Fundamental concepts of functional data analysis

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Examples of functional data

The horizontal component of the magnetic field measured at Honolulu magnetic observatory.
Average log–precipitation at St. Johns, Canada. The thick line shows a smooth.
Smoothed growth curves of 54 girls age 1–18.
Smoothing and dimension reduction have been major themes of FDA research.

Basis expansion:

\[ X_n(t) \approx \sum_{m=1}^{M} c_{nm} B_m(t), \quad 1 \leq n \leq N. \]

Derivatives can be computed.
Acceleration curves of 54 girls with the mean function in bold.
Continuously registered acceleration curves of the 54 girls.
Functional response model:

\[ Y_i = \int \psi(s)X_i(s)ds + \varepsilon_i, \quad 1 \leq i \leq N. \]

Function–on–function regression:

\[ Y_i(t) = \int \psi(t, s)X_i(s)ds + \varepsilon_i(t). \]

Autoregression:

\[ X_i(t) = \int \psi(t, s)X_{i-1}(s)ds + \varepsilon_i(t). \]

Dimension of functional parameters, e.g. \( \psi \), is larger than the sample size \( N \). Regularization has been a major theme of FDA research.
Octane ratings of 60 gasoline samples.
Left: Near infrared spectrum of a gasoline sample with index 1; Right: differences between the spectrum of the samples with indexes 2 and 1 (continuous) and 5 and 1 (dashed).
Time series of functions

USA: male death rates (2003)

Log death rate

Age

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Predicted US male log mortality rate curves in rainbow code; years close to 2011 are in red, those close to 2040 in violet.
Spatial functional data

35 Canadian weather stations, Calgary marked with a square.
True and predicted (kriged) temperature functions at Calgary.
Some new directions

• Clustering and classification of huge sets of functions (spectral profiles of stars, expression profiles of proteins)

• High dimensional functional regressions
  \( Y_i(t) \) protein expressions, \( X_g(s) \) gene expressions, \( 1 \leq g \leq G \), 
  \( G \) can be several thousand for one \( i \).

• Manifold domains (second generation FD) 
  (physical domains (brain), domains generated by restrictions)

• Multilevel dependence
  (patient, visit, records from brain regions, scalar covariates)

• Connectivity and Network identification (possibly multilevel)
  (Several types of measurements of brain activity, which regions interact at which levels?)

• Extreme events in set of functions indexed by time and space
  \( X_n(s, t) \), temperature in year \( n \) at location \( s \) on day \( t \),
  probabilities of heat waves or droughts).
Examples of asymptotic techniques

Functional observations are assumed to be functions in an $L^2$ space (so that variance-like objects can be defined).

$$\nu_p(X) = \left( E \left\{ \int X^2(t) dt \right\}^{p/2} \right)^{1/p} < \infty.$$ 

If $\nu_2^2(X) = E \int X^2(t) dt < \infty$, we say that $X$ is square integrable. We must often assume that $\nu_4(X) < \infty$. 

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Dependence in time series

Bernoulli shifts:

\[ X_n = f(\varepsilon_n, \varepsilon_{n-1}, \ldots), \]

the \( \varepsilon_i \) are iid elements taking values in a measurable space \( S \), and \( f \) is a measurable function \( f : S^\infty \rightarrow L^2 \).

Suppose \( \{\varepsilon'_i\} \) are independent copies of \( \{\varepsilon_i\} \) and set

\[ X_n^{(m)} = f(\varepsilon_n, \varepsilon_{n-1}, \ldots, \varepsilon_{n-m+1}, \varepsilon'_{n-m}, \varepsilon'_{n-m-1}, \ldots) \]

Approximability condition:

\[ \sum_{m=1}^{\infty} \nu_p(X_n - X_n^{(m)}) < \infty. \]
Estimation of second order structure

Covariance function \((EX_i = 0)\):

\[
c(t, s) = \text{cov}(X_1(t), X_1(s))
\]

Sample covariance function:

\[
\hat{c}(t, s) = \frac{1}{N} \sum_{n=1}^{N} (X_n(t) - \bar{X}_N(t))(X_n(s) - \bar{X}_N(s)).
\]

Under approximability with \(p = 4\),

\[
E\|\hat{C} - C\|_S^2 = O(N^{-1}).
\]

(Hilbert–Schmidt norm)

Consistency, with the same rate, of the eigenfunctions follows. These eigenfunctions, called functional principal components often form an optimal basis for expansions.
Invariance principle

Functional partial sum process:

\[ V_N(x, t) = \frac{1}{\sqrt{N}} \sum_{n=1}^{\lfloor Nx \rfloor} X_n(t), \quad 0 \leq t, x \leq 1. \]

Under approximability with \( p = 2 + \delta \), there are Gaussian processes \( \Gamma_N(x, t) \) such that for every \( N \)

\[ \sup_{0 \leq x \leq 1} \| V_N(x, \cdot) - \Gamma_N(x, \cdot) \|_2 = o_p(1). \]

\[ \{ \Gamma_N(x, t), 0 \leq x, t \leq 1 \} \overset{D}{=} \{ \Gamma(x, t), 0 \leq x, t \leq 1 \}, \]

\[ \Gamma(x, t) = \sum_{i=1}^{\infty} \lambda_i^{1/2} W_i(x) \phi_i(t). \]

The \( W_i \) are independent standard Wiener processes, \( \lambda_i \) and \( \phi_i \) are eigenvalues and the eigenfunction of the long–run covariance function (different from \( c \) above).