 0$$

- $|X_t|$ and X_t^2 have slowly decaying autocorrelations

$$\hat{\rho}_{|X|}(h) \text{ and } \hat{\rho}_{X^2}(h) \text{ converge to } 0 \text{ slowly as } h \text{ increases.}$$

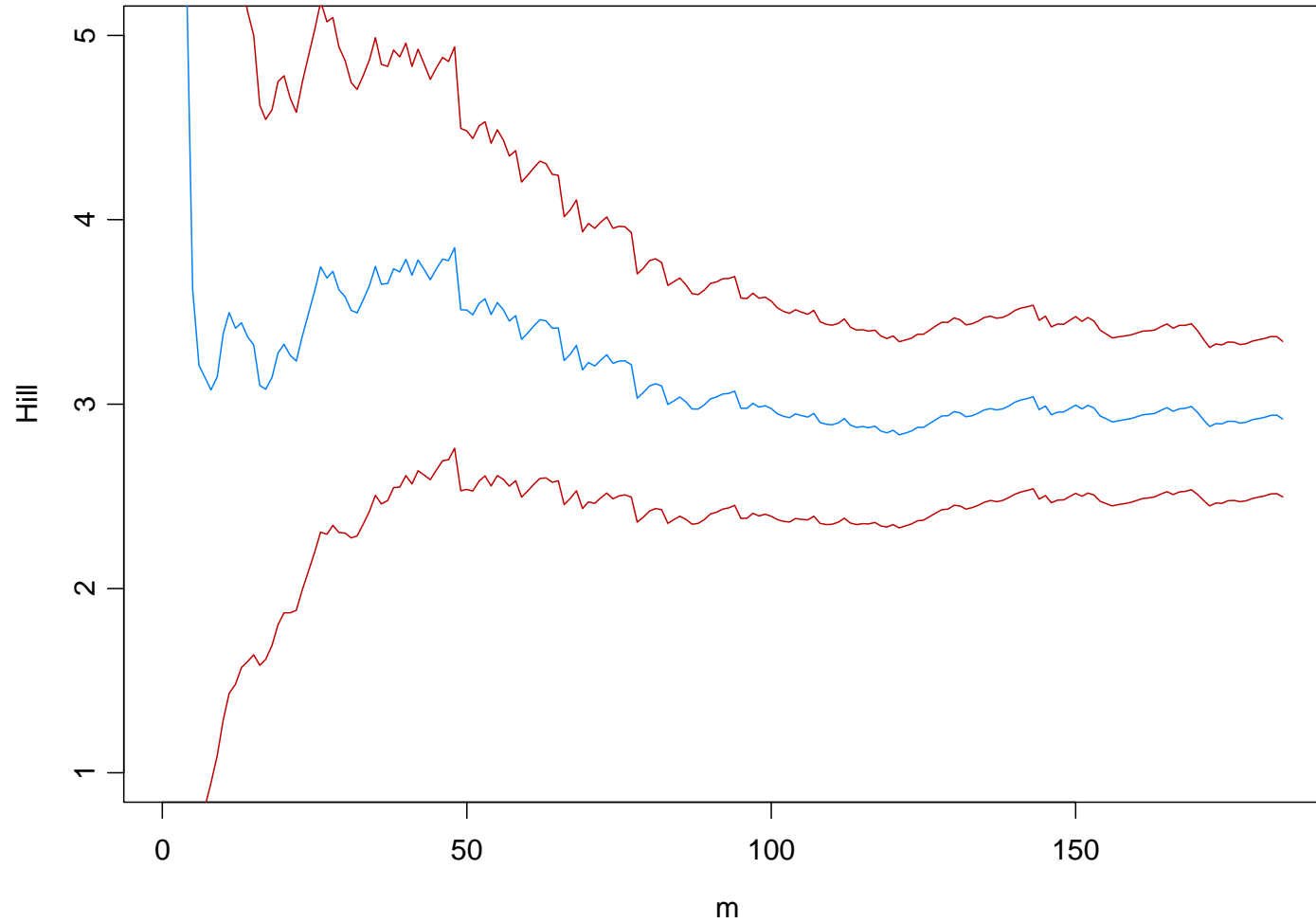
- process exhibits 'volatility clustering'.

Example: Pound-Dollar Exchange Rates (Oct 1, 1981 – Jun 28, 1985; Koopman website)



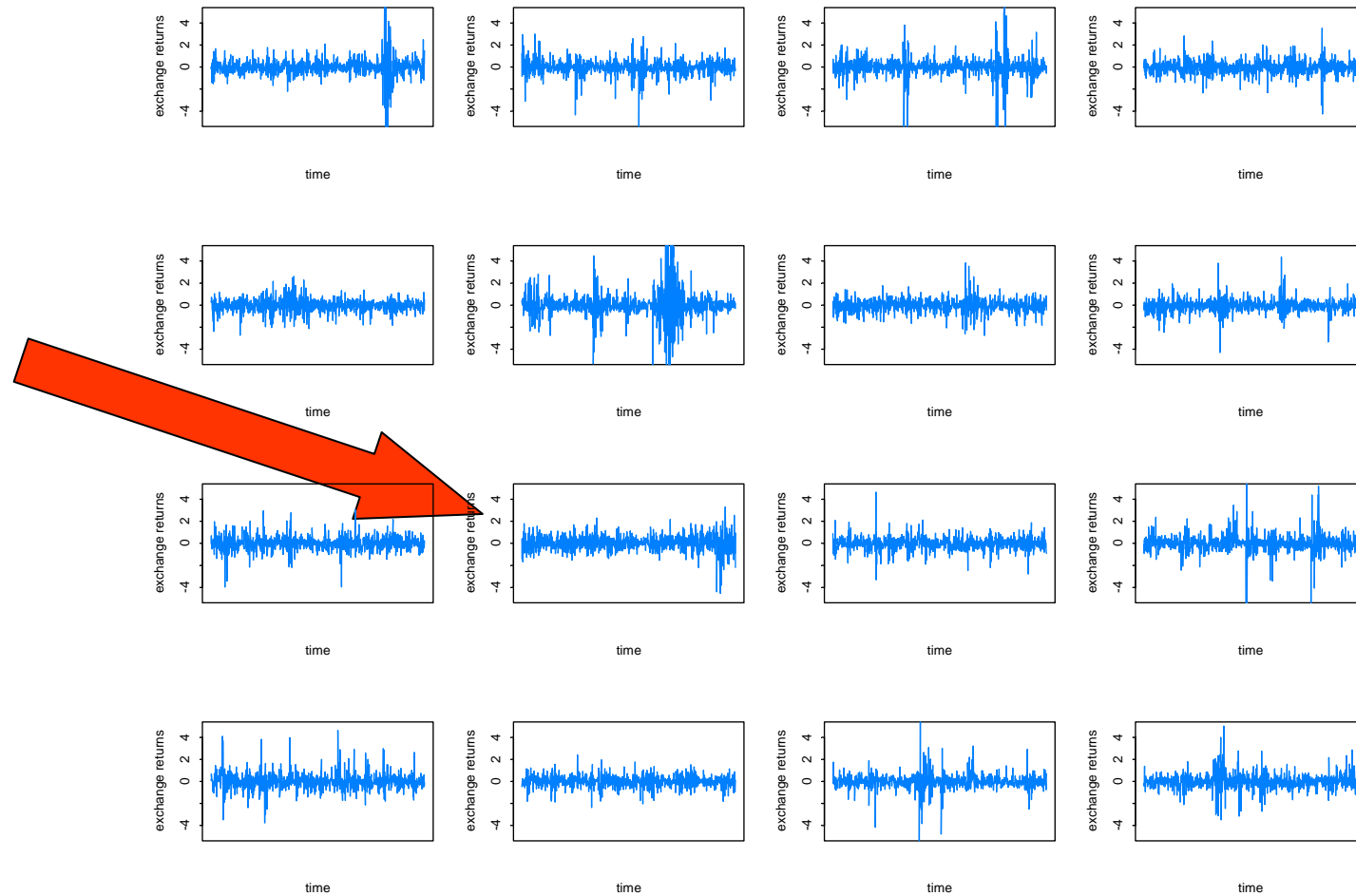
Example: Pound-Dollar Exchange Rates

Hill's estimate of alpha (Hill Horror plots-Resnick)



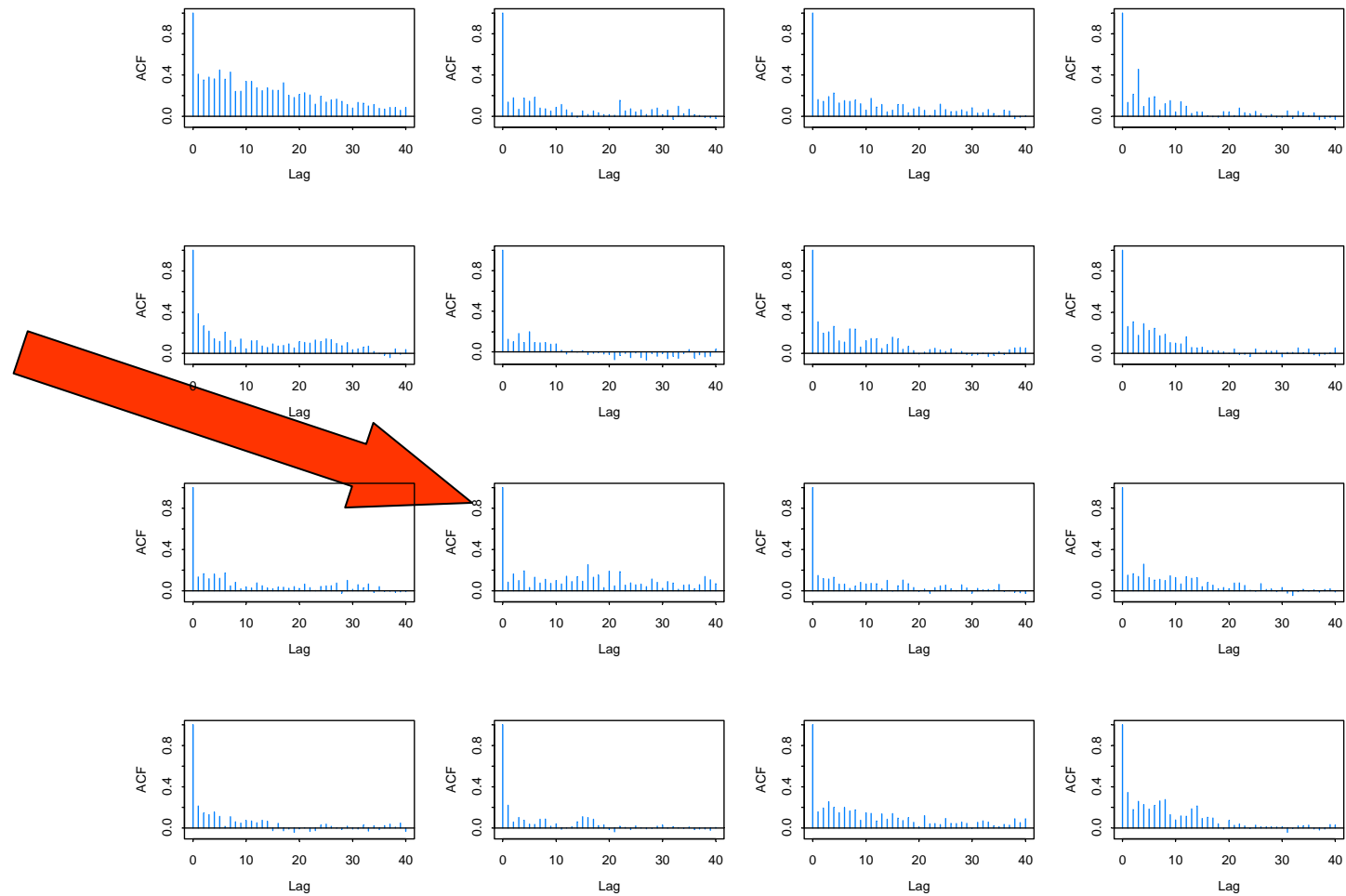
Stărică Plots for Pound-Dollar Exchange Rates

15 realizations from GARCH model fitted to exchange rates + exchange rate data. Which one is the real data?



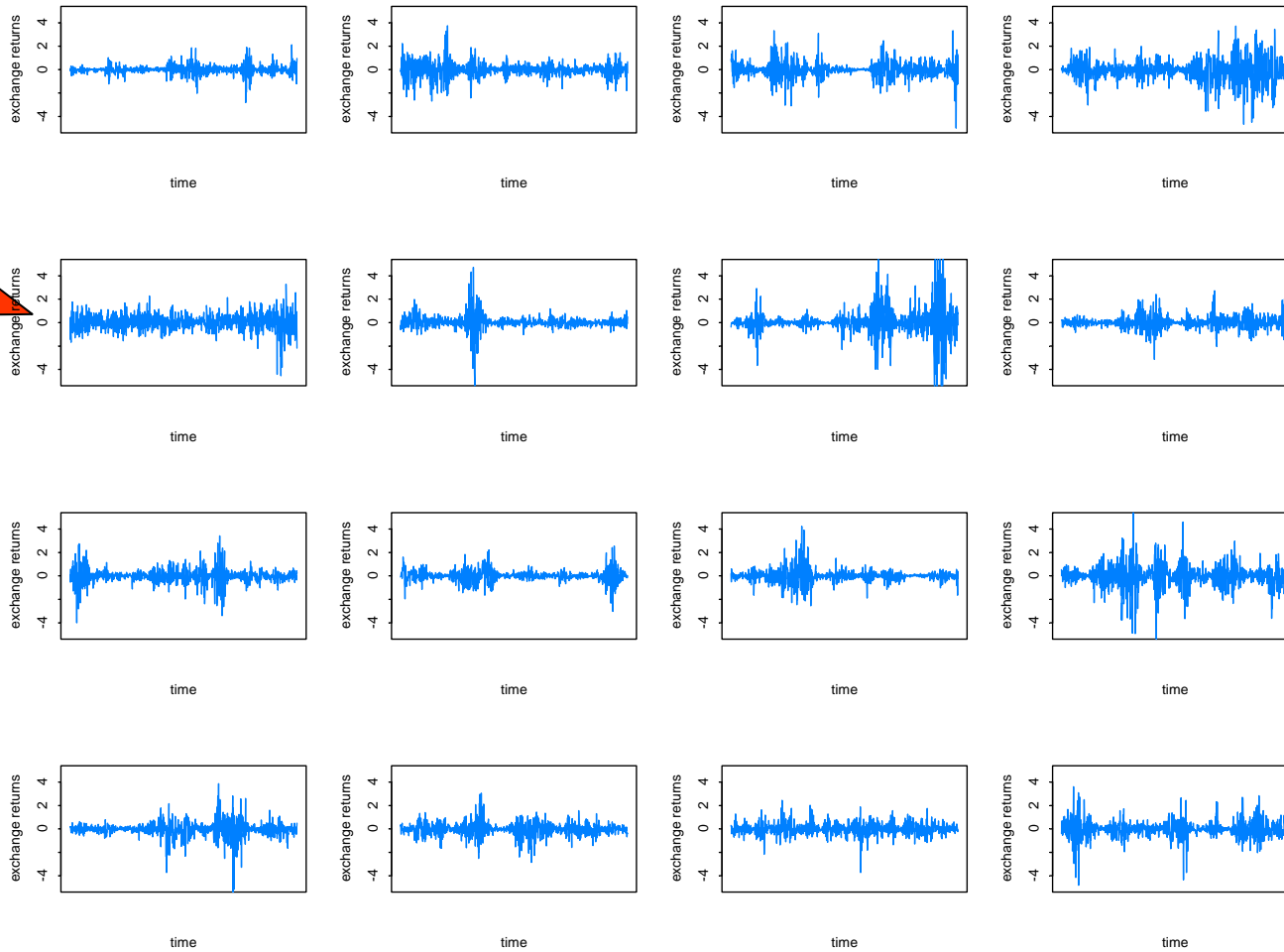
Stărică Plots for Pound-Dollar Exchange Rates

ACF of the squares from the 15 realizations from the GARCH model on previous slide.

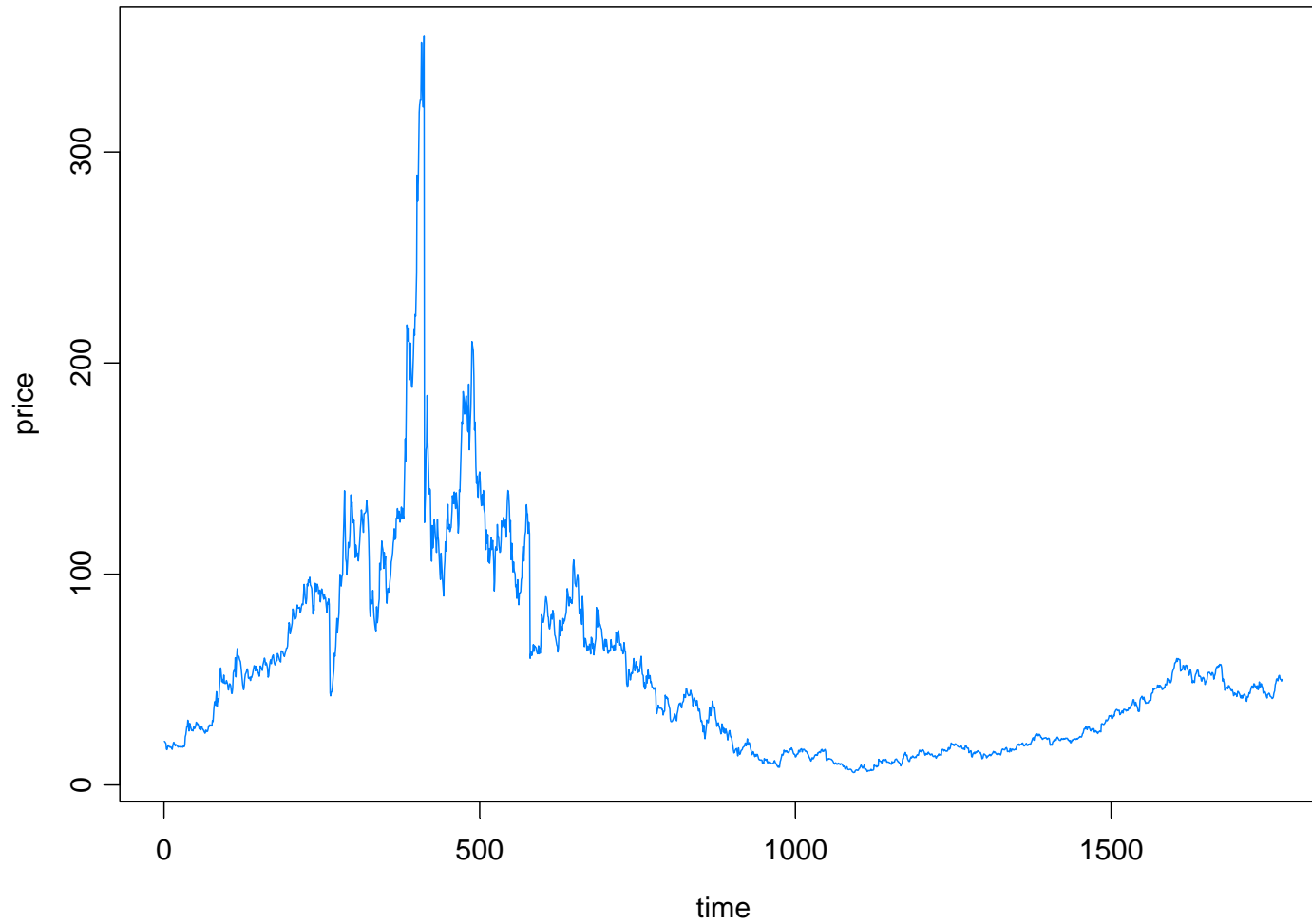


Stărică Plots for Pound-Dollar Exchange Rates

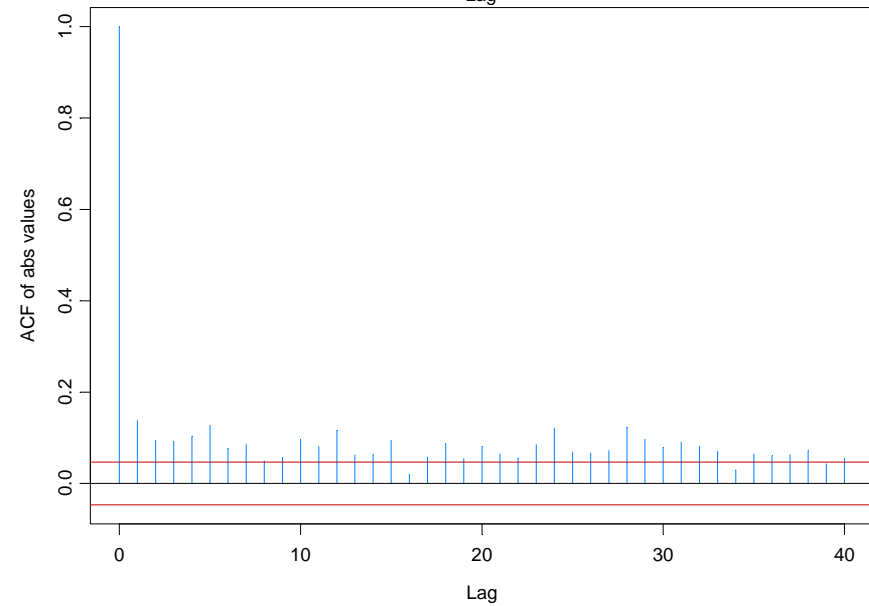
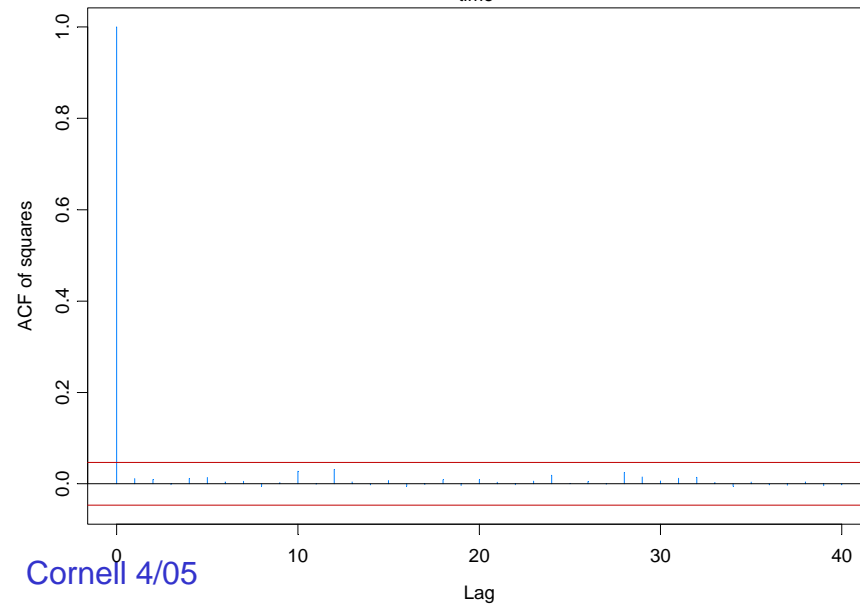
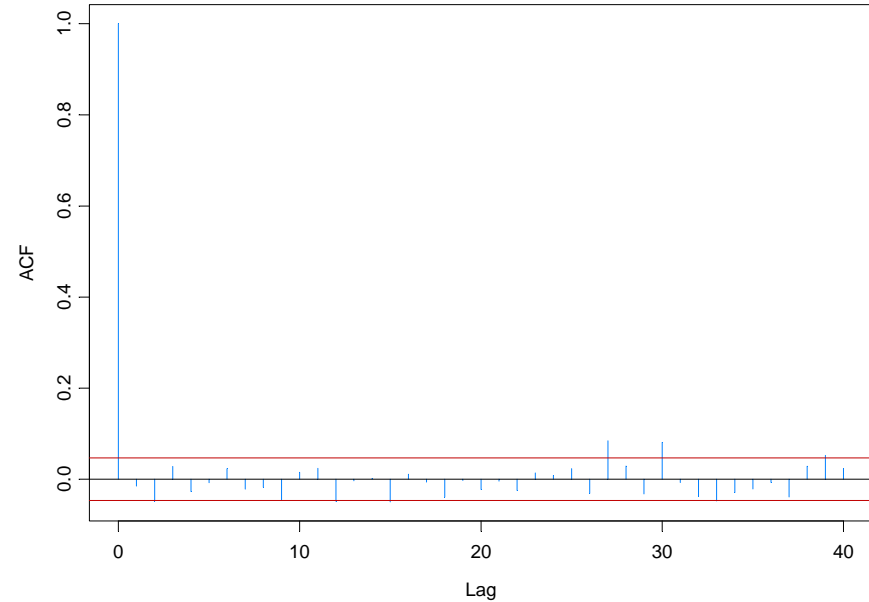
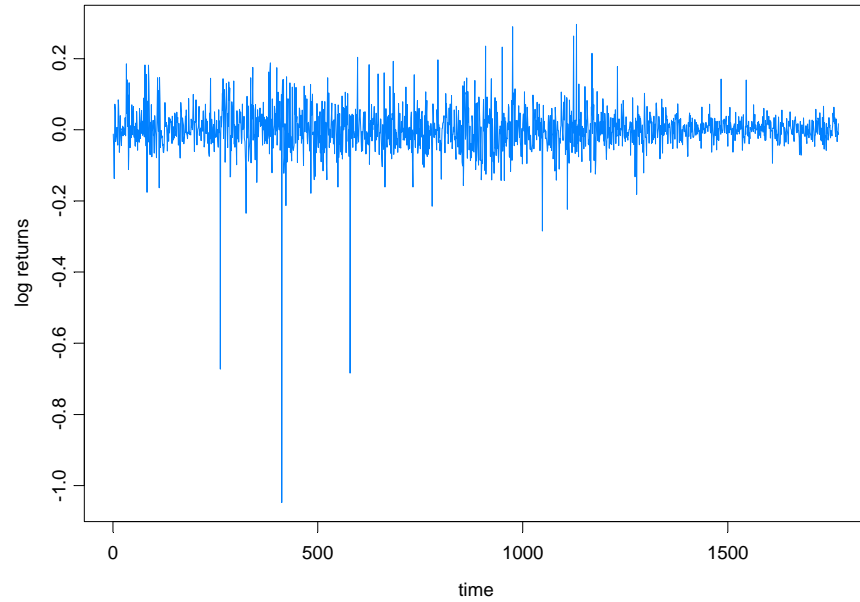
15 realizations from SV model fitted to exchange rates + real data. Which one is the real data?



Example: Amazon (May 16, 1997 – June 16, 2004)

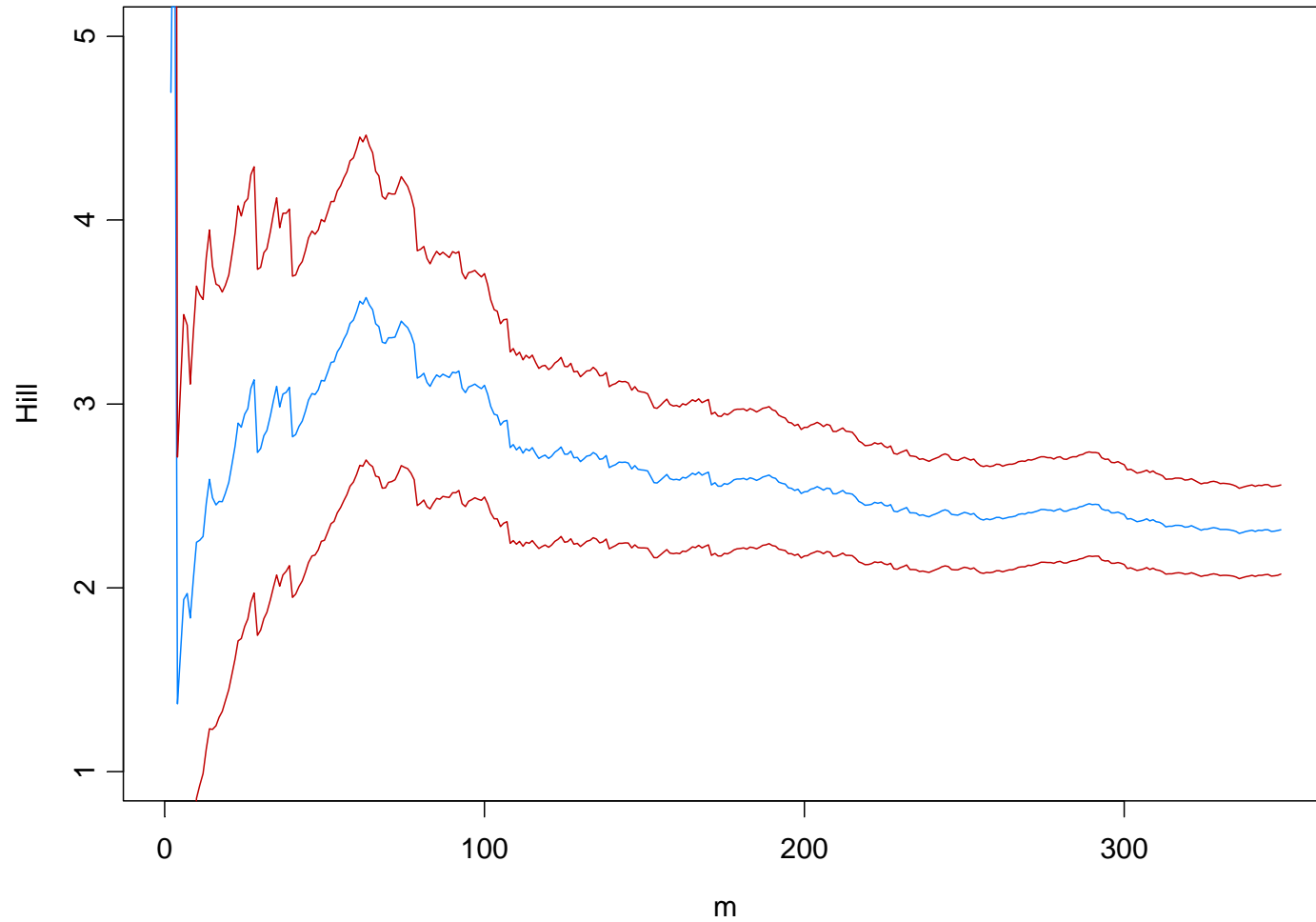


Example: Amazon-returns (May 16, 1997 – June 16, 2004)



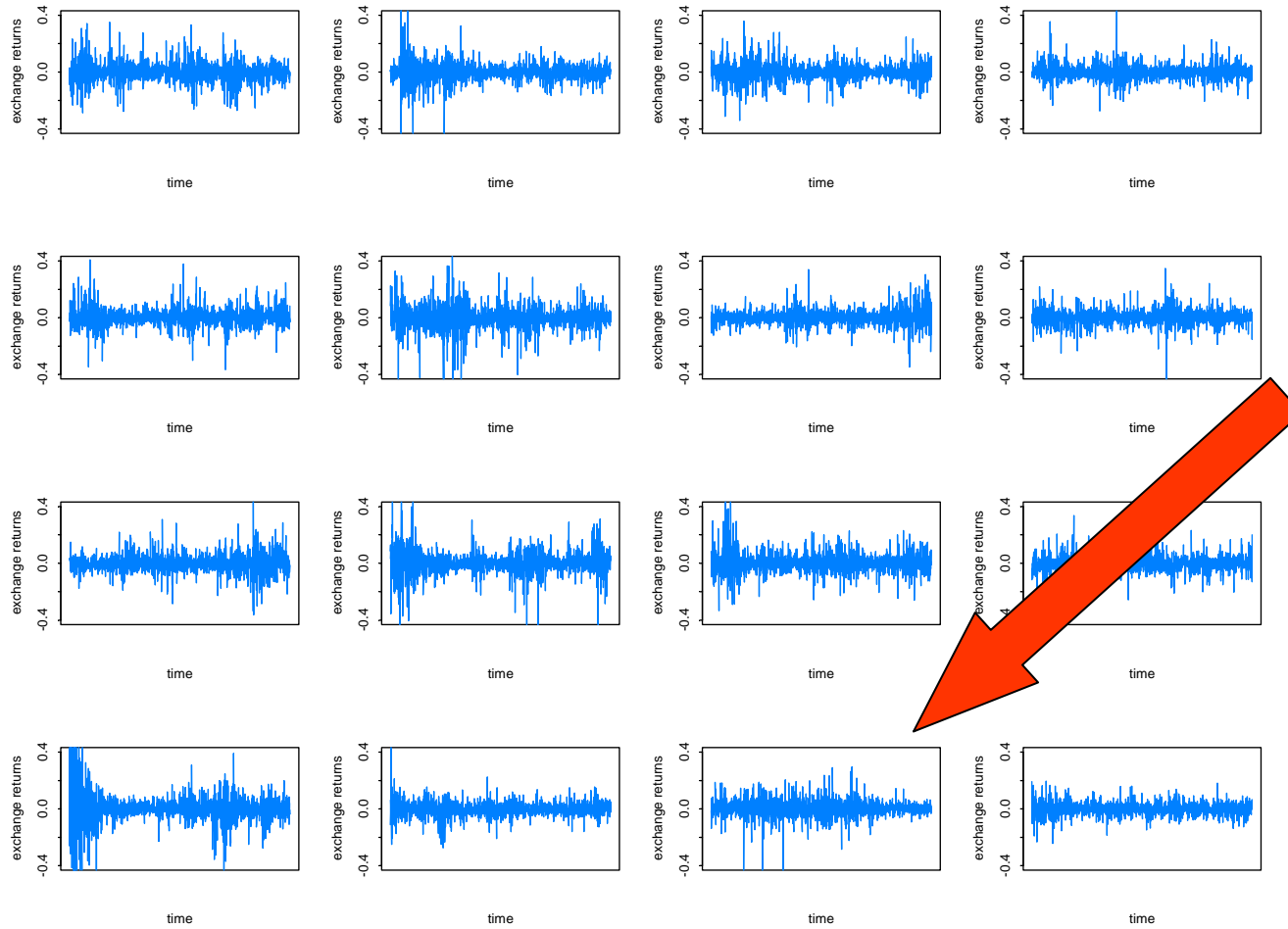
Example: Amazon-returns

Hill's estimate of alpha (Hill Horror plots-Resnick)



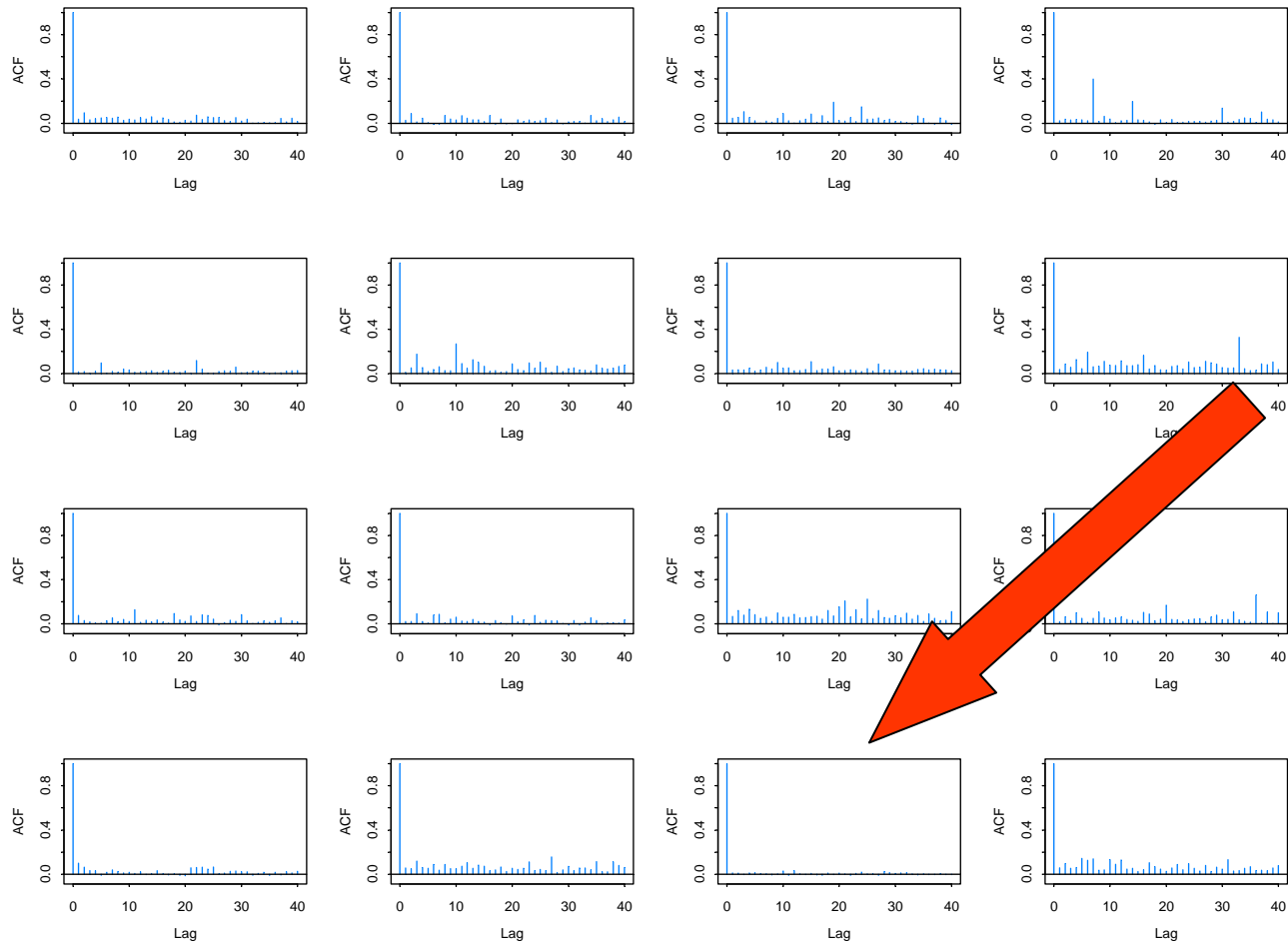
Stărică Plots for the Amazon Data

15 realizations from GARCH model fitted to Amazon + exchange rate data. Which one is the real data?



Stărică Plots for Amazon

ACF of the squares from the 15 realizations from the GARCH model on previous slide.



Multiplicative models for log(returns)

Basic model

$$\begin{aligned} X_t &= \ln(P_t) - \ln(P_{t-1}) \quad (\text{log returns}) \\ &= \sigma_t Z_t, \end{aligned}$$

where

- $\{Z_t\}$ is IID with mean 0, variance 1 (if exists). (e.g. $N(0,1)$ or a t -distribution with ν df.)
- $\{\sigma_t\}$ is the volatility process
- σ_t and Z_t are independent.

Properties:

- $EX_t = 0$, $\text{Cov}(X_t, X_{t+h}) = 0$, $h > 0$ (uncorrelated if $\text{Var}(X_t) < \infty$)
- conditional heteroscedastic (condition on σ_t).

Multiplicative models for log(returns)-cont

$$X_t = \sigma_t Z_t \quad (\text{observation eqn in state-space formulation})$$

Two classes of models for volatility:

(i) GARCH(p,q) process (General AutoRegressive Conditional Heteroscedastic-observation-driven specification)

$$\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \cdots + \alpha_p X_{t-p}^2 + \beta_1 \sigma_{t-1}^2 + \cdots + \beta_q \sigma_{t-q}^2 .$$

Special case: ARCH(1):

$$\begin{aligned} X_t^2 &= (\alpha_0 + \alpha_1 X_{t-1}^2) Z_t^2 \\ &= \alpha_1 Z_t^2 X_{t-1}^2 + \alpha_0 Z_t^2 \\ &= A_t X_{t-1}^2 + B_t \end{aligned} \quad (\text{stochastic recurrence eqn})$$

$$\rho_{X^2}(h) = \alpha_1^h, \text{ if } \alpha_1^2 < 1/3.$$

Multiplicative models for log(returns)-cont

GARCH(2,1): $X_t = \sigma_t Z_t$, $\sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \alpha_2 X_{t-2}^2 + \beta_1 \sigma_{t-1}^2$.

Then $\mathbf{Y}_t = (\sigma_t^2, X_{t-1}^2)'$ follows the SRE given by

$$\begin{bmatrix} \sigma_t^2 \\ X_{t-1}^2 \end{bmatrix} = \begin{bmatrix} \alpha_1 Z_{t-1}^2 + \beta_1 & \alpha_2 \\ Z_{t-1}^2 & 0 \end{bmatrix} \begin{bmatrix} \sigma_{t-1}^2 \\ X_{t-2}^2 \end{bmatrix} + \begin{bmatrix} \alpha_0 \\ 0 \end{bmatrix}$$

Questions:

- Existence of a unique stationary solution to the SRE?
- Regular variation of the joint distributions?

Multiplicative models for log(returns)-cont

$$X_t = \sigma_t Z_t \quad (\text{observation eqn in state-space formulation})$$

(ii) stochastic volatility process (parameter-driven specification)

$$\log \sigma_t^2 = \sum_{j=-\infty}^{\infty} \psi_j \varepsilon_{t-j}, \quad \sum_{j=-\infty}^{\infty} \psi_j^2 < \infty, \quad \{\varepsilon_t\} \sim \text{IIDN}(0, \sigma^2)$$

$$\rho_{X^2}(h) = \text{Cor}(\sigma_t^2, \sigma_{t+h}^2) / EZ_1^4$$

Question:

- Joint distributions of process regularly varying if distr of Z_1 is regularly varying?

Two models for log(returns)-cont

GARCH(1,1):

$$X_t = \sigma_t Z_t, \quad \sigma_t^2 = \alpha_0 + \alpha_1 X_{t-1}^2 + \beta_1 \sigma_{t-1}^2, \quad \{Z_t\} \sim \text{IID}(0,1)$$

Stochastic Volatility:

$$X_t = \sigma_t Z_t, \quad \log \sigma_t^2 = \phi_0 + \phi_1 \log \sigma_{t-1}^2 + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IIDN}(0, \sigma^2)$$

Main question:

What intrinsic features in the data (*if any*) can be used to discriminate between these two models?

Regular variation — univariate case

Def: The random variable X is *regularly varying with index* α if

$$P(|X| > tx) / P(|X| > t) \rightarrow x^{-\alpha} \text{ and } P(X > t) / P(|X| > t) \rightarrow p,$$

or, equivalently, if

$$P(X > tx) / P(|X| > t) \rightarrow px^{-\alpha} \text{ and } P(X < -tx) / P(|X| > t) \rightarrow qx^{-\alpha},$$

where $0 \leq p \leq 1$ and $p+q=1$.

Equivalence:

X is $RV(\alpha)$ if and only if $P(X \in t \bullet) / P(|X| > t) \rightarrow_v \mu(\bullet)$

(\rightarrow_v vague convergence of measures on $\mathbb{R} \setminus \{0\}$). In this case,

$$\mu(dx) = \left(p\alpha x^{-\alpha-1} I(x>0) + q\alpha (-x)^{-\alpha-1} I(x<0) \right) dx$$

Note: $\mu(tA) = t^{-\alpha} \mu(A)$ for every t and A bounded away from 0.

Regular variation — univariate case (cont)

Another formulation (polar coordinates):

Define the ± 1 valued rv θ , $P(\theta = 1) = p$, $P(\theta = -1) = 1 - p = q$.

Then

X is $RV(\alpha)$ if and only if

$$\frac{P(|X| > t x, X/|X| \in S)}{P(|X| > t)} \rightarrow x^{-\alpha} P(\theta \in S)$$

or

$$\frac{P(|X| > t x, X/|X| \in \bullet)}{P(|X| > t)} \rightarrow_v x^{-\alpha} P(\theta \in \bullet)$$

(\rightarrow_v vague convergence of measures on $S^0 = \{-1, 1\}$).

Regular variation — multivariate case

Multivariate regular variation of $\mathbf{X}=(X_1, \dots, X_m)$: There exists a random vector $\theta \in \mathbf{S}^{m-1}$ such that

$$P(|\mathbf{X}| > t x, \mathbf{X}/|\mathbf{X}| \in \bullet) / P(|\mathbf{X}| > t) \rightarrow_v x^{-\alpha} P(\theta \in \bullet)$$

(\rightarrow_v vague convergence on \mathbf{S}^{m-1} , unit sphere in \mathbf{R}^m).

- $P(\theta \in \bullet)$ is called the **spectral measure**
- α is the **index of \mathbf{X}** .

Equivalence:

$$\frac{P(\mathbf{X} \in t\bullet)}{P(|\mathbf{X}| > t)} \rightarrow_v \mu(\bullet)$$

μ is a measure on \mathbf{R}^m which satisfies for $x > 0$ and A bounded away from 0,

$$\mu(xB) = x^{-\alpha} \mu(xA).$$

Regular variation — multivariate case (cont)

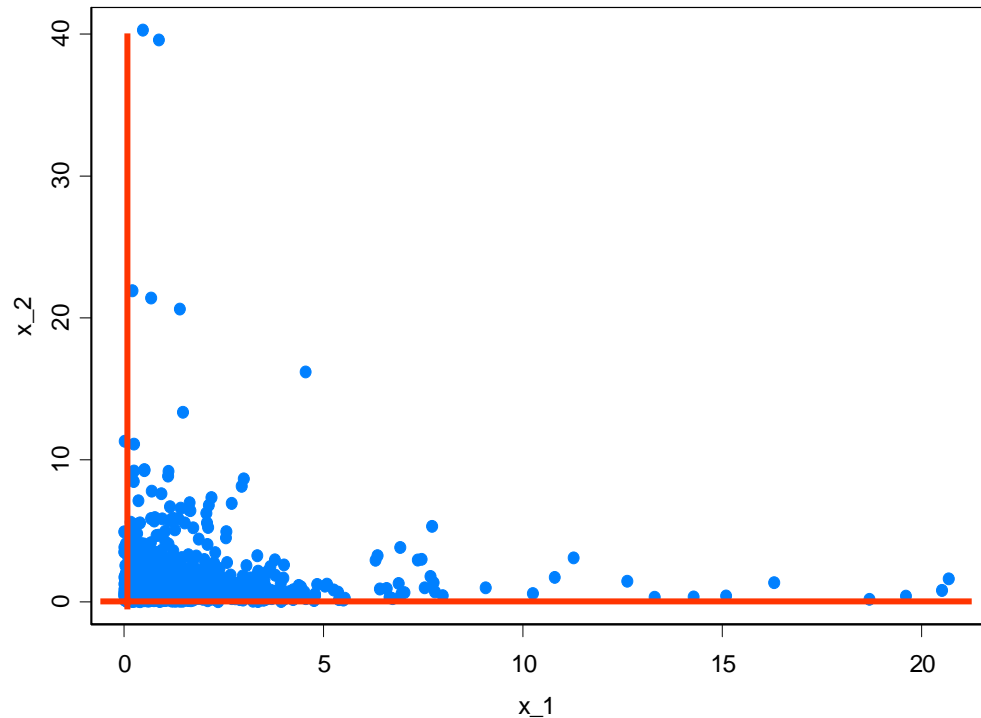
Examples:

1. If $X_1 > 0$ and $X_2 > 0$ are iid $RV(\alpha)$, then $\mathbf{X} = (X_1, X_2)$ is multivariate regularly varying with index α and *spectral distribution*

$$P(\theta = (0,1)) = P(\theta = (1,0)) = .5 \quad (\text{mass on axes}).$$

Interpretation: Unlikely that X_1 and X_2 are very large at the same time.

Figure: plot of (X_{t1}, X_{t2}) for realization of 10,000.



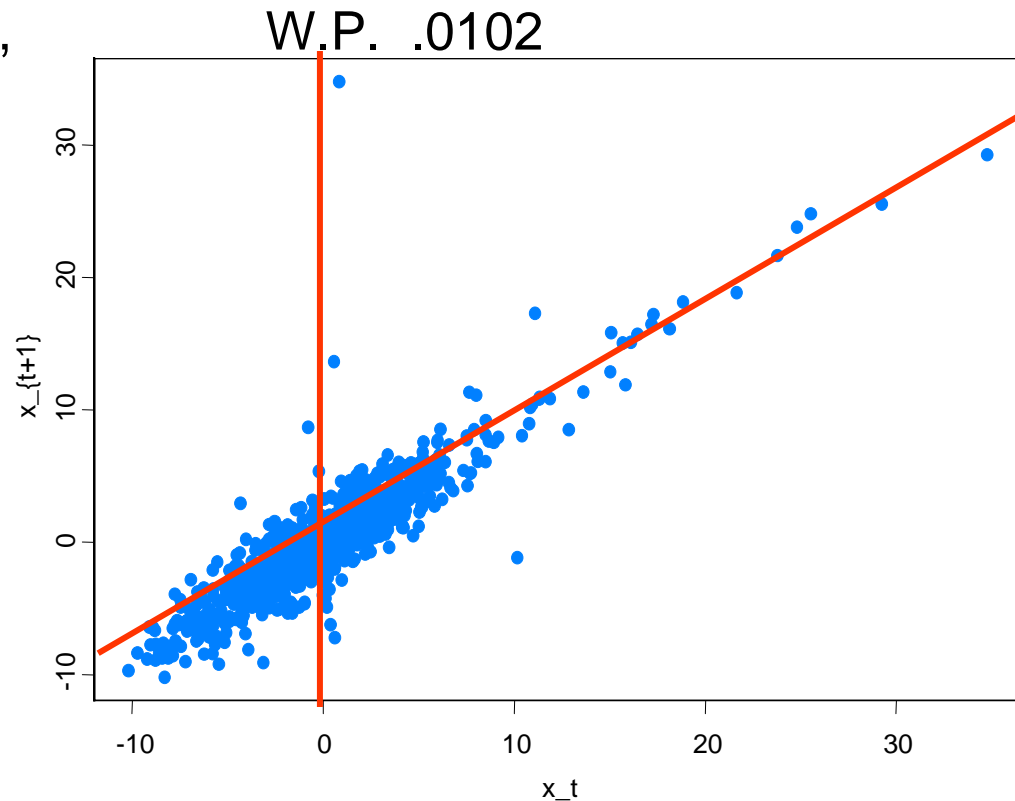
2. If $X_1 = X_2 > 0$, then $\mathbf{X} = (X_1, X_2)$ is multivariate regularly varying with index α and *spectral distribution*

$$P(\theta = (1/\sqrt{2}, 1/\sqrt{2})) = 1.$$

3. AR(1): $X_t = .9 X_{t-1} + Z_t$, $\{Z_t\} \sim \text{IID symmetric stable (1.8)}$

Distr of θ : $\begin{cases} \pm(1,.9)/\text{sqrt}(1.81), \text{ W.P. } .9898 \\ \pm(0,1), \\ \text{ W.P. } .0102 \end{cases}$

Figure: plot of (X_t, X_{t+1}) for realization of 10,000.



Applications of multivariate regular variation

- Domain of attraction for *sums of iid random vectors* (Rvaceva, 1962). That is, when does the partial sum

$$a_n^{-1} \sum_{t=1}^n \mathbf{X}_t$$

converge for some constants a_n ?

- *Spectral measure* of multivariate stable vectors.
- *Domain of attraction* for componentwise maxima of iid random vectors (Resnick, 1987). Limit behavior of

$$a_n^{-1} \bigvee_{t=1}^n \mathbf{X}_t$$

- Weak convergence of *point processes* with iid points.
- Solution to *stochastic recurrence equations*, $\mathbf{Y}_t = \mathbf{A}_t \mathbf{Y}_{t-1} + \mathbf{B}_t$
- Weak convergence of *sample autocovariances*.

Applications of multivariate regular variation (cont)

Linear combinations:

$\mathbf{X} \sim \text{RV}(\alpha) \Rightarrow$ all linear combinations of \mathbf{X} are regularly varying

i.e., there exist α and slowly varying fcn $L(\cdot)$, s.t.

$$P(\mathbf{c}^T \mathbf{X} > t) / (t^\alpha L(t)) \rightarrow w(\mathbf{c}), \text{ exists for all real-valued } \mathbf{c},$$

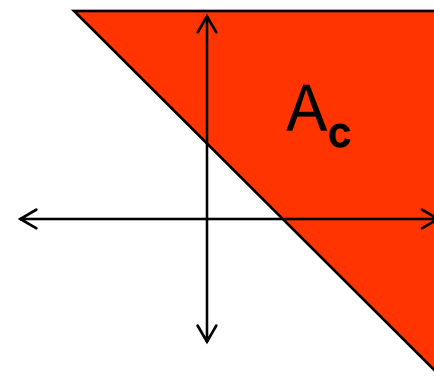
where

$$w(t\mathbf{c}) = t^{-\alpha} w(\mathbf{c}).$$

Use vague convergence with $A_{\mathbf{c}} = \{\mathbf{y} : \mathbf{c}^T \mathbf{y} > 1\}$, i.e.,

$$\frac{P(\mathbf{X} \in tA_{\mathbf{c}})}{t^{-\alpha} L(t)} = \frac{P(\mathbf{c}^T \mathbf{X} > t)}{P(|\mathbf{X}| > t)} \rightarrow \mu(A_{\mathbf{c}}) =: w(\mathbf{c}),$$

where $t^\alpha L(t) = P(|\mathbf{X}| > t)$.



Applications of multivariate regular variation (cont)

Converse?

$\mathbf{X} \sim \text{RV}(\alpha) \iff$ all linear combinations of \mathbf{X} are regularly varying?

There exist α and slowly varying fcn $L(\cdot)$, s.t.

(LC) $P(\mathbf{c}^T \mathbf{X} > t) / (t^\alpha L(t)) \rightarrow w(\mathbf{c})$, exists for all real-valued \mathbf{c} .

Theorem (Basrak, Davis, Mikosch, '02). Let \mathbf{X} be a random vector.

1. If \mathbf{X} satisfies (LC) with α non-integer, then \mathbf{X} is $\text{RV}(\alpha)$.
2. If $\mathbf{X} > 0$ satisfies (LC) for non-negative \mathbf{c} and α is non-integer, then \mathbf{X} is $\text{RV}(\alpha)$.
3. If $\mathbf{X} > 0$ satisfies (LC) with α an odd integer, then \mathbf{X} is $\text{RV}(\alpha)$.

Applications of multivariate regular variation (cont)

There exist α and slowly varying fcn $L(\cdot)$, s.t.

(LC) $P(\mathbf{c}^T \mathbf{X} > t) / (t^\alpha L(t)) \rightarrow w(\mathbf{c})$, exists for all real-valued \mathbf{c} .

1. If \mathbf{X} satisfies (LC) with α non-integer, then \mathbf{X} is $RV(\alpha)$.
2. If $\mathbf{X} > 0$ satisfies (LC) for non-negative \mathbf{c} and α is non-integer, then \mathbf{X} is $RV(\alpha)$.
3. If $\mathbf{X} > 0$ satisfies (LC) with α an odd integer, then \mathbf{X} is $RV(\alpha)$.

Remark: Hult and Lindskog (2005) show that .

- 1 cannot be extended to integer α .
- 2 cannot be extended to integer α .
- It is unknown if 3 can be extended to even integers.

Applications of theorem

1. **Kesten (1973)**. Under general conditions, **(LC)** holds with $L(t)=1$ for stochastic recurrence equations of the form

$$\mathbf{Y}_t = \mathbf{A}_t \mathbf{Y}_{t-1} + \mathbf{B}_t, \quad (\mathbf{A}_t, \mathbf{B}_t) \sim \text{IID},$$

\mathbf{A}_t $d \times d$ random matrices, \mathbf{B}_t random d -vectors.

It follows that the distributions of \mathbf{Y}_t , and in fact all of the finite dim'l distrs of \mathbf{Y}_t are regularly varying (if α is non-even).

2. **GARCH processes**. Since squares of a GARCH process can be embedded in a SRE, the *finite dimensional distributions* of a **GARCH** are regularly varying.

Examples

Example of ARCH(1): $X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} Z_t$, $\{Z_t\} \sim \text{IID}$.

α found by solving $E|\alpha_1 Z^2|^{\alpha/2} = 1$.

α_1	.312	.577	1.00	1.57
α	8.00	4.00	2.00	1.00

Distr of θ :

$$P(\theta \in \bullet) = E\{ \|(B,Z)\|^\alpha I(\arg((B,Z)) \in \bullet) \} / E\|(B,Z)\|^\alpha$$

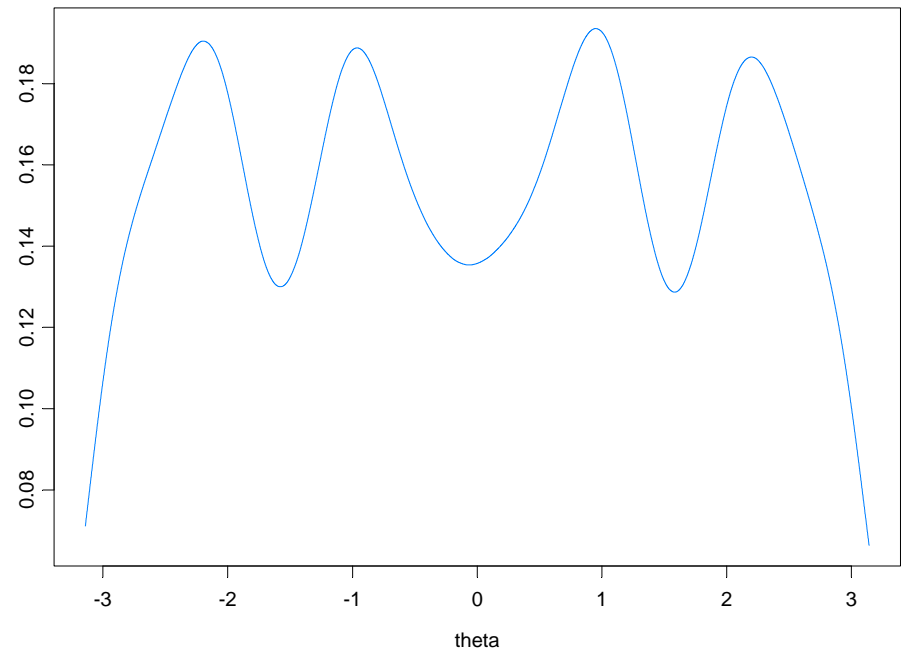
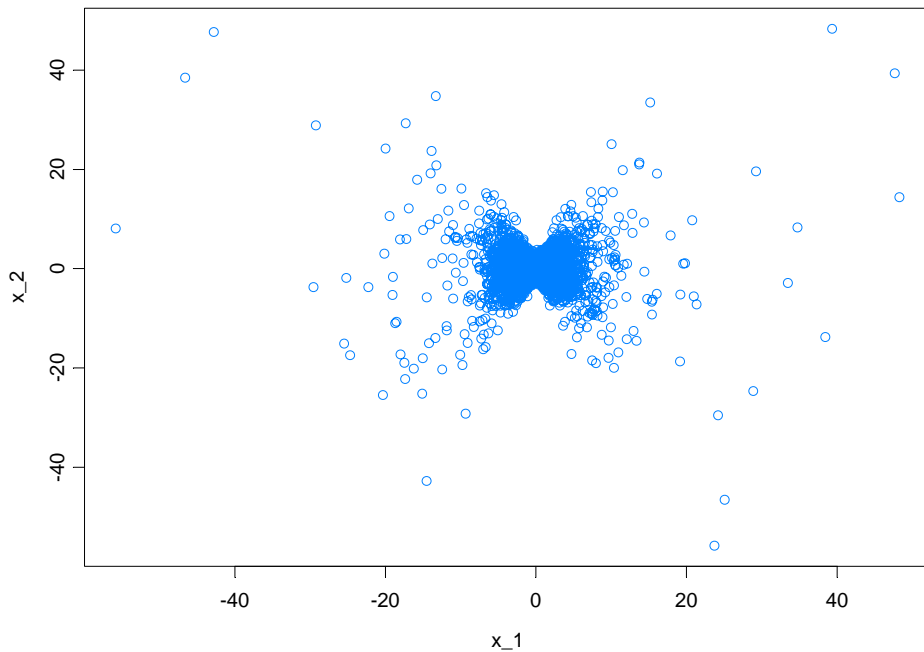
where

$$P(B = 1) = P(B = -1) = .5$$

Examples (cont)

Example of ARCH(1): $\alpha_0=1, \alpha_1=1, \alpha=2, X_t=(\alpha_0+\alpha_1 X_{t-1}^2)^{1/2}Z_t, \{Z_t\}\sim\text{IID}$

Figures: plots of (X_t, X_{t+1}) and estimated distribution of θ for realization of 10,000.



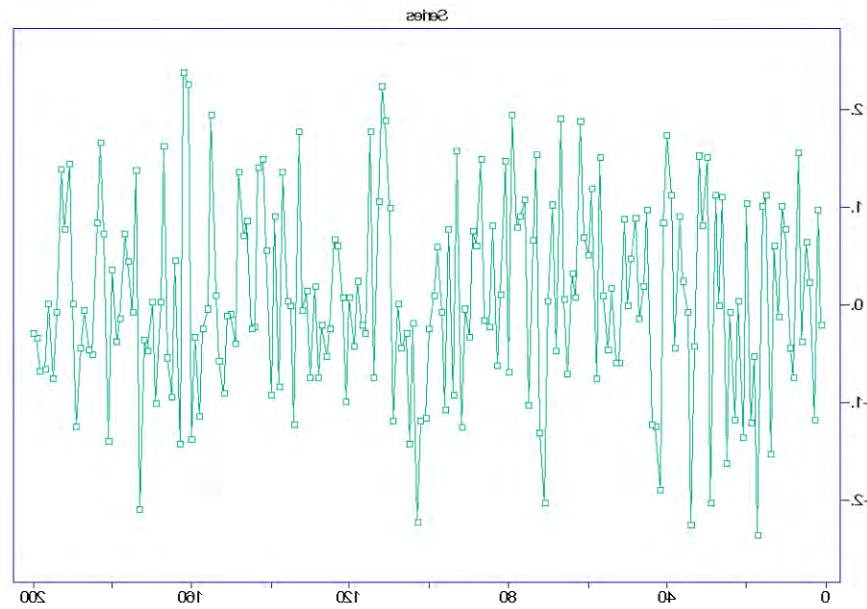
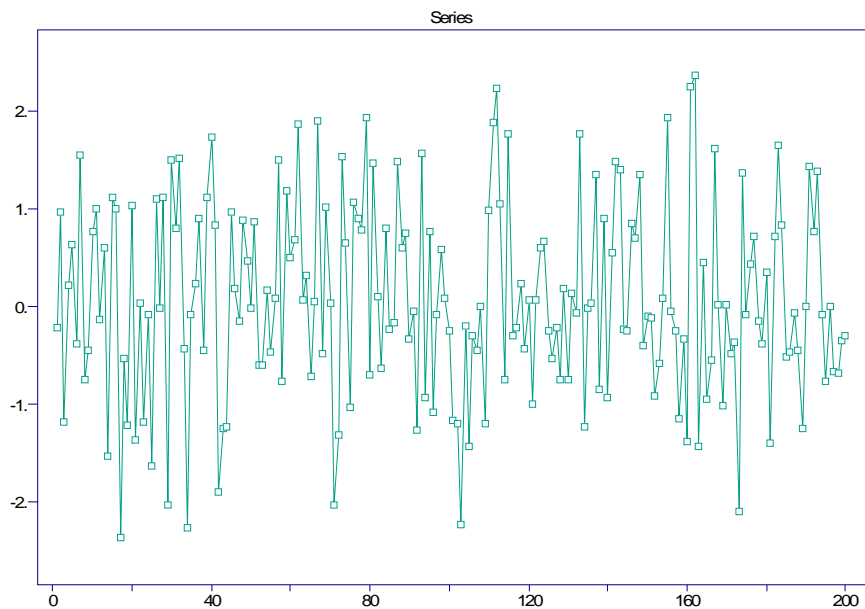
Excursion to time-reversibility

Reversibility. A stationary sequence of random variables $\{X_t\}$ is *time-reversible* if $(X_1, \dots, X_n) =_d (X_n, \dots, X_1)$ for all $n > 1$.

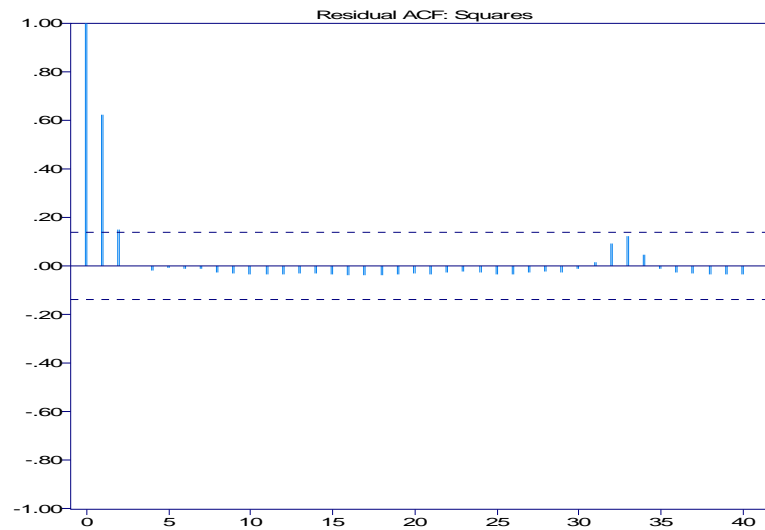
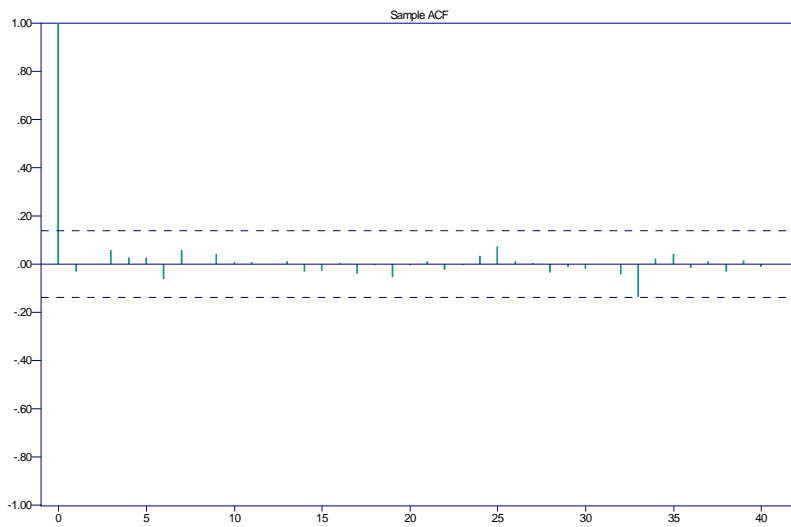
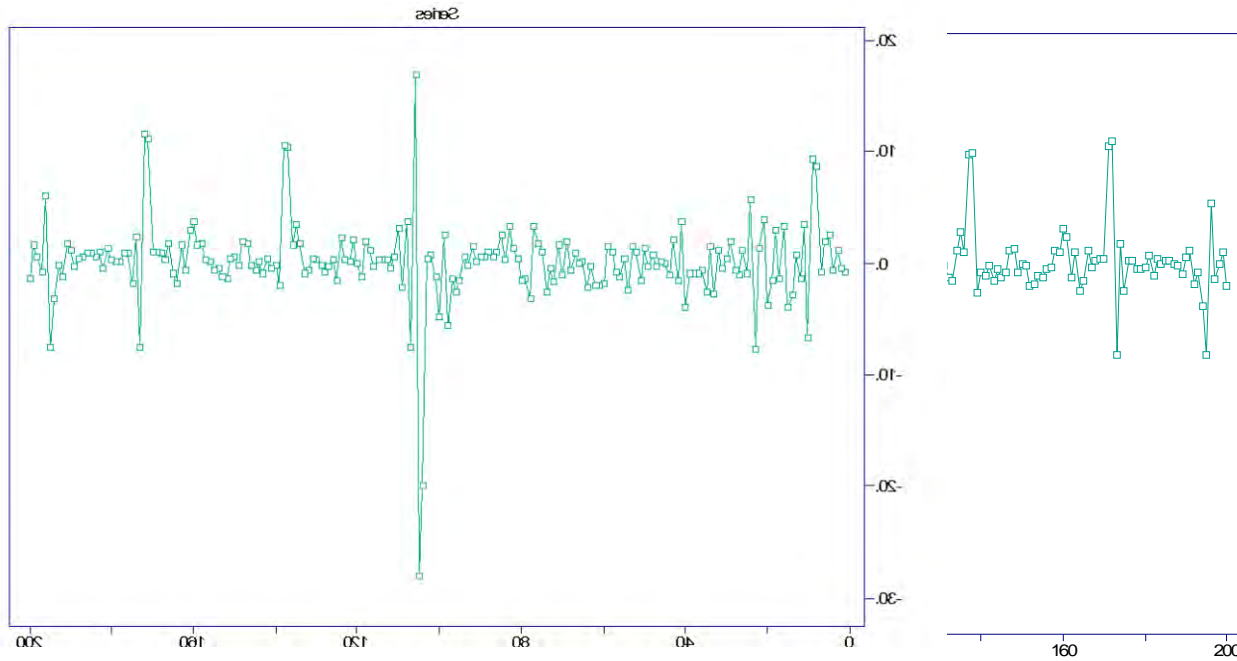
Results: i) IID sequences $\{Z_t\}$ are time-reversible.

ii) Linear time series (with a couple obvious exceptions) are time-reversible iff *Gaussian*. (Breidt and Davis `91)

Application: If plot of time series does not look time-reversible, then it *cannot* be modeled as IID or a Gaussian process. Use the “flip and compare” inspection test!



Reversibility. Does the following series look time-reversible?

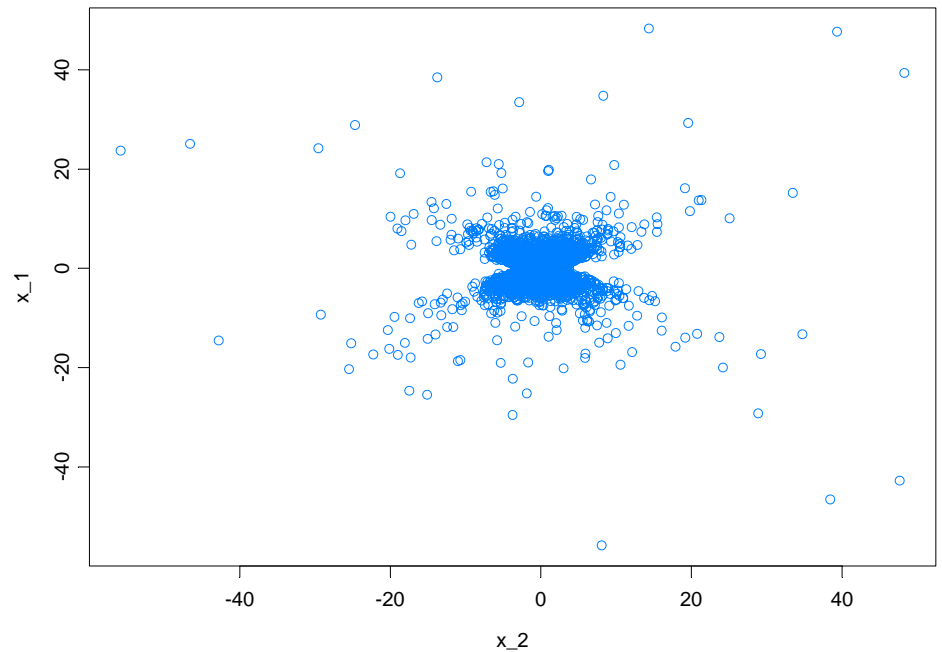
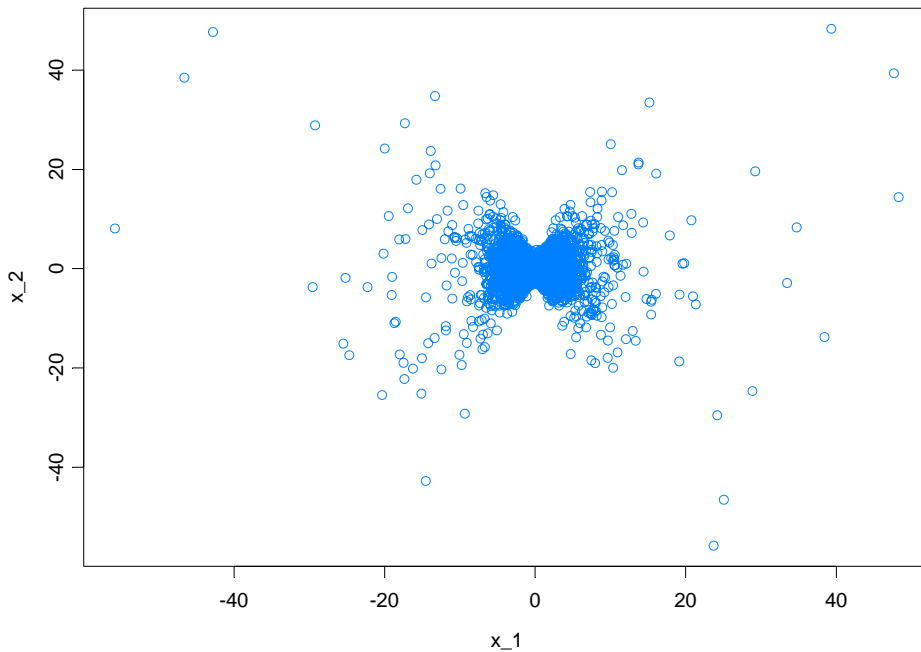


Examples (cont)

Example of ARCH(1): $\alpha_0=1, \alpha_1=1, \alpha=2, X_t=(\alpha_0+\alpha_1 X_{t-1}^2)^{1/2}Z_t, \{Z_t\}\sim\text{IID}$

Is this process time-reversible?

Figures: plots of (X_t, X_{t+1}) and (X_{t+1}, X_t) implies *non-reversible*.



Examples (cont)

Example: SV model $X_t = \sigma_t Z_t$

Suppose $Z_t \sim \text{RV}(\alpha)$ and

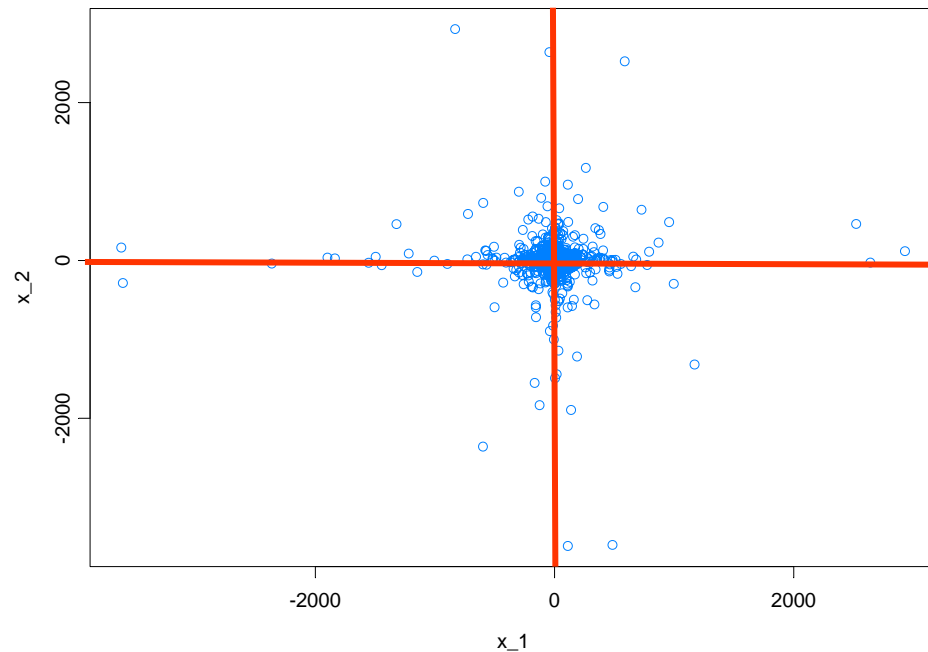
$$X_t = \sigma_t Z_t, \quad \log \sigma_t^2 = \phi_0 + \phi_1 \log \sigma_{t-1}^2 + \varepsilon_t, \quad \{\varepsilon_t\} \sim \text{IIDN}(0, \sigma^2)$$

Then $\mathbf{Z}_n = (Z_1, \dots, Z_n)'$ is regular varying with index α and so is

$$\mathbf{X}_n = (X_1, \dots, X_n)' = \text{diag}(\sigma_1, \dots, \sigma_n) \mathbf{Z}_n$$

with spectral distribution concentrated on $(\pm 1, 0)$, $(0, \pm 1)$.

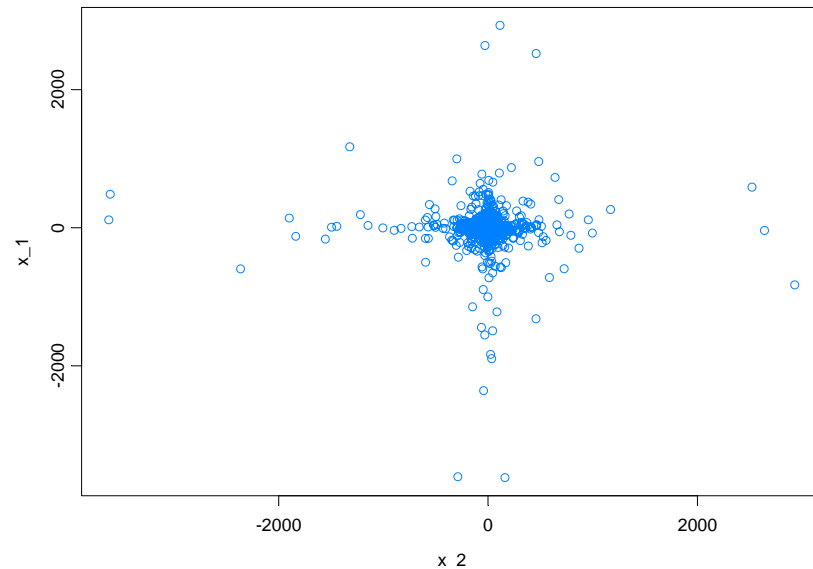
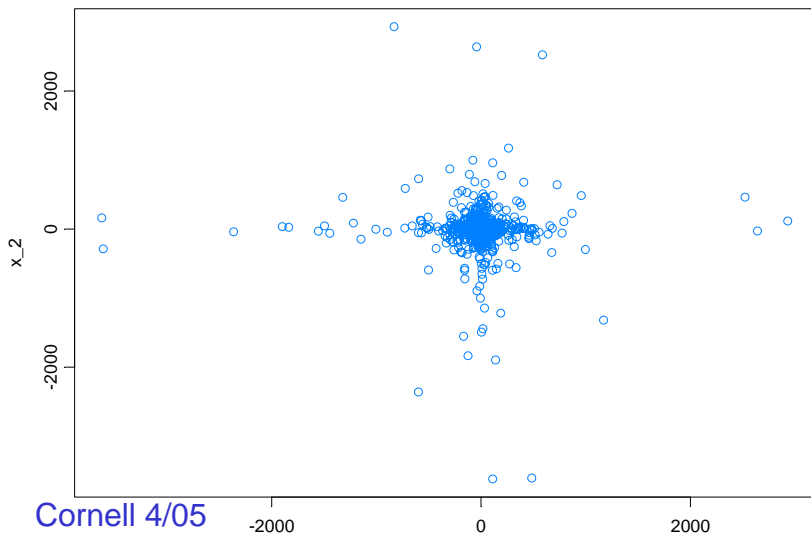
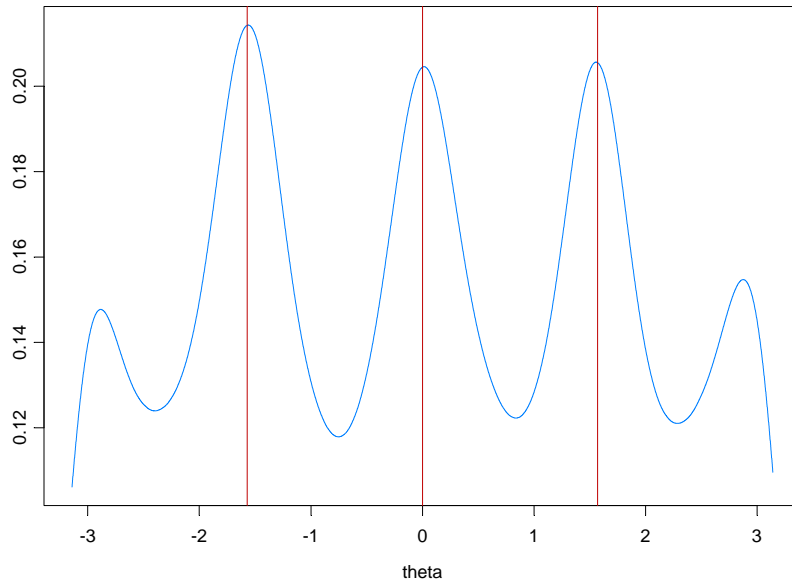
Figure: plot of
 (X_t, X_{t+1}) for
realization of 10,000.



Examples (cont)

Example: SV model $X_t = \sigma_t Z_t$

- SV processes are time-reversible if log-volatility is Gaussian.
- Asymptotically time-reversible if log-volatility is nonGaussian



Point process convergence

Theorem (Davis & Hsing `95, Davis & Mikosch `97). Let $\{\mathbf{X}_t\}$ be a stationary sequence of random m -vectors. Suppose

(i) finite dimensional distributions are jointly regularly varying (let $(\theta_{-k}, \dots, \theta_k)$ be the vector in $\mathcal{S}^{(2k+1)m-1}$ in the definition).

(ii) mixing condition $A(a_n)$ or strong mixing.

(iii) $\limlimsup_{k \rightarrow \infty} P(\bigvee_{k \leq |t| \leq r_n} |\mathbf{X}_t| > a_n y \mid |\mathbf{X}_0| > a_n y) = 0$.

Then

$$\gamma = \lim_{k \rightarrow \infty} E(|\theta_0^{(k)}|^\alpha - \bigvee_{j=1}^k |\theta_j^{(k)}|)_{+} / E|\theta_0^{(k)}|^\alpha \quad (\text{extremal index})$$

exists. If $\gamma > 0$, then

$$N_n := \sum_{t=1}^n \varepsilon_{\mathbf{X}_t/a_n} \xrightarrow{d} N := \sum_{i=1}^{\infty} \sum_{j=1}^{\infty} \varepsilon_{P_i Q_{ij}},$$

Point process convergence(cont)

- (P_j) are points of a Poisson process on $(0, \infty)$ with intensity function

$$v(dy) = \gamma \alpha y^{-\alpha-1} dy.$$

- $\sum_{j=1}^{\infty} \varepsilon_{Q_{ij}}$, $i \geq 1$, are iid point process with distribution Q , and Q is the weak limit of

$$\lim_{k \rightarrow \infty} E(|\theta_0^{(k)}|^\alpha - \bigvee_{j=1}^k |\theta_j^{(k)}|)_+ I.(\sum_{|t| \leq k} \varepsilon_{\theta_t^{(k)}}) / E(|\theta_0^{(k)}|^\alpha - \bigvee_{j=1}^k |\theta_j^{(k)}|)_+$$

Remarks:

1. GARCH and SV processes satisfy the conditions of the theorem.
2. Limit distribution for sample extremes and sample ACF follows from this theorem.

Extremes for GARCH and SV processes

Setup

- $X_t = \sigma_t Z_t$, $\{Z_t\} \sim \text{IID}(0,1)$
- X_t is RV (α)
- Choose $\{b_n\}$ s.t. $nP(X_t > b_n) \rightarrow 1$

Then

$$P^n(b_n^{-1} X_1 \leq x) \rightarrow \exp\{-x^{-\alpha}\}.$$

Then, with $M_n = \max\{X_1, \dots, X_n\}$,

(i) GARCH:

$$P(b_n^{-1} M_n \leq x) \rightarrow \exp\{-\gamma x^{-\alpha}\},$$

γ is extremal index ($0 < \gamma < 1$).

(ii) SV model:

$$P(b_n^{-1} M_n \leq x) \rightarrow \exp\{-x^{-\alpha}\},$$

extremal index $\gamma = 1$ no clustering.

Extremes for GARCH and SV processes (cont)

(i) GARCH: $P(b_n^{-1}M_n \leq x) \rightarrow \exp\{-\gamma x^{-\alpha}\}$

(ii) SV model: $P(b_n^{-1}M_n \leq x) \rightarrow \exp\{-x^{-\alpha}\}$

Remarks about extremal index.

(i) $\gamma < 1$ implies clustering of exceedances

(ii) Numerical example. Suppose c is a threshold such that

$$P^n(b_n^{-1}X_1 \leq c) \sim .95$$

Then, if $\gamma = .5$, $P(b_n^{-1}M_n \leq c) \sim (.95)^{.5} = .975$

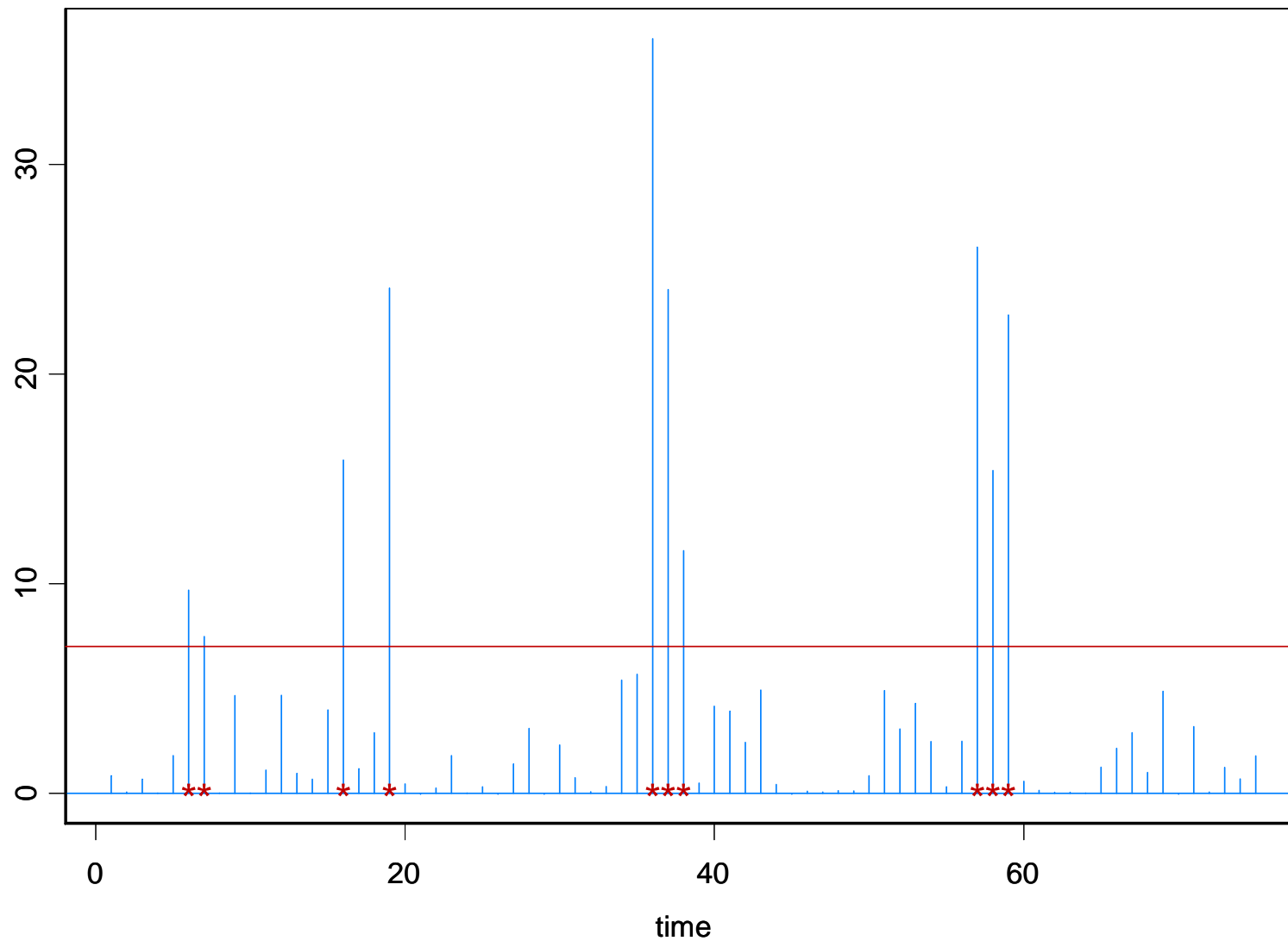
(iii) $1/\gamma$ is the *mean cluster size* of exceedances.

(iv) Use γ to *discriminate* between GARCH and SV models.

(v) Even for the light-tailed SV model (i.e., $\{Z_t\} \sim \text{IID } N(0,1)$), the *extremal index* is 1 (see Breidt and Davis '98)

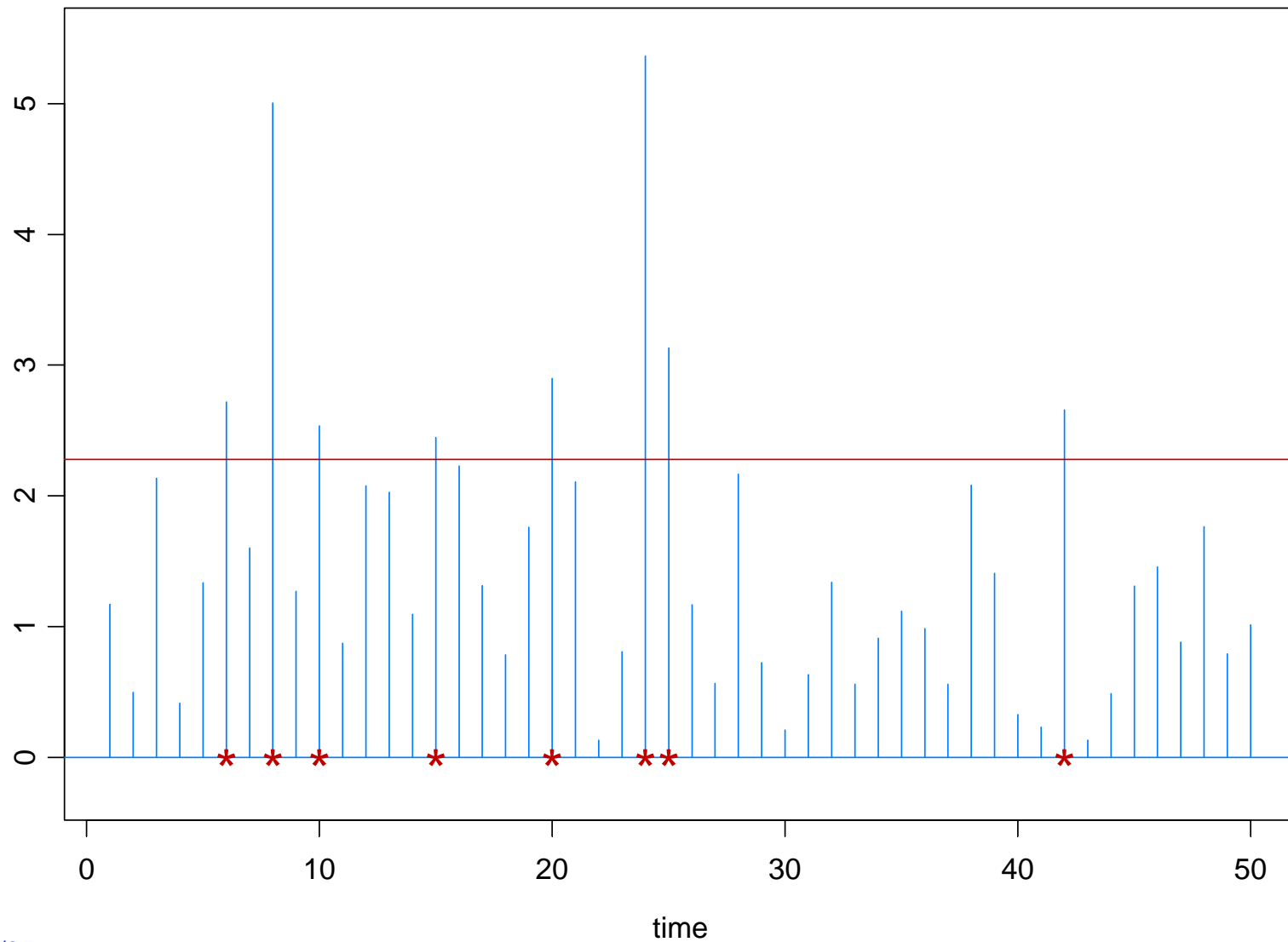
Extremes for GARCH and SV processes (cont)

Absolute values of ARCH

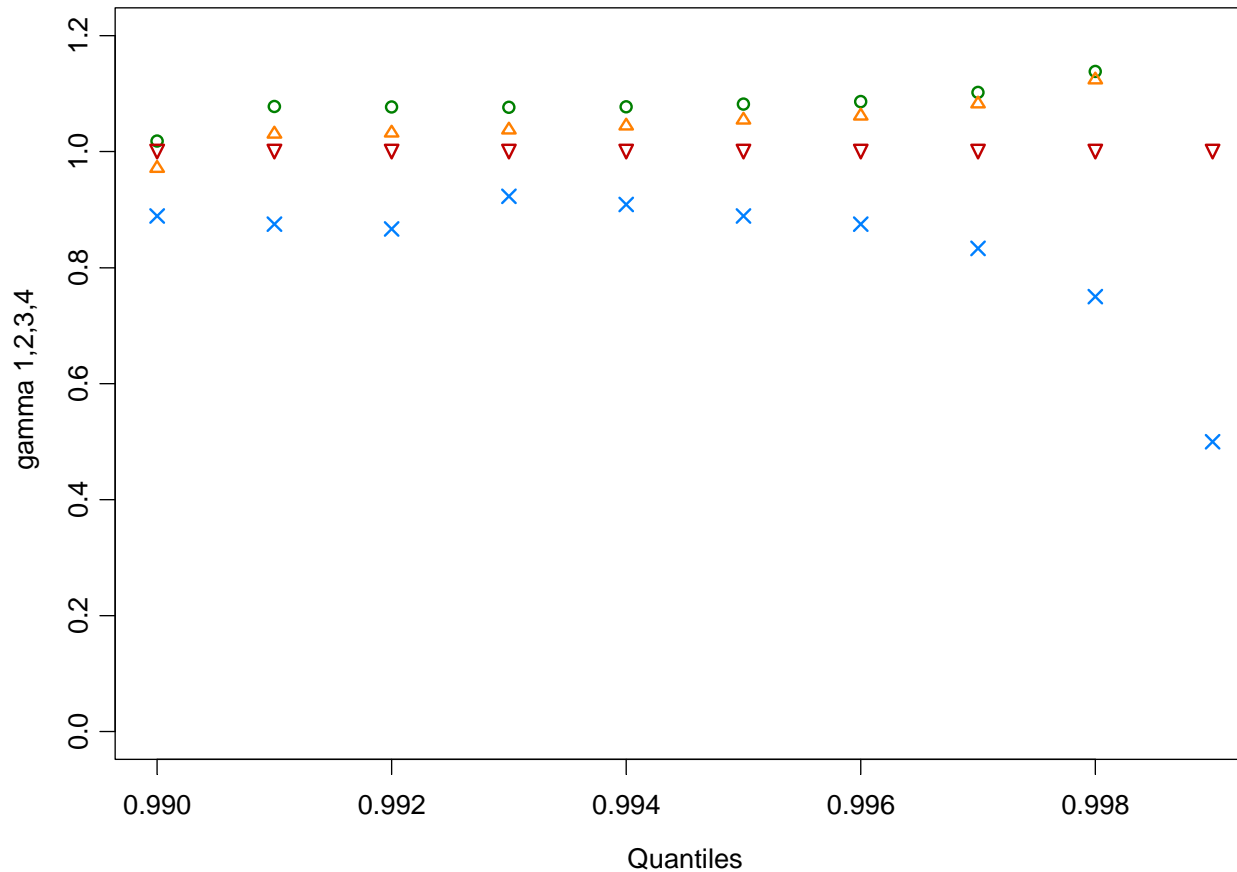


Extremes for GARCH and SV processes (cont)

Absolute values of SV process



Extremal Index Estimates for Amazon



γ_1 = block method

γ_2 = 1/(mean cluster size)

γ_3 = interval method (Ferro and Segers)

γ_4 = interval method (Ferro and Segers)

Summary of results for ACF of GARCH(p,q) and SV models

GARCH(p,q)

$\alpha \in (0,2)$:

$$(\hat{\rho}_X(h))_{h=1,\dots,m} \xrightarrow{d} (V_h / V_0)_{h=1,\dots,m},$$

$\alpha \in (2,4)$:

$$(n^{1-2/\alpha} \hat{\rho}_X(h))_{h=1,\dots,m} \xrightarrow{d} \gamma_X^{-1}(0)(V_h)_{h=1,\dots,m}.$$

$\alpha \in (4,\infty)$:

$$(n^{1/2} \hat{\rho}_X(h))_{h=1,\dots,m} \xrightarrow{d} \gamma_X^{-1}(0)(G_h)_{h=1,\dots,m}.$$

Remark: Similar results hold for the sample ACF based on $|X_t|$ and X_t^2 .

Summary of results for ACF of GARCH(p,q) and SV models (cont)

SV Model

$\alpha \in (0, 2)$:

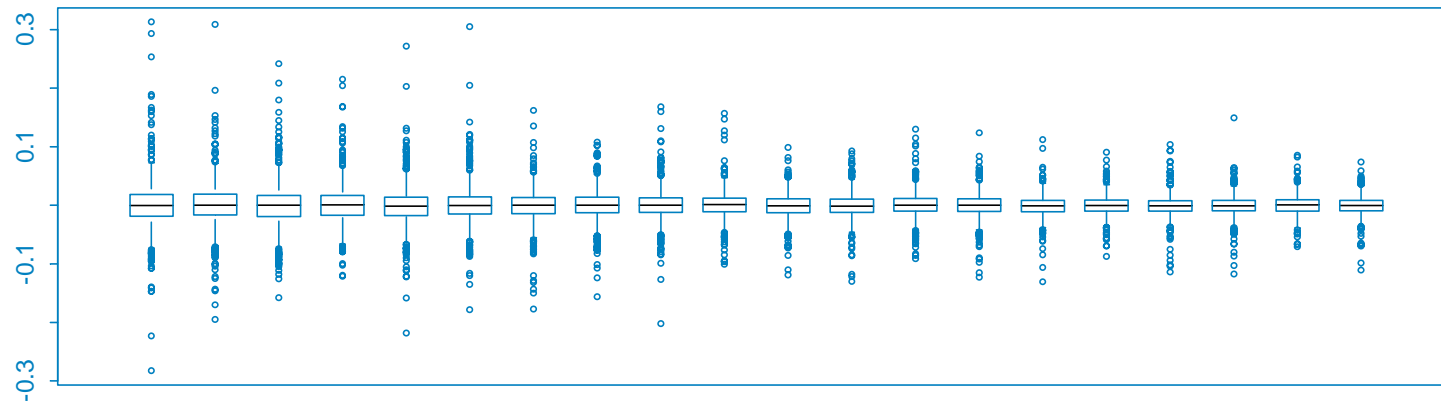
$$(n / \ln n)^{1/\alpha} \hat{\rho}_X(h) \xrightarrow{d} \frac{\|\sigma_1 \sigma_{h+1}\|_\alpha}{\|\sigma_1\|_\alpha^2} \frac{S_h}{S_0}.$$

$\alpha \in (2, \infty)$:

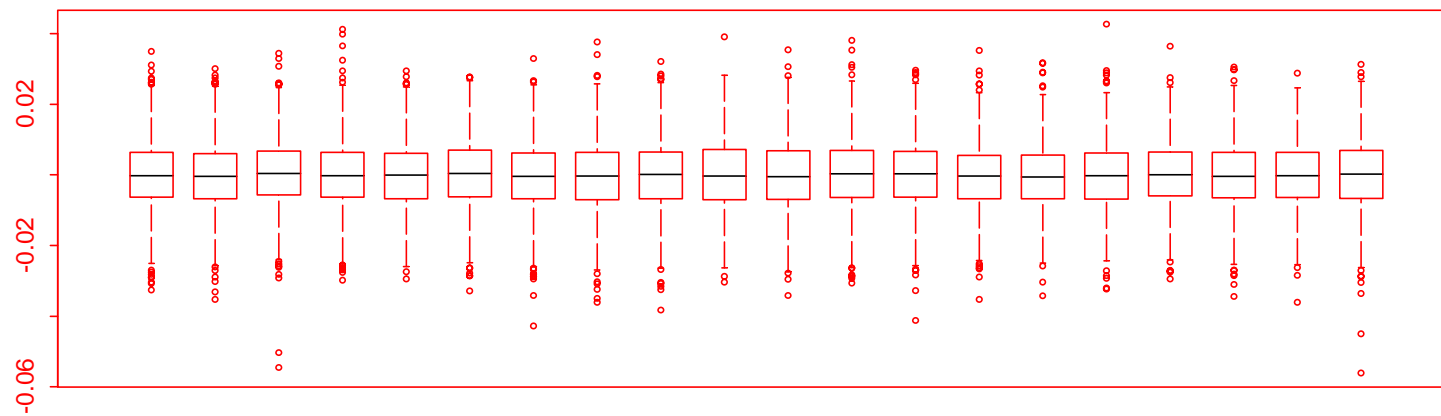
$$\left(n^{1/2} \hat{\rho}_X(h) \right)_{h=1, \dots, m} \xrightarrow{d} \gamma_X^{-1}(0) (G_h)_{h=1, \dots, m}.$$

Sample ACF for GARCH and SV Models (1000 reps)

(a) GARCH(1,1) Model, n=10000

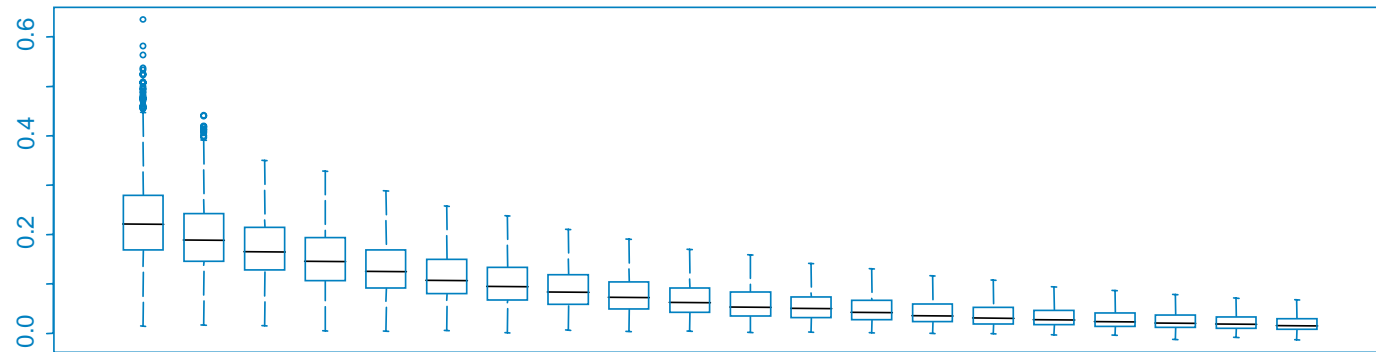


(b) SV Model, n=10000

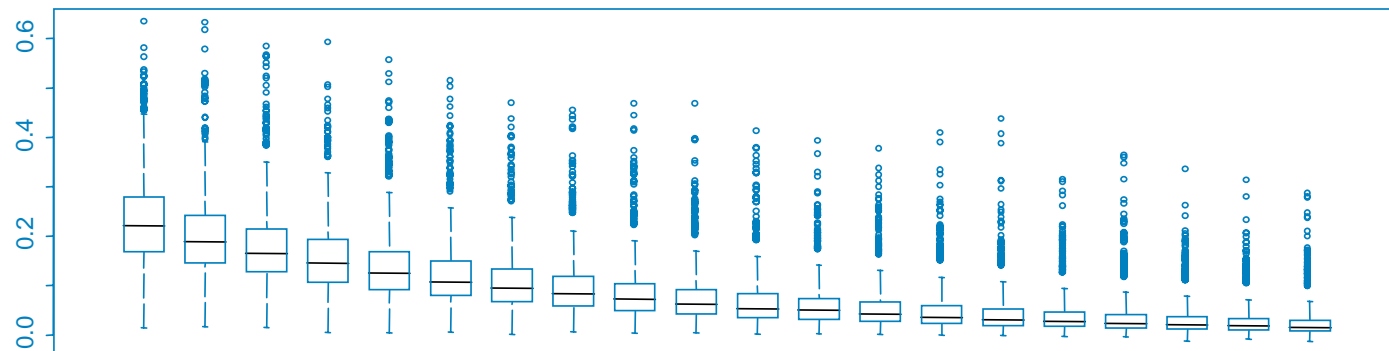


Sample ACF for Squares of GARCH (1000 reps)

(a) GARCH(1,1) Model, n=10000

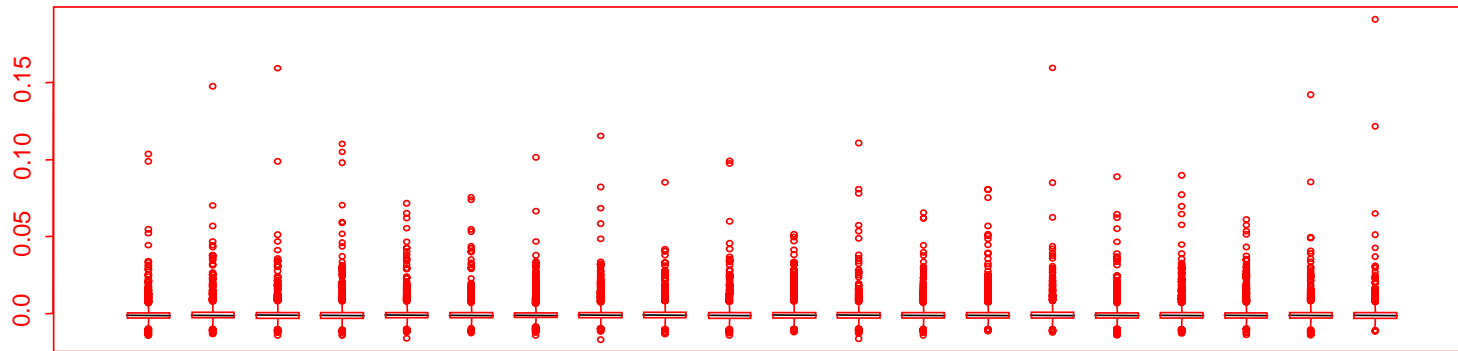


b) GARCH(1,1) Model, n=100000

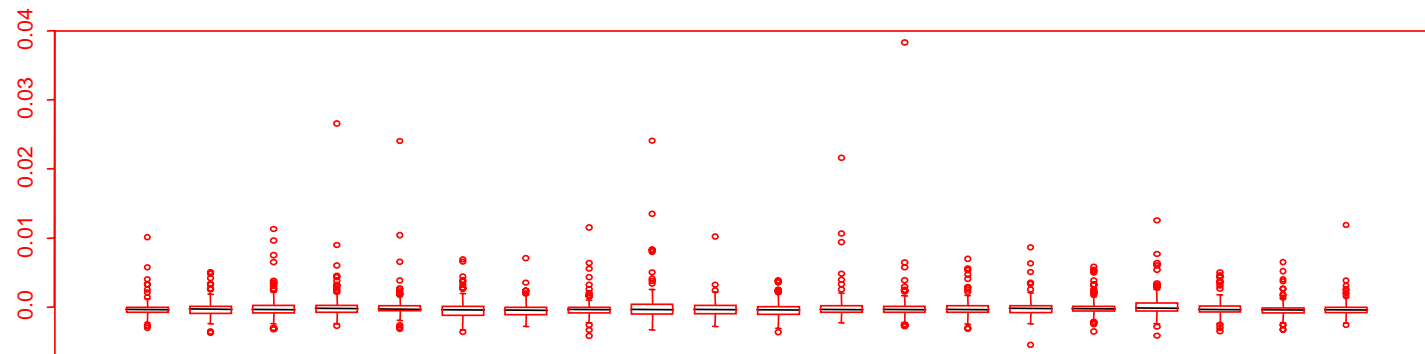


Sample ACF for Squares of SV (1000 reps)

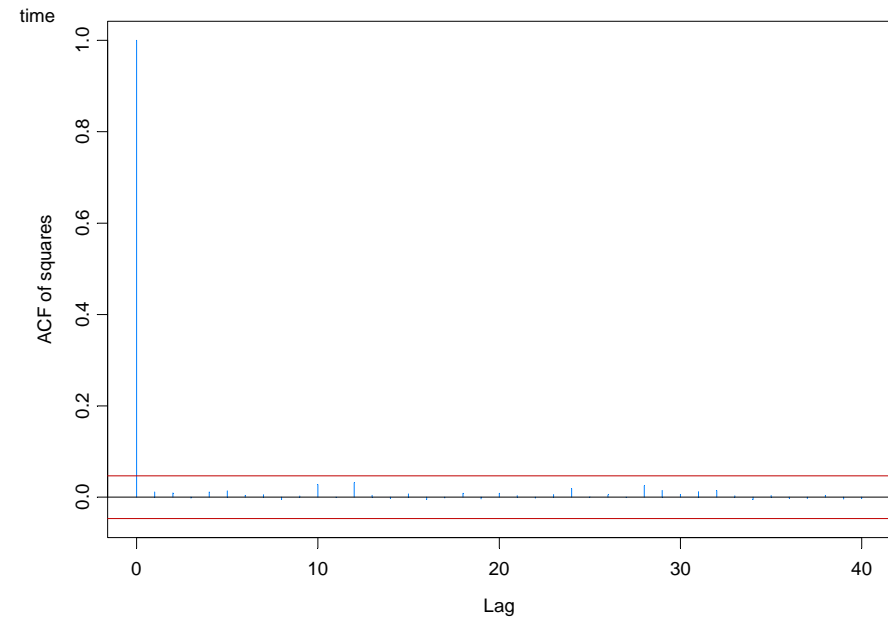
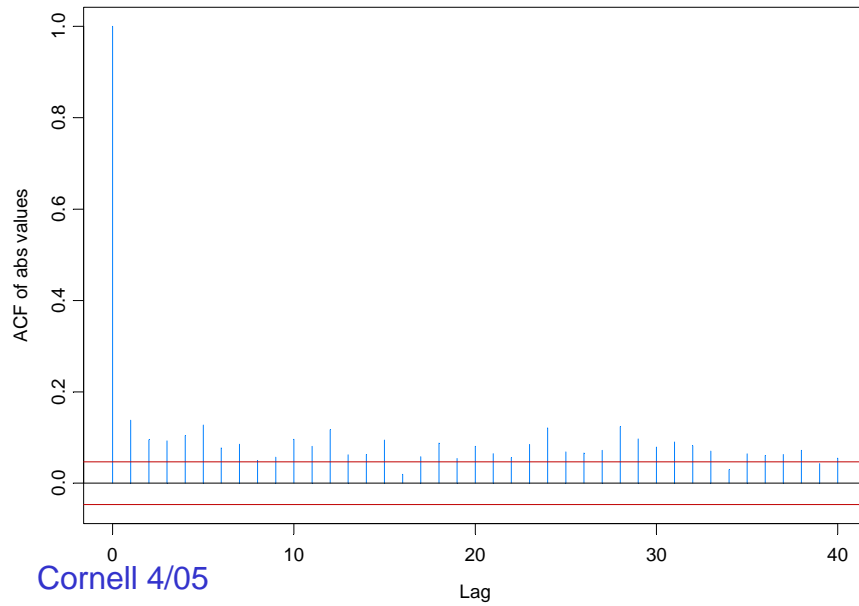
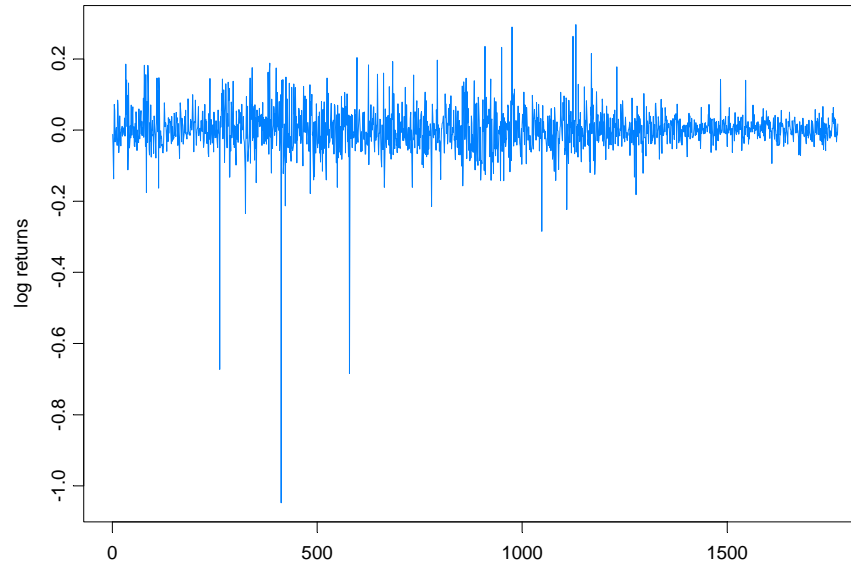
(c) SV Model, n=10000



(d) SV Model, n=100000



Example: Amazon-returns (May 16, 1997 – June 16, 2004)

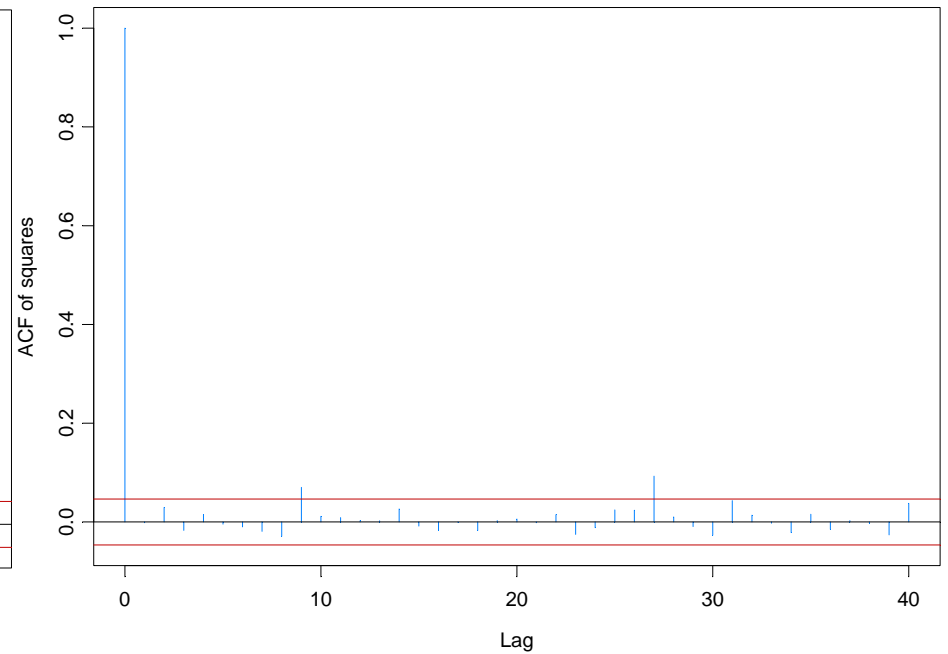
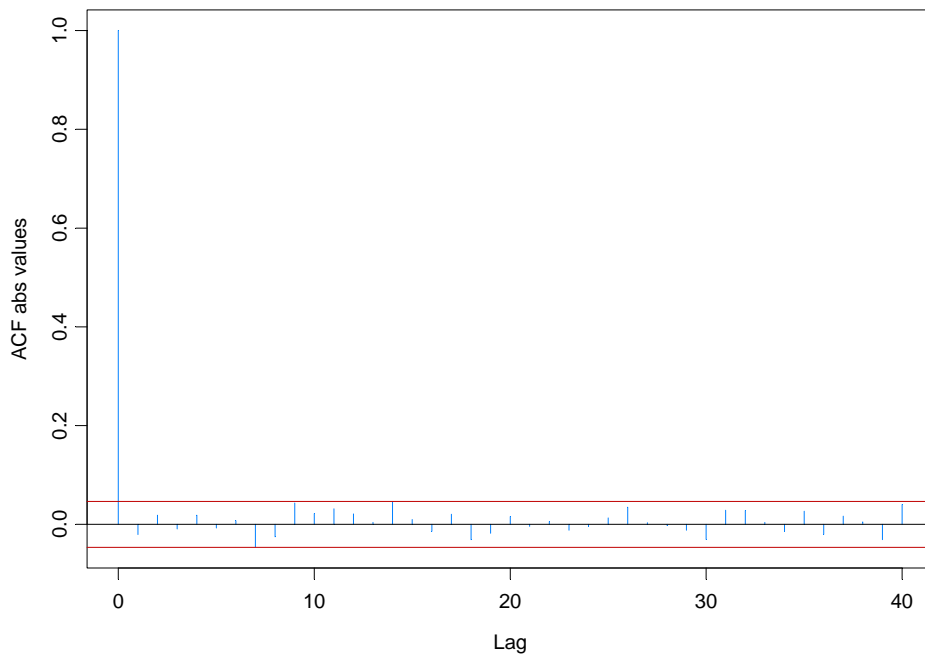


Amazon returns (GARCH model)

GARCH(1,1) model fit to Amazon returns:

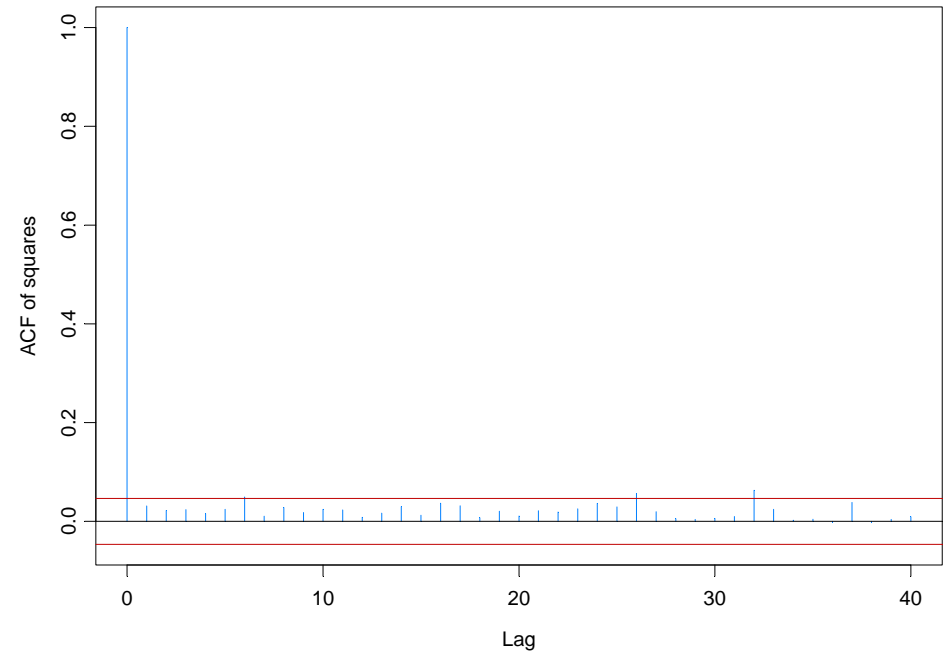
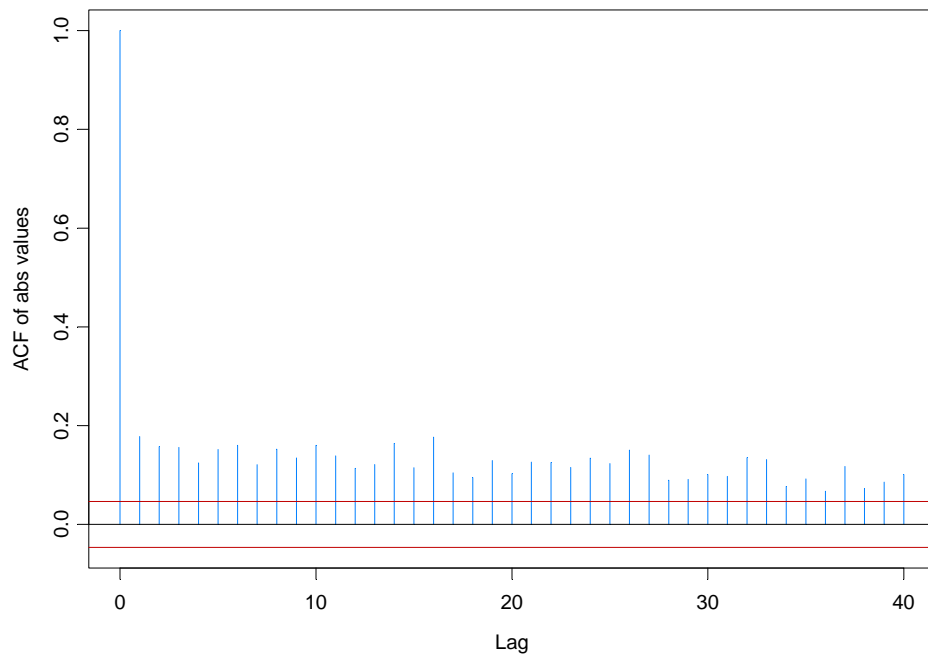
$$\alpha_0 = .00002493, \alpha_1 = .0385, \beta_1 = .957, X_t = (\alpha_0 + \alpha_1 X_{t-1}^2)^{1/2} Z_t, \{Z_t\} \sim \text{IID } t(3.672)$$

Simulation from GARCH(1,1) model



Amazon returns (SV model)

Stochastic volatility model fit to Amazon returns:



Wrap-up

- *Regular variation* is a flexible tool for modeling both *dependence* and *tail heaviness*.
- Useful for establishing *point process convergence* of heavy-tailed time series.
- *Extremal index* $\gamma < 1$ for GARCH and $\gamma = 1$ for SV.
- ACF has faster convergence for SV.