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Chaotic Maps with Slowly Decaying Correlations and Intermittency

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Dedicated to Miklos Csörgő on the occasion of his 70th birthday.

Abstract. Established stochastic models for the class of discrete-time long-memory stationary processes are linear and Gaussian. The chaotic intermittency maps provide alternative non-linear, non-Gaussian models for this class of processes. A formal algebraic definition of the intermittency maps, which allows for a distinction to be made between the regular and the cusp families of maps, is given. Examples of intermittency maps belonging to both these families are given, and their properties, especially the invariant density, correlation function, and the recurrence behaviour, are discussed. A discussion is also given of how intermittency maps are applied for modelling the Ethernet packet traffic.

1 Introduction

Consider a weakly stationary process, $\{x_t\}$ $(t = \dots, -1, 0, 1, \dots)$, with mean $\mu_x = E(x_t)$, covariance function $R_x(u) = E[\{x_t - \mu_x\}\{x_{t+u} - \mu_x\}]$ $(t, u = \dots, -1, 0, 1, \dots)$,

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correlation function $r_x(u) = R_x(u)/R_x(0)$ and spectral density function

$$f_x(\lambda) = (2\pi)^{-1} \sum_{u=-\infty}^{\infty} R_x(u) \exp(-iu\lambda). \tag{1.1}$$

Many of the established finite parameter stochastic models currently employed for describing the behaviour of $\{x_t\}$, for example the class of autoregressive-moving average models, may be said to possess short-memory, in the sense that they have an absolutely summable covariance function:

$$\sum_{u=-\infty}^{\infty} |R_x(u)| < \infty, \tag{1.2}$$

and a non-vanishing finite spectral density function:

$$0 < f_x(\lambda) < \infty, \text{ all } \lambda \in [-\pi, \pi]. \tag{1.3}$$

For this class of models, the correlation function, $r_x(u)$, tends to zero, as $u \to \infty$, extremely rapidly, often at an exponential rate, meaning that there exist constants, B and ρ such that $|r_x(u)| \leq B\rho^{|u|}$, $0 < \rho < 1$, as $u \to \infty$. It has however been recognised that many observed time series, especially those occurring in Hydrology, Telecommunication traffic and Econometrics, display "long-memory" in the sense that they have correlations which decay to zero at a sub-exponential, polynomial or slower, rate. Many of the suggested stochastic models for characterizing this behaviour are however linear and related to the popular class of non-stationary integrated processes, see Box and Jenkins [10], since they typically postulate that the observed long memory process, $\{y_t\}$, say, is such that the process, $\{x_t\}$, obtained by taking its dth fractional difference, $0 < d < 0.5$,

$$x_t = (1-z)^d y_t, \tag{1.4}$$

where z denotes the backshift operator, $z^j y_t = y_{t-j}$, has short-memory and its covariance and spectral density functions satisfy conditions (1.2) and (1.3), respectively.

The process, $\{y_t\}$, so defined is stationary and possesses long-memory in the following sense:

a) its correlation function, $r_y(u)$, decreases to zero as $u \to \infty$ at a polynomial rate, $r_y(u) \sim Bu^{2d-1}$;
b) the covariance function, $R_y(u)$, is not summable:

$$\sum_{u=-\infty}^{\infty} R_y(u) = \infty; \tag{1.5}$$

c) the spectral density function, $f_y(\lambda)$, has a singularity at the origin,

$$f_y(\lambda) \sim |\lambda|^{-2d} \Delta(1/|\lambda|), \quad \text{as } \lambda \to 0, \tag{1.6}$$

where $\Delta(\lambda)$ is a slowly-varying function at infinity.

If $-0.5 < d < 0$, then the process is stationary and has an absolutely-summable covariance function and it is often called an intermediate-memory or antipersistent process, see McLeod [35].

Typically, $\{x_t\}$, is assumed to be a Gaussian process in the sense that its innovations process, $\{\epsilon_t\}$, is assumed to be Gaussian; this assumption may however be relaxed by allowing $\{\epsilon_t\}$ to have a distribution belonging to the domain of attraction of an infinite variance stable law, see Kokoszka and Taqqu [24], and Bhansali and Kokoszka [9], and the references therein.

A related yet slightly different variation of the long-memory model has been considered in Giraitis, Robinson and Surgailis [16], where the problem of modelling the empirically-observed long-memory in the volatility of financial returns time series, that is, in the absolute values, or the squares, of the returns series, is considered. These authors extend the well-known ARCH family of models by introducing and providing a rigorous treatment of the LARCH, long-memory ARCH, models; see also Berkes and Horváth [8] for further analysis of this model.

In this paper, a new and an entirely different approach to modelling long-memory is considered, namely by a deterministic sequence, $\{w_n\}(n = 0, 1, \dots)$, generated by iteratively applying a one-dimensional map of the following form:

$$w_{n+1} = \zeta(w_n) \ (n = 0, 1, \dots), \tag{1.7}$$

where w_0 is real-valued and specifies the *initial condition* for the iterative scheme specified by (1.7), $\zeta : J \to J$ is a non-linear map (function), J denotes a bounded closed interval of the real line, $\mathbb{R}$.

It is recognized, see Rasband [41], that such maps can generate *chaotic dynamics* and although the successive values of w_n evolve in a deterministic way from the immediately preceeding one, they could still be *chaotic* and sensitive to the choice of the initial condition, w_0. This is in contrast to a *random system* in which the present observation does not entirely depend on the previous observation.

In section 2, the relationship between a weakly stationary process and the deterministic sequence, $\{w_n\}$, generated by the map (1.7) is discussed and we explain why $\{w_n\}$ may nevertheless define a weakly stationary process.

Many of the established chaotic maps, however, still imply that the generating sequence has a short memory and has a correlation function, $r_y(u)$, which decays exponentially to 0, as $u \to \infty$. This point is discussed further in section 3 where we review the behaviour of the correlation function for three well known chaotic maps, namely the Tent, Logistic, and Bernoulli maps.

There has lately been much development on devising chaotic *intermittency* maps exhibiting long or intermediate memory, that is, maps exhibiting correlation functions which decay at a polynomial rate, $r_y(u) \sim Bu^{-\alpha}$, $0 < \alpha < 2$, or even more slowly. A main objective of this paper is to introduce an algebraic definition of the intermittent family of maps; we also demonstrate how the various known examples of intermittency maps with slowly decaying correlations may be brought into the framework of the definition we introduce.

The paper concludes in section 6, where a brief account is given of how intermittency maps are applied for modelling the Ethernet packet traffic.

The paper is intended for mathematical statisticians and especially time series analysts who may not necessarily be familiar with the recent developments in the *Dynamical Systems* literature on the intermittent family of maps. We assume also that the reader is familiar with classical mathematical analysis, as available in Rudin [42], for example.

2 Chaotic maps and ergodic theory

Suppose J is a closed interval and μ a probability measure on the Borel subsets of J. We say that a map $\zeta : J \to J$ is measure preserving if $\mu(\zeta^{-1}A) = \mu(A)$ for all Borel sets $A \subset J$. In this case, we say that μ is a ζ-invariant measure. A measure preserving transformation, $\zeta(w)$, is called ergodic if, for every Borel set $A \in J$, $\zeta^{-1}A = A$ implies $\mu(A) = 0$ or $\mu(A) = 1$. There are many equivalent characterizations of ergodicity, see e.g. Section 1.5 of Walters [46],

In this section, we assume that the map, $\zeta : J \to J$, occurring in (1.7) is ergodic and admits an invariant measure, μ. Birkhoff's Ergodic Theorem, see e.g. Theorem 1.14 Walters [46], implies that:

(1) for a continuous function $\varphi : J \to \mathbb{R}$,

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=0}^{n-1} \varphi(\zeta^k(w)) = \int_J \varphi d\mu, \quad \mu - \text{a.e.} \tag{2.1}$$

(2) for all subintervals $A = [a, b] \subset J$, such that

$$\mu(\{a\} \cup \{b\}) = 0, \tag{2.2}$$

$$\frac{1}{n}\big(\sharp\{k \in [0, n-1] : \zeta^k(w) \in A\}\big) \to \mu(A), \quad (n \to \infty) \quad \mu - \text{a.e.} \tag{2.3}$$

Note that if (2.2) does not hold, then (2.3) is not necessarily true. In (2.1) and (2.3), $\zeta^n(w) = \zeta(\zeta(\dots\zeta(w)\dots))$ denotes an n-fold application of the transformation $\zeta(w)$. We shall mainly be concerned with absolutely continuous invariant probability measures, μ, such that $d\mu(w) = \chi(w)dw$; $\chi(w)$ is called the time invariant density for $\zeta(w)$. In many examples, it is possible to obtain a precise formula for the invariant density, $\chi(w)$, but this need not necessarily be so and $\chi(w)$ does not always admit a closed form, analytic expression.

Ergodicity enables probability to be assigned, in a relative frequency sense, to the event that a typical orbit, $\{w_n, n \in \mathbb{N}\}$, visits a given set $A \subset J$, where $\mathbb{N} = \{0, 1, \dots\}$ is the set of all non-negative integers, and makes it possible to replace the "time average" occurring on the left of (2.1) by the "ensemble average" defined on its right.

On letting $\zeta'(w)$ denote the first derivative of $\zeta(w)$ at w, a critical point, c, of $\zeta(w)$ is a solution of the equation $\zeta'(c) = 0$. We say that a critical point is of order ℓ, with $\ell > 1$, if $\zeta'(w) = O(|w - c|^{\ell-1})$ in a neighbourhood of c.

A fixed point, $\tilde{w}$, say, of the map $\zeta(w)$ is a solution of the equation $\zeta(\tilde{w}) = \tilde{w}$. We say that $\tilde{w}$ is an attracting fixed point if $|\zeta'(\tilde{w})| < 1$, a repelling fixed point if $|\zeta'(\tilde{w})| > 1$, and an indifferent, or a neutral fixed point, if $|\zeta'(\tilde{w})| = 1$. More generally, $\tilde{w}$ is a fixed point of *prime* period p, if $\zeta^p(\tilde{w}) = \tilde{w}$, but $(\zeta^j)'(\tilde{w}) \neq \tilde{w}$, $(j = 1, 2, \dots p - 1)$.

For a map, $\zeta(w)$, the Lyapunov exponent evaluated at some point w is given by

$$L(w) = \lim_{n\to\infty} \frac{1}{n} \log |(\zeta^n)'(w)|. \tag{2.4}$$

Positive Lyapunov exponents indicate sensitive dependence of $\{w_n, n \in \mathbb{N}\}$ on the initial condition, w_0, and hence would imply that the system is chaotic. It follows from (2.1) that if $\chi(w)$ is an invariant density for $\zeta(w)$, the Lyapunov exponent is given by

$$L = \int_J \log |\zeta'(w)| \chi(w) dw, \tag{2.5}$$

at all points w for which equation (2.1) holds.

Let $\varphi : J \to \mathbb{R}$ be a continuous function. The expectation and variance of $\varphi(w_n)$, for each $n \in \mathbb{N}$, are given by

$$E(\varphi(w_n)) = \int_J \varphi(w)\chi(w)dw \quad \text{and} \quad V(\varphi(w_n)) = E(\varphi(w_0)^2) - E(\varphi(w_0))^2.$$

For two continuous functions $\varphi, \psi : J \to \mathbb{R}$ and each $n \in \mathbb{N}$, the covariance, $R_{\varphi,\psi}(n)$, between $\varphi(w_n)$ and $\psi(w_0)$ is defined as follows:

$$\begin{aligned} R_{\varphi,\psi}(n) &= E\{\varphi(w_n)\psi(w_0)\} - E\{\varphi(w_n)\}E\{\psi(w_0)\} \\ &= \int_J \varphi(\zeta^n(w))\psi(w)\chi(w)dw - \int_J \varphi(w)\chi(w)dw \int_J \psi(w)\chi(w)dw. \end{aligned} \tag{2.6}$$

If $\psi(w) = \varphi(w) = w$, then $R_{\varphi,\psi}(n) = R_w(n)$ is the autocovariance function of the weakly stationary process $\{w_n, n \in \mathbb{N}\}$. Ergodic theory however enables us to consider also more general functions of w_n and w_0; for example, $\varphi(w_n) = w_n^2$ and $\psi(w_0) = w_0^2$, Lawrence [26].

In what follows, we restrict our attention to the class, $\mathcal{H}$, of Hölder continuous functions, φ, defined by the property that there exist $\gamma \in (0,1)$ and $C < \infty$, independent of x, y, such that

$$|\varphi(x) - \varphi(y)| \leq C|x-y|^\gamma, \quad \forall\, x, y \in J.$$

Let L^1 be a Banach space of integrable functions with respect to the Lebesgue measure and let $\|\cdot\|_1$ denote the standard L^1 norm. For evaluating the expectations occuring on the right of (2.6), and for also finding the invariant density, $\chi(w)$, associated with the map, $\zeta(w)$, a useful mathematical tool is the *Perron-Frobenius* operator, $(P_\zeta\varphi)(w)$, where $(P_\zeta\varphi)(w)$ denotes the operation of applying P_ζ to φ first, and then evaluating the result at the point w. If the *domain* and *range* of this operator are contained in L^1, then $(P_\zeta\varphi)(w)$ may be defined as follows:

$$(P_\zeta\varphi)(w) = \sum_{t:\zeta(t)=w} \frac{\varphi(t)}{|\zeta'(t)|}, \tag{2.7}$$

see Baladi [2] for a more general definition of this operator, which for ease of exposition is not given here. As shown by Lawrance [26], the invariant density, $\chi(w)$, is a fixed point, $(P_\zeta\chi)(w) = \chi(w)$, of the Perron Frobenius operator and satisfies

$$\sum_{t:\zeta(t)=w} \frac{\chi(t)}{|\zeta'(t)|} = \chi(w).$$

A map is said to be unimodal if it has only a single critical point. For a twice differentiable unimodal map admitting a critical point $c \in J$ of order $\ell > 1$, set $D_n(c) = |(\zeta^n)'(w)|_{w=\zeta(c)}$. A result of Nowicki and van Strien [37] shows that if

$$\sum_{n\geq 0} D_n(c)^{-1/\ell} < \infty, \tag{2.8}$$

then $\zeta(w)$ is ergodic and admits an invariant density $\chi(w)$.

For the class of twice differentiable unimodal maps, bounds on the rate of decay of correlations may also be written down, see Bruin, Luzzatto and van Strien [11], but the details are omitted as these bounds do not apply to the intermittent family of maps considered in Section 4.

For the latter family of maps, a method for obtaining an upper bound for the rate of decay of correlations is given below. The method applies to all functions, $\varphi, \psi \in \mathcal{H}$ such that $\int_J \psi(w)dw \neq 0$. On using the identity, see Lawrance et al [29],

$$\int_J \varphi(w)(P_\zeta \psi)(w)dw = \int_J \varphi(\zeta(w))\psi(w)dw,$$

it follows that

$$\begin{aligned} R_{\varphi,\psi}(n) &= \int_J \varphi(\zeta^n(w))\psi(w)\chi(w)dw - \int_J \varphi(w)\chi(w)dw \int_J \psi(w)\chi(w)dw \\ &= \int_J \varphi(w)\big((P_\zeta^n(\psi\chi))(w)\} - \chi(w)\int_J \psi(w)dw\big)dw \qquad (2.9) \\ &\leq \{\max_J |\varphi(w)|\}\big\{\|(P_\zeta^n(\psi\chi))(w) - \chi(w)\int_J \psi(t)dt\|_1\big\} \end{aligned}$$

where $P_\zeta^n\{\cdot\}(w) = P_\zeta\{P_\zeta\{\cdots\}\}(w)$ denotes denotes an $n - fold$ application of the Perron-Frobenius operator, evaluated at the point w. The result (2.9) is used in the appendix for obtaining a bound on the rate of decay of correlations. The approach used recognises the fact that $\psi \in \mathcal{H}$ is bounded, and hence it is possible to normalize it so that $\int_J \psi(w)dw = 1$. Moreover, since the invariant density, $\chi(w)$, is a fixed point of P_ζ, an estimate for the upper bound on the rate of decay of the correlation function may be obtained by finding how fast $\|(P_\zeta^n(\psi(w)\chi))(w) - \chi(w)\|_1$ converges to zero as $n \to \infty$; this last convergence rate is often called the *speed of convergence to equilibrium.*

If an explicit expression for the correlation function is unavailable, it would in addition be desirable to obtain also a lower bound on the decay of the correlation function. However, while within the class of all Hölder continuous functions, $\mathcal{H}$, it is possible to specify an upper bound for the rate of decay of the correlations which is independent of the given function, φ, this is not so for the lower bound, and the latter in general depends on φ (see Sarig [44] for details).

A concept related to the rate of decay of correlations is *the speed of mixing*, and for some special functions, φ, ψ, these two concepts coincide. Thus, for two intervals, $A, B \subset J$, let

$$M_n(A,B) = |\,\mu(A \cap \zeta^{-n}B) - \mu(A)\mu(B)|.$$

We say that the map $\zeta(w)$ is mixing if, for all $A, B \subset J$,

$$M_n(A,B) \to 0, \qquad (2.10)$$

as $n \to \infty$, and the rate at which this last convergence occurs is called the *speed of mixing.* If $\varphi = \mathcal{X}_A$ and $\psi = \mathcal{X}_B$, where $\mathcal{X}$ denotes the indicator function, then $M_n(A,B) = R_{\varphi,\psi}(n)$. However, since $\mathcal{X}_A$ and $\mathcal{X}_B$ are not Hölder continuous, a knowledge of the rate of decay of correlations does not provide information about the speed of mixing, and vice-versa.

3 Three well-known chaotic maps

As a prelude to our discussion of intermittent chaotic maps, it will be useful to discuss three of the widely-studied chaotic maps and to outline briefly some of the known results concerning their invariant densities and the rate of decay of correlations.

3.1 Beta-transformations. For $J = [0,1]$ and a $\beta > 1$, the map

$$B_\beta(w) = \beta w \mod 1,$$

defines the *beta-transformation* family of maps. For integer values of the parameter, β, the corresponding maps are ergodic and admit an invariant density, $\chi(w) = 1, \forall w \in J$, which coincides with the probability density function of a uniform distribution on $[0,1]$. Let $w_n = B_\beta^n(w)(n \in \mathbb{N})$, where $\beta > 1$ is an integer. The correlation function associated with this map can be computed explicitly. We have

$$\begin{aligned} R_{w,w}(n) &= \int_0^1 w B_\beta^n(w)\chi(w)dw - E\{w\}^2 = \int_0^1 w P_{B_\beta}^n\{w\chi(w)\}dw - \frac{1}{4} \\ &= \int_0^1 \frac{w^2}{\beta} + \frac{w}{2}\left(1 - \frac{1}{\beta}\right)dw - \frac{1}{4} = \frac{1}{12}\frac{1}{\beta^n}; \end{aligned}$$

and so $r_{w,w}(n) = \beta^{-n}$ decays exponentially to zero, as $n \to \infty$. Moreover, for $\beta = 2$, see Lawrence, Tsuneda and Kohda [29],

$$r_{w^2,w^2}(n) = \frac{1}{2^{2n+4}} + \frac{15}{2^{n+4}}, \quad n = 1, 2, \ldots$$

and which also decays exponentially to zero as $n \to \infty$.

When $\beta > 1$ is not an integer, an analytic closed-form expression for the invariant density is difficult to obtain. It is however shown in Parry [39] that the density is bounded both from above and below, and, in fact, it can be expressed as a β-expansion:

$$\chi_\beta(w) = \frac{1}{C_\beta} \sum_{B_\beta^n(1) > w}' \frac{1}{\beta^n},$$

where C_β is the normalizing constant, and $\sum'$ denotes the sum over all n such that $B_\beta^n(1) > w$ (see Parry [39] and Flatto and Lagarias [25]). Computational methods for approximating the invariant densities, applicable for all $\beta > 1$ and with fast convergence rates, are considered in Pollicott and Jenkinson [40]. In addition, although precise decay rates for the correlation functions are difficult to obtain, as they depend on the algebraic properties of β, Flatto and Lagarias [25], it is known that the rates of decay are exponentially fast, see Young [48].

The Lyapunov exponents for the beta transformations are easy to compute and equal $\log \beta$, implying that the map $B_\beta(w)$ becomes more and more chaotic as β increases.

3.2 Logistic maps. For $J = [0,1]$, the map defined by

$$L_\theta(w) = \theta w(1-w), \quad \theta \in [0,4]$$

defines the *logistic family* of maps. Each map in this family is unimodal and the case $\theta = 4$ has been widely studied, Devaney [15]. The existence of an invariant density for this map follows from the result (2.8) discussed in Section 2; in fact, the invariant density is given by

$$\chi_4(w) = \frac{1}{\pi\sqrt{w(1-w)}},$$

and it coincides with the density of a Beta$(\frac{1}{2}, \frac{1}{2})$ distribution, see [13] for the definition of the latter.

For the stationary sequence defined by $w_n = L_4^n(w)$, $(n \in \mathbb{N})$, it is also known that, see Lawrence, Tsuneda and Kohda [29], $r_{w,w}(n) = 0$, for all $n > 0$. Moreover for a wide class of polynomial functions, $p(w)$ and $q(w)$, it is also known that $r_{p(w),q(w)}(n) = 0, n > 0$, see Hall and Wolff [20].

It is relevant to note that the logistic map with $\theta = 4$ is related to the beta-transformation with $\beta = 2$ discussed in subsection 3.1. Thus let $\xi(w) = \sin^2(\frac{\pi}{2}w)$, with $w \in [0, 1]$. Then it is readily verified that $L(\xi(w)) = \xi(B_2(w))$ and the dynamics of the logistic maps with $\theta = 4$ are related to those of the $B_2(w)$ map in an elementary manner, see [15]

The behaviour of the invariant density and the correlation function for other values of θ is complex and sensitive to the choice of θ itself; the details are omitted to save space. Suffice to say that there is a set of values of θ of positive Lebesgue measure for which the correlations decay exponentially fast, see Young [47], and there is also a subset of values of θ for which the rate of decay of correlations is subexponential, see Bruin, Luzzatto and van Strien [11]. Hall and Wolff [20] investigate, by computer simulations, the empirical behaviour of the correlations for a range of values of θ.

3.3 Tent-maps. For this family of maps also $J = [0, 1]$ is the unit interval and it is defined by

$$T_\nu(w) = 1 - |2w - 1|^\nu,$$

where $\nu > 0$ is a parameter of the map. If $\nu \geq 2$, this family of maps is twice differentiable and unimodal.

For $\nu \in (0, 1/2)$, it is known that there can be no absolutely continuous invariant measure, μ, supported on the whole of the interval $[0, 1]$. In fact, 0 is an attracting fixed point, meaning that for all initial conditions, w_0, in a neighbourhood of 0, w_n converges to 0, as $n \to \infty$. In addition, computer simulations, see Hall and Wolff [21], suggest that this behaviour may hold for all initial conditions in J.

When $\nu = 1/2$, the invariant density, $\chi_{1/2}(w)$, is given by $\chi_{1/2}(w) = 2(1 - w)$. The resulting map, however, is a special case of the class of cusp intermittency maps discussed by Balakrishnan and Theunissen [6], see also Section 4.

For $\nu \in (1/2, 1)$, Hall and Wolff [21] study the tail behaviour of the invariant density, $\chi_\nu(w)$. However, a precise expression for the correlation function in this case, including that concerning the rate of decay, is as yet unavailable.

For $\nu = 1$, the map simplifies to the beta transformation with $\beta = 2$.

If $\nu > 1$, the invariant density is supported on the whole of $[0, 1]$ and is regularly varying, see Hall and Wolff [21]. Moreover, it follows from the results of Young [47] that, for all $\nu \geq 1$, the correlations functions, $r_\nu(n)$, decay exponentially fast, as $n \to \infty$.

4 The intermittent family of maps

This family of maps was first studied, see Manneville and Pomeau [34], in connection with the continuous-time Lorenz system, see [32] for a discussion of the latter. The main motivation was to construct a mathematical model for intermittent chaos, that is, phenomena exhibiting long periods of laminar behaviour together with short bursts of erratic behaviour, and which is also known to provide a route to chaos. The maps discussed earlier in section 3 are unsuitable for modelling this phenomenon; for example the Logistic map with $\theta = 4$ leads to chaos through bifurcations, see Rasband [41].

The intermittent family of maps also differs from the maps discussed in section 3 as regards its recurrence properties. Suppose that the initial condition, w_0, is such that $w_0 \in C$, where $C \subset J$ is some fixed interval. The recurrence property that we consider is the time, $T_C(w_0)$, say, for the orbit, $\{w_n, n \in \mathbb{N}\}$, of the map, $\zeta(w)$, to return back to C for the first time, having started from within this interval. To this end, define the following event:

$$S_C(m) = \{w_m \in C, w_{m-1} \notin C, \ldots, w_1 \notin C | w_0 \in C\} \quad (m = 1, 2, \ldots). \tag{4.1}$$

Note that $S_C(m) \subset J$, and hence the probability of $S_C(m)$ may be found from the invariant density, $\chi(w)$. Moreover, the probability distribution of $T_C(w_0)$ is given by:

$$P(T_C(w_0) = n \mid w_0 \in C) = K_C \int_{S_C(n)} \chi(w) \mathcal{X}_C(w) dw, \quad (n = 1, 2, \ldots)$$

where $K_C = 1/\int_C \chi(w)dw$ is a constant that depends on C, but not on n or w_0. Now, let

$$H_C(n) = P(T_C(w_0) \leq n \mid w_0 \in C) \quad (n = 1, 2, \ldots) \tag{4.2}$$

denote the distribution function of $T_C(w_0)$. Clearly, $H_C(n) \to 1$, as $n \to \infty$. The speed at which this last convergence occurs may be used for distinguishing a short-memory map from other maps and enables the former group of maps to be characterized in terms of this convergence rate. Indeed, a characteristic property of the short memory maps discussed in Section 3 is that $H_C(n) \to 1$, as $n \to \infty$, at an exponential rate, and moreover this property holds for *every* subinterval $C \subset J$. It has now been recognized that there exist maps for which this last result does not hold for every $C \subset J$, and there may exist intervals for which the rate of decay is subexponential. Intermittent family of maps represents one group of such maps. Below, we introduce a formal mathematical definition of this family of maps and discuss in general terms how this family of maps provides a suitable mathematical model for describing intermittent chaos as well as the sub-exponential decay of the recurrence times outlined above. Note that this definition is new and hitherto unavailable in the published accounts of intermittency maps, where the treatment is largely graphical. Moreover, we also provide a formal algebraic treatment of *Type I* intermittency, as discussed in Manneville and Pomeau [34]; we do not however discuss the associated *Type II* and *Type III* intermittency as these do not give rise to long-memory chaotic maps.

Let $\zeta_\theta(w)$ denote a parametric family of maps. The nature of fixed points for this family would depend on the values of θ. Suppose that there is a value, θ_0, say, of θ at which $\zeta_\theta(w)$ has a neutral fixed point, $\tilde{w}(\theta_0)$, say. Below, in Definition 4.3, we assume that for a given θ_0, $\tilde{w}(\theta_0)$ is unique, but the possibility that for a $\theta = \theta_1 \neq \theta_0$, there may exist another neutral fixed point, $\tilde{w}(\theta_1)$, say, has been allowed for, provided θ_1 is not "too close to" θ_0.

For a fixed $\delta_0 > 0$ and $\delta_1 > 0$, let

$$A_{\delta_0}(\tilde{w}) = \{w : w \in [\tilde{w} - \delta_0, \tilde{w} + \delta_0]\} \cap J \tag{4.3}$$

be a set of values of w in a neighbourhood of the neutral fixed point, and also let

$$\Theta_{\delta_1}(\theta_0) = \{\theta : \theta \in [\theta_0 - \delta_1, \theta_0 + \delta_1]\} \tag{4.4}$$

be a set of values of θ in a neighbourhood of θ_0. It will be convenient to introduce the set

$$A_{R\delta_0}(\tilde{w}) = \{w : w \in [\tilde{w}, \tilde{w} + \delta_0]\} \cap J, \tag{4.5}$$

for denoting the set of values of w in $A_{\delta_0}(\tilde{w})$ which are greater than $\tilde{w}$, and the set

$$A_{L\delta_0}(\tilde{w}) = \{w : w \in [\tilde{w} - \delta_0, \tilde{w}]\} \cap J, \tag{4.6}$$

for denoting the values of w in $A_{\delta_0}(\tilde{w})$ which are less than $\tilde{w}$. Analogously, let

$$\Theta_{R\delta_1}(\theta_0) = \{w : w \in [\theta_0, \delta_1 + \delta_0]\}, \tag{4.7}$$

be the values θ in $\Theta_{\delta_1}(\theta_0)$ which are greater than θ_0. In addition, for any interval $A = [a, b] \subset J$, we let $\mathrm{int}(A) = (a, b)$ denote the corresponding open interval.

The definitions given below of an intermittent family of maps depend upon the nature of the neutral fixed point, $\tilde{w}(\theta_0)$, and in particular on the following three-fold classification of neutral fixed points:

a) a weakly attracting neutral fixed point;
b) a weakly repelling neutral fixed point;
c) an attracting/repelling neutral fixed point.

Intermittency occurs only when the situations b) and c) above hold, and, furthermore, long-memory can only occur if b) holds. It would accordingly be useful to give at the outset a precise definition of these three types of neutral fixed points.

Definition 4.1 a) For a map, $\zeta(w)$, We say that a neutral fixed point, $\tilde{w}$, is *weakly attracting,* if there exists a $\delta > 0$ such that for all $w \in A_\delta(\tilde{w})$, $\zeta(w)$ admits the following representation:

$$\zeta(w) = w + K(w), \tag{4.8}$$

and where $K(w)$ satisfies the following regularity conditions: i) $K(w) < 0$ for $w \in [A_{R\delta}(\tilde{w}) - \{\tilde{w}\}]$, provided this set is nonempty, ii) $K(w) > 0$ on $[A_{L\delta}(\tilde{w}) - \{\tilde{w}\}]$, provided also that this set is nonempty, iii) $K(w)$ is differentiable on $A_\delta(\tilde{w})$, and iv) $K(\theta, w) = o(|w - \tilde{w}|)$, as $w \to \tilde{w}$.

b) We say a neutral fixed point, $\tilde{w}$, is *weakly repelling,* if there exists a $\delta > 0$ such that for all $w \in A_\delta(\tilde{w})$, $\zeta(w)$ admits the following representation:

$$\zeta(w) = w + K(w), \tag{4.9}$$

and where $K(w)$ satisfies the following regularity conditions: i) $K(w) > 0$ for $w \in [A_{R\delta}(\tilde{w}) - \{\tilde{w}\}]$, provided this set is nonempty, ii) $K(w) < 0$ on $[A_{L\delta}(\tilde{w}) - \{\tilde{w}\}]$, again provided that this set is nonempty, iii) $K(w)$ is differentiable on $A_\delta(\tilde{w})$, and iv) $K(\theta, w) = o(|w - \tilde{w}|)$ as $w \to \tilde{w}$.

c) We say a neutral fixed point, $\tilde{w}$, is *attracting/repelling,* if there exists a $\delta > 0$ such that for all $w \in A_\delta(\tilde{w})$, $\zeta(w)$ admits the following representation:

$$\zeta(w) = w + K(w), \tag{4.10}$$

and where $K(w)$ satisfies the following regularity conditions: i) K(w) is a positive, or a negative, function of w for all $w \in A_\delta(\tilde{w}) - \{\tilde{w}\}$, ii) $K(w)$ is differentiable on $A_\delta(\tilde{w})$, and iii) $K(\theta, w) = o(|w - \tilde{w}|)$ as $w \to \tilde{w}$.

Remark 4.2 If $\tilde{w}$ is a left endpoint of J, then condition a)iii), or b)iii), is redundant, and similarly if $\tilde{w}$ is a right endpoint of J, then condition a)ii), or b)iii), is redundant.

If $\zeta''(w)$ exists and is non-zero at $w = \tilde{w}$ then it can be shown, see[15], that $\tilde{w}$ is an attracting/repelling fixed point, and it is also known as a *saddle-node.* By contrast, if $\zeta(w)$ is twice differentiable, but $\zeta''(w) = 0$ at $w = \tilde{w}$, then the nature of the neutral fixed point is indeterminate, and an examination of the third order derivative (if it exists) is necessary to determine which of the three categories it falls into. An advantage of the definition given above is that it does not require the existence of the higher order derivatives of $\zeta(w)$.

Definition 4.3 (Saddle-node intermittency) We say that $\zeta_\theta(w)$ is an intermittent family of maps if the following conditions hold:

(1) There exists a $\theta = \theta_0$, such that the map, $\zeta_{\theta_0}(w)$, admits a unique neutral fixed point, $\tilde{w}(\theta_0) \in J$. Moreover there exists a $\delta_1 > 0$ such that $\forall \theta \in [\Theta_{\delta_1}(\theta_0) - \{\theta_0\}]$, there are no neutral fixed points for the map $\zeta_\theta(w)$ inside J.

(2) There exists a collection, $\{I_j : j = 1, \dots, m\}$, with $m > 1$, of closed intervals, with disjoint interiors, satisfying the following properties:

a) $J = \cup_{j=1}^m I_j$,

b) for all $(\theta, w) \in \Theta_{\delta_1}(\theta_0) \times J$, the map $\zeta_\theta(w)$ is a continuous function of both θ and w on the interior of each I_j,

c) for $\theta = \theta_0$, $\zeta_\theta(w)$ satisfies an irreducibility condition, namely, for each $i = 1, \dots, m$, we can find an $n(i)$ such that $\zeta_\theta^{n(i)}(I_i) \supset J$.

(3) There exists a j, call it j^*, and a $\delta_0 > 0$, such that (i) $\tilde{w}(\theta_0) \in I_{j^*}$, (ii) $A_{\delta_0}(\tilde{w}(\theta_0)) \subset I_{j^*}$; (iii) for all $\theta \in \Theta_{\delta_1}(\theta_0)$, and $w \in A_{\delta_0}(\tilde{w}(\theta_0))$, $\zeta_\theta(w)$ may be represented as:

$$\zeta_\theta(w) = (\theta - \theta_0) + w + K(\theta, w), \tag{4.11}$$

where $K(\theta, w)$ satisfies the following conditions: (iv) $K(\theta, w) > 0$ is a positive function of θ and w for $w \neq \tilde{w}$, and (v) $K(\theta, w)$ is a differentiable function for all $(\theta, w) \in \Theta_{\delta_1}(\theta_0) \times A_{\delta_0}(\tilde{w})$ and such that:

$$\left.\frac{\partial}{\partial \theta} K(\theta, w)\right|_{(\theta_0, \tilde{w})} = 0, \tag{4.12}$$

and

$$K(\theta_0, w) = o(|w - \tilde{w}|), \tag{4.13}$$

as $w \to \tilde{w}$,

(4) For all $w \notin A_{\delta_0}(\tilde{w})$ and all $\theta \in \Theta_{\delta_1}(\theta_0)$ the map is uniformly expanding in the sense that $|\zeta_\theta(w)| \geq c_0 > 1$.

(5) a) for all $\theta > \theta_0$ there are no fixed points in $A_{\delta_0}(\tilde{w})$; b) for all $\theta < \theta_0$ there exist an attracting *and* a repelling fixed point in $A_{\delta_0}(\tilde{w})$.

Remark 4.4 If $\zeta_\theta(w)$ admits continuous partial derivatives with respect to both θ and w inside $\Theta_{\delta_1}(\theta_0) \times A_{\delta_0}(\tilde{w})$, then an explicit expression for $K(\theta, w)$ may be given as follows:

$$K(\theta, w) = \Big(\left.\frac{\partial}{\partial w} \zeta_\theta(w)\right|_{(\theta^*, w^*)} - 1\Big)(w - \tilde{w}(\theta_0)) + \left.\frac{\partial}{\partial \theta} \zeta_\theta(w)\right|_{(\theta^*, w^*)} (\theta - \theta_0), \tag{4.14}$$

where $w^* = \alpha_1 \tilde{w} + (1 - \alpha_1) w, \theta^* = \alpha_2 \theta_0 + (1 - \alpha_2)\theta$, for some $\alpha_1, \alpha_2 \in (0, 1)$, and now condition (4.13) is satisfied if

$$\frac{d}{dw} \zeta_{\theta_0}(w)|_{(\theta^*, w^*)} = 1 + o(1),$$

as $w \to \tilde{w}(\theta_0)$.

Remark 4.5 Definition 4.3 as stated above includes as a special case intermittent behaviour through saddle-node bifurcation, which is much discussed in Physics, and it enables a precise mathematical formulation of this type of intermittent behaviour to be given: Thus, suppose that $\zeta_\theta(w)$ satisfies Definition 4.3 and, in addition, suppose that $\tilde{w}(\theta_0)$ is such that $\zeta''_{\theta_0}(\tilde{w}(\theta_0)) \neq 0$. The resulting family of maps is said to be *saddle-like* and for $\theta = \theta_0$ the corresponding map is said to undergo a saddle-node bifurcation and it will display intermittent behaviour through a saddle-node bifurcation. To understand the terminology, note that by condition (5) the resulting family of maps admits both a repelling fixed point, called a saddle, and an attracting fixed point, called a node, and as θ approaches θ_0 from below, the repelling fixed point collides with the attracting fixed point and this collision gives rise to a saddle-node bifurcation (see [41] page 27).

Remark 4.6 The family of maps satisfying Definition 4.3 will display intermittent behaviour for all $\theta \geq \theta_0$, but not for $\theta < \theta_0$. Moreover, if $\theta = \theta_0$, the corresponding map is not ergodic because the neutral fixed point, $\tilde{w}(\theta_0)$, is weakly attracting, implying that if $w_0 \in A_{L\delta_0}(\tilde{w})$, then $w_n \to \tilde{w}$, as $n \to \infty$.

As an example of a family of maps satisfying Definition 4.3, consider the following modification of the beta-transformation with $\beta = 2$, but the branch for $w \in [0, 1/2]$ modified so as to allow a neutral fixed point at $\theta = 0$.

Example 4.7 For $J = [0,1]$ and $\theta \in (-1/30, 1/30)$, consider the family of maps defined by:

$$\zeta_\theta(w) = \begin{cases} \theta + w - \frac{25}{8}(1-2\theta)(w - \frac{1}{10})^2 & \text{if } 0 \leq w \leq 1/2, \\ 2w - 1 & \text{if } 1/2 < w \leq 1. \end{cases} \tag{4.15}$$

To show that $\zeta_\theta(w)$ as defined by (4.15) satisfies Definition 4.3, we note that the neutral fixed point, $\tilde{w} = 1/10$, occurs only for $\theta = 0$, and hence any choice of $\delta_1 \in (0, 1/30)$ accords with condition (1) of Definition 4.3. For verifying condition (2a), we take $m = 2$ and set $I_1 = [0, 1/2]$ and $I_2 = [1/2, 1]$. Condition (2b) may similarly be verified by observing that $\zeta_\theta(w)$ is continuous and differentiable on $\text{int}(I_1)$ and $\text{int}(I_2)$. Moreover the irreducibility condition, (2c), may also be seen to hold with $n(1) = 2$ and $n(2) = 1$. Next, consider condition (3). The representation on the right of (4.11) is seen to hold with $j^* = 1$, $\delta_0 = 2/5$, and

$$K(\theta, w) = \frac{25}{8}(1-2\theta)(w - \frac{1}{10})^2. \tag{4.16}$$

Hence, conditions (4.12) and (4.13) are readily seen to hold. Verification of condition (4) maybe accomplished by noting that $\zeta_\theta(w)$ is an expanding map on $[1/2, 1]$. Similarly, to verify (5), note that for all $\theta > \theta_0$ and all $w \in [0, 1/2]$, $\zeta_\theta(w) > w$ which implies that there are no fixed points in this interval. Also for $\theta < \theta_0$, $\zeta_\theta(0) > 0$, $\zeta_\theta(\tilde{w}) > \tilde{w}$, $\zeta_\theta(1/2) > 1/2$ and hence it follows by the intermediate value theorem that there exists an attracting and a repelling fixed point inside $[0, 1/2]$.

The definition of an intermittent family of maps applicable to maps admitting a weakly repelling neutral fixed point is given below. We distinguish between the case in which the neutral fixed point is an endpoint of J and when it is an interior point of J. In the former situation, for ensuring that $\zeta_\theta(w)$ maps J into itself, the admissible of θ must be restricted to $\theta \geq \theta_0$, and hence $\Theta_{\delta_1}(\theta_0)$ is replaced by $\Theta_{R\delta_1}(\theta_0)$, and this restriction in turn gives rise to what we call the *Type I repelling fixed point intermittency.* By contrast, *Type II repelling fixed point intermittency* occurs when $\tilde{w}(\theta_0) \in J$ is an interior point of J.

Definition 4.8 (Type II repelling fixed point intermittency) Suppose that $\zeta_\theta(w)$ is a parametric family of maps admitting a unique weakly repelling neutral fixed point, $\tilde{w}(\theta_0) \in \mathrm{int}(J)$, at $\theta = \theta_0$. We say that $\zeta_\theta(w)$ is an intermittent family of maps if conditions (1), (2) and (4) of Definition 4.3 together with parts (i), (ii), (iii) and (v) of condition (3) are satisfied, but condition (5) and part (iv) of condition (3), are replaced by their alternative counterparts given below: $(3(iv)')$ $K(\theta, w) > 0$ is a positive function of w if $w > \tilde{w}$, and $K(\theta, w) < 0$ is a negative function of w if $w < \tilde{w}$, $(5')$ For all $\theta \neq \theta_0$, there exists a repelling fixed point in $A_\delta(\tilde{w})$, but no attracting fixed points.

Definition 4.9 (Type I repelling fixed point intermittency) Suppose that $\zeta_\theta(w)$ is a parametric family of maps admitting a unique weakly repelling neutral fixed point which is an endpoint of J. We say that $\zeta_\theta(w)$ is an intermittent family of maps if conditions (1), (2) and (4) of Definition 4.3 hold but with $\Theta_{\delta_1}(\theta_0)$ replaced by $\Theta_{R\delta_1}(\theta_0)$, and in addition the following replacements of conditions (3) and (5) of Definition 4.3 are satisfied:

(3") If $\tilde{w}$ is a left endpoint of J, then condition (3) of Definition 4.3 holds as stated, but if $\tilde{w}$ is a right endpoint of J, then the representation (4.11) is replaced by the following:

$$\zeta_\theta(w) = (\theta_0 - \theta) + w - K(\theta, w), \tag{4.17}$$

where $K(\theta, w)$ satisfies the conditions 3(iv) and 3(v) of Definition 4.3.

(5") Condition 5(a) of Definition 4.3 holds, but condition 5(b) is deleted.

Henceforth, for convenience in reference, it will be useful to define an intermittent family of maps as a family of maps satisfying any one of the Definitions 4.3, 4.9,4.8. We have the following definition.

Definition 4.10 We say that $\zeta_\theta(w)$ is an intermittent family of maps if any one of the Definitions 4.3, 4.8 or 4.9 is satisfied.

Furthermore, it is also useful to introduce an intermittency map as a member of the intermittent family of maps as follows:

Definition 4.11 A map belonging to the intermittent family but defined only for a specific value of θ is called an "intermittency map".

We next classify an intermittent family of maps into two distinct categories, according to the behaviour of $\zeta_\theta(w)$ outside the region $A_{\delta_0}(\tilde{w})$. We have the following definition.

Definition 4.12 An intermittent family of maps is said to be "regular" if, in addition, the following condition holds:

(A) $\zeta'_\theta(w)$ is bounded for all w, and there is a constant c_1 such that $|\zeta'_\theta(w)| \leq c_1 < \infty$,

and is called "cusp" if, instead, the following condition holds:

(B) There exists a w^* such that $\lim_{w \to w^*} |\zeta'_\theta(w)| = \infty$.

In Sections 5 and 6, we give specific examples of intermittent family of maps satisfying Definition 4.9. As an example of a family of maps satisfying Definition 4.8, consider the following, which, like Example 4.7, is a modification of the Beta-transformation, $B_\beta(w) = \beta w \mod 1$, but now $\beta = 3$, and the branch $B_\beta(w) = 3w - 1, w \in [1/3, 2/3]$ is modified so as to allow for a weakly repelling neutral fixed point.

Example 4.13 For $J = [0,1]$ and $\theta \in (-1/3, 1/3)$ consider the family of maps defined by:

$$\zeta_\theta(w) = \begin{cases} 3w & \text{if } 0 \le w \le 1/3, \\ \theta + w - 6^{1+\alpha}(\frac{1}{3}+\theta)(\frac{1}{2}-w)^{1+\alpha} & \text{if } 1/3 \le w \le 1/2, \\ \theta + w + 6^{1+\alpha}(\frac{1}{3}-\theta)(w-\frac{1}{2})^{1+\alpha} & \text{if } 1/2 \le w \le 1/3, \\ 3w-2 & \text{if } 2/3 < w \le 1, \end{cases} \tag{4.18}$$

where $\alpha \in (0,1)$ is fixed. This family of maps has a unique neutral fixed point $\tilde{w} = 1/2$ with $\theta = 0$. As this fixed point is weakly repelling, all the conditions of Definition 4.8 are readily seen to hold.

Next we explain in qualitative terms how for $\theta = \theta_0$ the orbit $\{w_n, n \in \mathbb{N}\}$ of a family of maps satisfying Definitions 4.10 will be intermittent and display both laminar and chaotic behaviour. We however first give precise definitions of these two concepts.

Definition 4.14 Suppose that the orbit $\{w_n, n \in \mathbb{N}\}$ of a map $\zeta(w)$ is such that for a set $A \subset J$,$w_n \in A$, for all $n \in [N_1, N_2]$, where $N_2 > N_1 \ge 1$, are known integers. We say that the map $\zeta(w)$ displays chaotic behaviour on the set $A \subset J$, if we can find some constants $C > 0$ and $c_0 > 1$ such that

$$\frac{\partial}{\partial w}\zeta^{(N_2-N_1)}(w)\bigg|_{w=w_{N_1}} \ge C c_0^{N_2-N_1}$$

where C and c_0 are independent of $(N_2 - N_1)$ and the point $w_{N_1} \in A$.

Remark 4.15 The map $\zeta(w)$ can display chaotic behaviour on a set $A \subset J$, but the map need not be chaotic over the whole of J. Hence only the value of the Lyapunov exponent evaluated in the chaotic region of J will be positive and that in the non-chaotic region will vanish, and the average value of this exponent evaluated as in (2.4) could vanish.

Remark 4.16 The behaviour of the orbit inside A is sensitively dependent on the value w_{N_1} in the sense that if $\Omega_\epsilon(w_{N_1}) = \{w : |w - w_{N_1}| < \epsilon\}$ denotes an ϵ-neighbourhood of a $w_{N_1} \in A$, and if $\zeta^n(\Omega_\epsilon(w_{N_1})) \subset A$ for all $n \in [0, N_2 - N_1]$, then

$$|\zeta^n(w) - \zeta^n(w_{N_1})| \ge C c_0^n |w - w_{N_1}|,$$

for all $w \in \Omega_\epsilon(w_{N_1})$ and all $n \in [0, N_2 - N_1]$.

The notion of laminar behaviour of $\{w_n\}$, unlike that of chaotic behaviour, is however defined with respect to an intermittent family of maps satisfying Definitions 4.10. Loosely speaking, the term laminar behaviour refers to a "relatively static" behaviour of w_n inside a given set $A \subset J$, for some consecutive values of n. The following definition provides a precise mathematical formulation of this phenomenon.

Definition 4.17 Let $\zeta_\theta(w), \omega \in J$, be an intermittent family of maps according to Definition 4.10. For $\theta = \theta_0$ and a set $A \subset J$, $\zeta_\theta(w)$ is said to exhibit laminar behaviour if, for an initial condition $w_0 \in A$, and an $N > 1$, $w_n = \zeta^n(w_0) \in A$ for $(n = 1, \ldots, N)$ and the following conditions hold: 1) w_n, $(n = 1, \ldots, N)$ are either monotonically increasing or decreasing; 2) for some $\gamma > 0$, $\frac{\partial}{\partial w}\zeta^n(w)\Big|_{w=w_0} = O(n^\gamma), n = 1, \ldots, N$.

The intermittent behaviour of w_n may be characterised by distinguishing between the following three scenarios:

Scenario A. Assume that the following conditions hold:
(i) $w_0 \in A_\delta(\tilde{w})$; (ii) for some $N_0 \geq 1$ and all $n = 1, 2, \ldots, N_0$, $w_n \in A_\delta(\tilde{w})$; (iii) $w_n \notin A_\delta(\tilde{w})$ for $n = N_0 + 1$.
We first consider the behaviour of the orbit according to Definition 4.3. Since, by Condition (3iv) of this definition, $K(\theta, w)$ is a positive function, $\zeta_\theta(w) \geq w$, $\forall w \in A_{\delta_0}(\tilde{w})$, and $w_n = \zeta_\theta^n(w)$ $(n = 1, 2, \ldots, N_0)$ are monotonically increasing and display laminar behaviour. Next consider the behaviour of the orbit when Definition 4.8 holds instead. Condition (3iv') of this Definition, however. implies that if $w_0 > \tilde{w}$, w_n, $(n = 1, 2, \ldots, N_0)$ are monotonically increasing, but if if $w_0 < \tilde{w}$, w_n, $(n = 1, 2, \ldots, N_0)$ are monotonically decreasing. Finally, consider the behaviour of the orbit when Definition 4.9 applies. The behaviour of w_n will depend upon whether $\tilde{w}$ is a left endpoint or a right endpoint of J. If the former holds, the w_n are monotonically increasing, but if the latter holds, the w_n are monotonically decreasing.

Scenario B. Assume that the following conditions hold:
(i) $w_0 \in A_\delta(\tilde{w})$; (ii) for some N_1, N_2, such that $N_2 \geq N_1 \geq 1$, $w_n \notin A_\delta(\tilde{w})$, for all $n = N_1, N_1 + 1, \ldots, N_2$; (iii) $w_n \in A_\delta(\tilde{w})$ at $n = N_2 + 1$.
Now, by Condition (4) of Definition 4.3, which also holds under Definitions 4.8 and 4.9, the values of w_n for $N = N_1, \ldots, N_2$ will display chaotic behaviour, and will be sensitive to the value of w_n for $n = N_1$.

Scenario C. Assume that $w_0 \notin A_\delta(\tilde{w})$, and thus the initial condition falls in the chaotic region. Now, as long as the successive iterates, w_n, remain in the chaotic region, that is, $w_n \notin A_\delta(\tilde{w})$, they will display chaotic behaviour according to all three definitions. If, however, for some $n = N_2$, $w_n \in A_\delta(\tilde{w})$, the successive iterates, w_n, for $n \geq N_2$ will display laminar behaviour as described in Scenario A above, provided $w_n \in A_\delta(\tilde{w})$, but this will revert to chaotic behaviour if $w_n \notin A_\delta(\tilde{w})$. In this sense, the orbit, $\{w_n\}$, of the intermittency map for $\theta = \theta_0$ could alternate between chaotic and laminar regions, and this map accordingly provides a mathematical model for describing such behaviour.

For $\theta > \theta_0$, Scenarios (A), (B) and (C) discussed above continue to hold in a weaker sense provided the definition of the laminar behaviour is modified as follows:

Definition 4.18 The family of maps satisfying Definition 4.17 is said to exhibit laminar behaviour for $\theta > \theta_0$, if for an initial condition $w_0 \in A$, and an $N > 1$, $w_n = \zeta^n(w_0) \in A$ for $(n = 1, \ldots, N)$ and the following conditions hold: 1) w_n, $(n = 1, \ldots, N)$ are either monotonically increasing or decreasing; 2) there exists a $c = c(\theta) > 1$ such that $\frac{\partial}{\partial w}\zeta^n(w_0) = O(c(\theta)^n), n = 1, \ldots, N$ where $c(\theta)$ is monotonically increasing in θ, and $c(\theta) \to 1$ as $\theta \to \theta_0$.

We next consider the intermittency maps with $\theta = \theta_0$ and examine the extent to which the description given in Scenarios A, B and C above applies according to whether Definition 4.3 or Definitions 4.8, 4.9 hold, that is according to whether $\tilde{w}(\theta_0)$ is an attracting/repelling fixed point, or whether it is a weakly repelling fixed point. We therefore consider the following two situations separately.

Situation (1): Attracting/Repelling fixed point. Now, $\tilde{w}$ lies in the interior of J, and as stated previously the corresponding map is not ergodic, and indeed, for all $w_0 \in J$, $w_n = \zeta_{\theta_0}^n(w_0) \to \tilde{w}$, with probability one, as $n \to \infty$. But the behaviour of w_n for finite n depends on the position of w_0 in relation to $\tilde{w}$. If, $w_0 \in A_{R\delta_0}(\tilde{w})$ and thus the initial condition is such that it lies in the laminar region to the right of $\tilde{w}$, then Scenario A describes the behaviour of the orbit, w_n, but only as long as it remains there, and once the orbit escapes this part of the laminar region and enters the chaotic region, that is, $w_n \in J - A_{\delta_0}(\tilde{w})$, Scenario B describes its behaviour until it remains there. In this way, the orbit will continue to alternate between these two regions until it reaches the laminar region to the left of $\tilde{w}$, that is, $w_n \in A_{L\delta_0}$, and this is an event of probability one, it will remain in $A_{L\delta_0}$ and converge to $\tilde{w}$ also with probability one.

Situation (2): Weakly repelling fixed point. The ergodic properties of the map depend upon the relationship between the initial condition, w_0, and the fixed point, $\tilde{w}$ and we distinguish between the following three cases: (i) $w_0 = \tilde{w}$; (ii) $w_0 \in A_{\delta_0}(\tilde{w})$; (iii) $w_0 \in J - A_{\delta_0}(\tilde{w})$. The case (i) is rather special and gives rise to a constant orbit, $w_n = \tilde{w}$, $(n \geq 0)$. The case (ii) corresponds to Scenario A described above and now there is a finite integer, k_{w_0}, dependent on w_0, such that for some $n > k_{w_0}$, $w_n \notin A_{\delta_0}(\tilde{w})$, and hence the orbit will escape the laminar region. Notice however, that the set $N^+ = \{k_{w_0} : w_0 \in A_{\delta_0}(\tilde{w}) - \{\tilde{w}\}\}$ is not compact, and hence k_{w_0} can take arbitrarily large values. Nevertheless, with probability one, the orbit will escape the laminar region, $A_{\delta_0}(\tilde{w})$, and enter the chaotic region, $J - A_{\delta_0}(\tilde{w})$. But once the orbit, w_n, is inside the chaotic region, it can escape this region with a nonzero probability at any time. The case (iii) corresponds to Scenario C described above, and since the orbit may escape the chaotic region for any $n \geq 1$, it will enter the laminar region with a non-zero probability, where it will behave in accordance with case (ii), or exceptionally case (i).

It may be gleaned from the discussion above that only the maps described in Situation (2) can be ergodic and for such maps a pertinent recurrence property that we consider is the event that the orbit, w_n, returns back to the chaotic region, $J - A_{\delta_0}(\tilde{w})$ for the first time at time $n = m\,(m = 1, 2, \ldots)$, having started from this region. Thus for a set $C \in J - A_{\delta_0}(\tilde{w})$, let $S_C(m)$ be defined by (4.1).

The behaviour of $S_C(m)$ may be characterized by two different methods:

(1) by the probability distribution of $S_C(m)$, namely $F(m, C) = P(S_C(m))$, where the probability is evaluated with respect to the invariant density, $\chi(w)$;

(2) by $G(m, C)$, the Lebesgue measure assigned to the set of points returning back to C at time m having started in C.

Hence,

$$F(m, C) = \frac{1}{K_C} \int_{S_{C(m)}} \chi(w) dw, \tag{4.19}$$

$$G(m, C) = \frac{1}{K_C} \int_{S_{C(m)}} dw, \tag{4.20}$$

where K_C is a normalization constant. Note that $G(m, C)$ is well-defined even when an invariant density does not exist and it may be evaluated in the same way as $F(m, C)$, but with the probability evaluated with respect to the uniform density on J.

A brief account of how to find $F(m,C)$ and $G(m,C)$ is given in the appendix. As shown there, and see also Young [49] among others, the rate of decay of correlations for the intermittency map with $\theta=\theta_0$, maybe determined from the tail behaviour of $G(m,C)$, and long memory occurs if, for example, $G(m,C)=O(n^{-2-\epsilon})$, for some $\epsilon>0$.

It may be observed that the cusp intermittency maps considered in Section 6 are a special case, with $\theta=\theta_0$, of an intermittent family of maps admitting two neutral fixed points, $\tilde{w}_1$ and $\tilde{w}_2$, say, at a unique value, $\theta=\theta_0$, of θ. To understand why Definition 4.9 applies to this map, we have the following extension.

Extension 1. Suppose that a parametric family of maps, $\zeta_\theta(w)$, admits two neutral fixed points, $\tilde{w}_1=\tilde{w}_1(\theta_0)$ and $\tilde{w}_2=\tilde{w}_2(\theta_0)$, at a unique value, $\theta=\theta_0$, of the parameter, θ, and that with either of the neutral fixed points, $\zeta_\theta(w)$ is an intermittent family of maps in accordance with Definition 4.10. Then, it may readily be gleaned from the discussion given in Situations 1 and 2 above that the intermittency map corresponding to $\theta=\theta_0$ can display ergodic behaviour provided both neutral fixed points are weakly repelling. The details of this extension are omitted to save space.

In our Definition 4.10, the behaviour of the map in the chaotic region, $J-A_\delta(\tilde{w})$, is prescribed by requiring that the map is uniformly expanding in this region, see condition (4). It is possible to construct intermittent families of maps for which the behaviour in this region is specified differently, see for example [23]. Also, for convenience, the domain and range of $\zeta_\theta(w)$ coincide in our Definition 4.10,for all θ. However it is possible to construct families of intermittent map where this is not so, the details are omitted to save space.

We conclude this section by briefly explaining why the logistic, tent and beta transformation families of maps discussed in Section 3, do not belong to the intermittent family of maps satisfying Definitiondefintfam. For the logistic family, $\zeta_\theta'(w)=0$ for $w=1/2$ and hence condition (4) is violated for all values of θ. Similarly, the tent family of maps does not admit the representation given in condition (3) of Definitions 4.3-4.9. Finally for the beta transformation family of maps, $\zeta_\theta'(w)>1$ for all w, and this family of maps does not admit a neutral fixed point. It should be recognized nevertheless that both the logistic and tent family of maps can display intermittent chaos for certain values of θ. For example, if $\theta=\theta_0=1+2\sqrt{2}$, the former is intermittent with a neutral fixed point of prime period three, see Rasband [41]. For values of $\theta>\theta_0$, but close to θ_0, the laminar and chaotic regions for this map can be identified, but the intermittency associated with this neutral periodic point does not persist for other values of θ, and as a consequence condition (2) of Definition 4.3 is violated.

5 Regular intermittency maps with long memory

5.1 Polynomial maps. A celebrated example, with $J=[0,1]$, is:

$$\zeta_\alpha(w)=\begin{cases} w(1+2^\alpha w^\alpha) & \text{if } 0\le w\le 1/2,\\ 2w-1 & \text{if } 1/2<w\le 1,\end{cases} \tag{5.1}$$

where $\alpha>0$ (see Liverani, Saussol and Vaienti [31] and Young [49]). For each α, $\zeta_\alpha(w)$ has a neutral fixed point, $w_0=0$, but its second derivative, $\zeta_\alpha''(w)$, behaves like $w^{\alpha-1}$, and it is therefore unbounded, as $w\to 0$, for $\alpha\in(0,1)$, and bounded for

$\alpha \geq 1$. When $\alpha = 0$, $\zeta_\alpha(w)$ reduces to the Bernoulli map, the beta transformation with $\beta = 2$.

To see why, for each $\alpha > 0$, the map (5.1) defines an intermittency map, we consider the following parametric family of maps, parametrized by two parameters, θ and α, such that $\theta \in [0, 1/2)$ and $\alpha > 0$, and verify that this family satisfies Definition 4.9:

$$\zeta_{\theta,\alpha}(w) = \begin{cases} \zeta_{\theta,\alpha,L}(w) = \theta + w + 2^\alpha(1-2\theta)w^{\alpha+1} & \text{if } 0 \leq w \leq 1/2, \\ \zeta_{\theta,\alpha,R}(w) = 2w - 1 & \text{if } 1/2 < w \leq 1. \end{cases} \tag{5.2}$$

It is readily seen that for all $\theta \in [0, 1/2)$ and for a given $\alpha > 0$, the fixed point, $\tilde{w}$, is a solution of the following equation: $w^{\alpha+1} = -\theta/\{2^\alpha(1-2\theta)\}$. However, since $w \in [0, 1/2]$ and $\theta \in [0, 1/2)$, the map admits only a single fixed point at $\theta = 0 = \theta_0$ and $w = 0 = \tilde{w}$, which is a weakly repelling neutral fixed point of Type I. Condition (1) of Definition 4.9 is readily seen to hold when $\delta_1 < 1/2$, where the admissible values of θ belong to the set $\Theta_{R\delta_1}(\theta_0) = \{\theta : \theta \in [\theta_0, \theta_0 + \delta_1]\}$. Condition (2) of Definition 4.9 is satisfied with $I_1 = [0, 1/2], I_2 = [1/2, 1]$ and $n(i) = 1$ for $i = 1, 2$. For a given $\alpha > 0$, $\zeta_{\theta,\alpha}(w)$ may be represented in the form given on the right of equation (4.11), but with $\theta_0 = \tilde{w} = 0$ and $K(\theta, w) = 2^\alpha(1-2\theta)w^{\alpha+1}$. Clearly, for all $\theta \in [0, 1/2)$ and all $w \in J$, $K(\theta, w)$ is a positive and differentiable function of θ and w. Also, $|K_2(w)/w| \to 0$ as $w \to 0$, and, consequently, condition (3") of Definition 4.9 is satisfied for all $\alpha > 0$. Moreover, since there are no fixed points for $\theta \in (0, 1/2)$, condition (5") of this definition is also satisfied. Finally, since $\tilde{w} = 0$ is the left end point of J, $A_{\delta_0}(\tilde{w}) = A_{R\delta_0}(\tilde{w})$, with $\delta_0 = 1/2$. It is readily verified that condition (4) of Definition 4.9 is also satisfied, for example with $c_0 = 2$, and thus $\zeta_{\theta,\alpha}(w)$, as defined by equation 5.1, is an intermittent family of maps. Note that for $w \in [1/2, 1]$, the orbit $w_n = \zeta_\theta^n(w)$ will be chaotic, and for $w \in [0, 1/2]$ the orbit will display laminar behaviour.

On taking $\theta = \theta_0 = 0$, $\zeta_{\theta,\alpha}(w)$ reduces to the map defined on the right of equation (5.1) and, in accordance with Definition 4.9, it is an intermittency map .

When $\alpha \in (0, 1)$, an explicit expression for the invariant density, $\chi_\alpha(w)$, is as yet unknown, but it is known that, see Thaler [45], $\chi_\alpha(w)$ is of the following form:

$$\chi_\alpha(w) = \frac{V(w)}{w^\alpha},$$

where $V(w)$ is a uniformly bounded piecewise continuous function of w satisfying

$$\frac{1}{B} \leq V(w) \leq B$$

and $B > 1$ is a constant.

For $\alpha \geq 1$, the invariant density $\chi_\alpha(w)$ is singular - concentrated at $w = 0$- and if, for all n, $X_\alpha(w) = P(w_n \leq w)$, $w \in \mathbb{R}$, denotes the corresponding distribution function of w_n, then

$$X_\alpha(w) = \begin{cases} 0 & \text{if } w < 0, \\ 1 & \text{if } w \geq 0. \end{cases} \tag{5.3}$$

Next, consider the recurrence properties of the map (5.1). For this map, as noted earlier, the sets $A_{\delta_0}(\tilde{w})$ and $A_{R\delta_0}(\tilde{w})$, as defined by (4.3) and (4.5) coincide. Let $C \subset J - A_{R\delta_0}(\tilde{w})$, be a set inside the chaotic region of the map. Then, with

$G(n, C)$ defined by in Section 4, it is known that, for all $\alpha \geq 0$,

$$G(n, C) = O\big(n^{-\frac{1}{\alpha}-1}\big) \quad (n \to \infty), \tag{5.4}$$

see Aaronson [1] and Young [49], for example. However, since the invariant density is singular for $\alpha \geq 1$, $F(n, C)$ is only defined for $\alpha \in (0, 1)$ and we may show that

$$F(n, C) = O\big(n^{-\frac{1}{\alpha}-1}\big) \quad (n \to \infty),\ \alpha \in (0, 1). \tag{5.5}$$

The results (5.5) and (5.4) imply that the distribution of the recurrence time to C has a power law (Pareto) tail rather than the usual exponential fall off.

The behaviour of the correlation function for the map (5.1) has been studied by several authors. As in Section 4, let $\varphi, \psi \in \mathcal{H}$ be two Hölder continuous functions. The first known result concerning the behaviour of the correlation function was established in Liverani, Saussol and Vaienti [31], where it is shown that, with $B > 0$ denoting a constant:

$$R_{\varphi,\psi}(n) \leq B((\log n)^{1/\alpha} n^{1-1/\alpha}), \quad (n \to \infty).$$

This bound was however later improved, see Young [49], to:

$$R_{\varphi,\psi}(n) \leq Bn^{1-1/\alpha}, \quad (n \to \infty), \tag{5.6}$$

and the improvement was made possible because of the development of a sharp bound

$$\|P^n_{\zeta_\alpha}\{\varphi(w)\chi_\alpha(w)\} - \chi_\alpha(w)\|_1 = O(n^{1-1/\alpha}) \quad (n \to \infty), \tag{5.7}$$

on the *speed of convergence to equilibrium*, as given by the second term to the right of the inequality (2.9).

Note that the bound(5.6), while valid for all Hölder continuous functions of w_n, is from above, and for some Hölder continuous functions, the covariance function might decay at a faster rate. A more precise result has recently been developed, see Sarig [44] and Gouëzel [17]. It follows from the work of these authors that

$$\lim_{n\to\infty} n^{(1/\alpha-1)} R_{\varphi,\psi}(n) = U, \tag{5.8}$$

where, for $\alpha \in (0, 1/2)$,

$$U = \frac{1}{4}\chi_\alpha\left(\frac{1}{2}\right)\alpha^{-1/\alpha}\left(\frac{1}{\alpha} - 1\right)^{-1}\int \varphi(w)\chi_\alpha(w)dw \int \psi(w)\chi_\alpha(w)dw,$$

and it depends on the invariant density and on the nature of the functions φ, ψ, and for $\alpha \in (1/2, 1)$, U is known to be a constant. Hence, for this family of maps, the process, $\{w_n : n \in \mathbb{N}\}$, has long memory provided $\alpha \in (1/2, 1)$, and, if $\alpha \in (0, 1/2)$, it has intermediate memory.

In addition to the results described above, a Central Limit Theorem for the map (5.1) is also known for $\alpha \in (0, 1/2)$. Let φ be a continuous function satisfying $\int_J \varphi(w)\chi_\alpha(w)dw = 0$, and let $S_n(\varphi)(w) = \frac{1}{n}\sum_{k=0}^{n-1}\varphi(w_k)$. Young [49] shows that, as $n \to \infty$, $\sqrt{n}S_n(\varphi)(w)$ converges in law to a Gaussian random variable with zero mean and finite variance, σ^2, given by

$$\sigma^2 = \int \varphi(w)^2\chi_\alpha(w)dw + 2\sum_{n=1}^{\infty}\int \varphi(w)\varphi(\zeta^n_\alpha(w))\chi_\alpha(w)dw,$$

and $\sigma^2 = 0$ if and only if there exists a square integrable function, $\psi(w)$, such that $\varphi(w) = \psi(\zeta_\alpha(w)) - \psi(w)$.

5.2 Logarithmic maps. A second class of regular intermittency maps has recently been introduced by Holland [22], and it is defined on $J = [0,1]$ by:

$$\zeta_\beta(w) = \begin{cases} \zeta_{\theta,\alpha,L}(w) = w(1+Y(\beta)w\log(1/w)^{1+\beta}) & \text{if } 0 \le w \le 1/2, \\ \zeta_{\theta,\alpha,R}(w) = 2w-1 & \text{if } 1/2 < w \le 1, \end{cases} \tag{5.9}$$

where $Y(\beta) = 2(\log 2)^{-(1+\beta)}$ is chosen to ensure that $\lim_{w\to 1/2^-} \zeta_\beta(w) = 1$. For $w \in (1/2,1]$, the right hand side of (5.9) coincides with that of (5.1) given above. For $w \in [0,1/2]$, however, the multiplying factor of $(1+2^\alpha w^\alpha)$ occuring on the right of the latter has been replaced by the factor $(1+Y(\beta)w\log(1/w)^{1+\beta})$. Although both these maps admit neutral fixed points at $w=0$, for the logarithmic map, (5.9), the second derivative, $\zeta''_\beta(w) = O\big((\log(1/w))^{1+\beta}\big)$, as $w \to 0$, grows at a slower rate than for the map (5.1), whose second derivative, $\zeta''_\alpha(w) \sim 2^\alpha\alpha(\alpha+1)w^{\alpha-1}$, as $w \to 0$, and grows at a polynomial rate.

One consequence of this modification is that the invariant density, $\chi_\beta(w)$, for the logarithmic map, (5.9), exists , for all $\beta \in (0, 2\log 2 - 1)$, and takes the following form on $(0,1/2)$:

$$\chi_\beta(w) = \frac{V(w)}{w\log(1/w)^{\beta+1}}, \quad \forall\, w \in J,$$

where $V(w)$ is a uniformly bounded piecewise continuous function, which is also bounded away from zero. On $[1/2,1]$, the invariant density is a bounded, piecewise continuous function. By contrast, for the polynomial map, (5.1), the invariant density exists only if $\alpha \in (0,1)$, and it diverges to infinity as $w \to 0$ at a much faster rate than $\chi_\beta(w)$.

A similar comparison, also applies to the behaviour of the recurrent time distributions, $F(n,C)$ and $G(n,C)$, which for the logarithmic map take the following form:

$$G(n,C) = \left(\frac{1}{n^2(\log n)^{1+\beta}}\right), \quad F(n,C) = \left(\frac{1}{n^2(\log n)^{1+\beta}}\right),$$

and those for the polynomial map are given by equations (5.4) and (5.5). It follows from a comparison of these expressions that the variance of the recurrent time distribution for the logarithmic map does not exist for any $\beta > 0$, but that for the polynomial maps does exist for $\alpha \in (0,1/2)$, though the latter also does not exist for $\alpha \in [1/2,1)$.

For the logarithmic map, Holland [22] also examines the rate of decay of the correlation function of $\{w_n : n \in \mathbb{N}\}$ and shows that since now the *speed of convergence to equilibrium* is slower, $\|P^n_{\zeta_\beta}\{\varphi(w)\chi_\beta(w)\} - \chi_\beta(w)\|_1 = O\big((\log n)^{-\beta}\big)$,

$$R_{\varphi,\psi}(n) \le B(\log n)^{-\beta}, \tag{5.10}$$

where B is a uniform constant, and within the class of all Holder continuous functions, the correlations for this map could also decay at a slower rate than for the polynomial map. It should be emphasized, nevertheless that (5.10) only provides an upper bound on the rate of decay of the correlations. A lower bound analogous to (5.8) has recently been obtained in [18].

A comparison of the functional forms of the polynomial and logarithmic maps suggests that the class of regular intermittency maps may be extended by defining a map of the following general form:

$$\zeta_\alpha(w) = \begin{cases} \zeta_{\theta,\alpha,L}(w) = w(1 + Y_\alpha w^\alpha \Delta(w)) & \text{if } 0 \le w \le 1/2, \\ \zeta_{\theta,\alpha,R}(w) = 2w - 1 & \text{if } 1/2 < w \le 1, \end{cases} \tag{5.11}$$

where $\alpha > 0$ and $Y_\alpha = 2^\alpha/\Delta(1/2)$ and $\Delta(1/w)$ is a slowly varying function at infinity such that $\Delta(1/w)$ is twice differentiable for all $w \in [M, \infty)$, and M is a fixed constant. Holland [22] also investigates the rate of decay of correlations for this extended class of maps and shows that

$$R_{\varphi,\psi}(n) \le Bn^{1-1/\alpha}\tilde{\Delta}(n), \tag{5.12}$$

where $B > 0$ is a uniform constant, and $\tilde{\Delta}(1/w)$ is another slowly varying function at infinity, and its precise functional form may be derived from that of $\Delta(w)$. This result shows that, by choosing an appropriate functional form for $\Delta(w)$, it is possible to construct regular intermittency maps with correlation functions whose rate of decay is bounded above by a function that is decaying at a prescribed rate. We emphasize once again, however, that (5.12) gives only an upper bound on the rate of decay of the correlations, as $n \to \infty$. Lower bounds are proved in [18].

For ease of exposition, the account given above of the regular intermittency maps was restricted to maps admitting a single neutral fixed point; however, as remarked by Young [49], for example, many of the results generalize to maps with multiple neutral fixed points.

6 The cusp intermittency maps

This class of maps has been investigated by Balakrishnan et. al. [6, 5, 4]. A specific example of the maps they consider, with $J = [-1, 1]$, is as follows:

$$\zeta(w) = \begin{cases} \zeta_L(w) = 1 - 2\sqrt{(-w)} & \text{if } w \in [-1, 0], \\ \zeta_R(w) = 2\sqrt{w} - 1 & \text{if } w \in [0, 1]. \end{cases} \tag{6.1}$$

It admits two neutral fixed points, namely $w = \pm 1$, and since $|\zeta''(w)| = 1/2$ for $w = \pm 1$ its second derivative is bounded at these fixed points. The map is discontinuous at $w = 0$, where its first derivative diverges to infinity.

For verifying that $\zeta(w)$, as given by equation (6.1), is an intermittency map, we appeal to Extension 1 discussed in section 4 and consider a parametric family of maps, $\zeta_\theta(w)$, defined for all $\theta \in [1, \tau]$, and $J = [-1, 1]$, where $\tau \in (1, 16/15)$ is a fixed constant:

$$\zeta_\theta(w) = \begin{cases} \zeta_{La\theta}(w) = \theta - 2\sqrt{(-w)} & \text{if } w \in [-1, -1/2], \\ \zeta_{Lb\theta}(w) = \theta - 2\sqrt{(-w)} + 8(\theta - 1)(1/2 + w)^3 & \text{if } w \in [-1/2, 0], \\ \zeta_{Rb\theta}(w) = 2\sqrt{w} - \theta + 8(\theta - 1)(1/2 - w)^3 & \text{if } w \in [0, 1/2], \\ \zeta_{Ra\theta}(w) = 2\sqrt{w} - \theta & \text{if } w \in [1/2, 1]. \end{cases} \tag{6.2}$$

It is readily seen that the map admits fixed points, $\tilde{w} = \pm 1$, only when $\theta = 1$. For demonstrating that with $\tilde{w} = -1$, the family of maps defined by (6.2) satisfies Definition 4.9 and it is thus intermittent, we note that $\tilde{w} = -1$, is a left-end point of $J = [-1, 1]$ and hence $A_{\delta_0}(\tilde{w}) = A_{R\delta_0}(\tilde{w})$, and we need only show that $\zeta_{La\theta}(w)$ satisfies this condition. Now, $\zeta_{La\theta}(w) = (\theta - 1) + w + K(\theta, w)$, where, for all $w \in [-1, 1/2]$, $K(\theta, w) = 1 - w - 2\sqrt{-w} = (1 - \sqrt{-w})^2$ is a positive function of w and independent of θ. Hence $\zeta_{La\theta}(w)$ has a representation given by equation

(4.17), and moreover. $K(\theta, w+1) = \frac{1}{4}(1+w)^2 + O((w+1)^3)$ satisfies condition (4.13), and Condition (3") is verified. By an argument analogous to that used in Section 5.1, conditions (1), (2), (4) and (5) of this definition may also be verified. Similarly, for $\tilde{w} = 1$, the representation given by equation (4.17) holds. That is for all $w \in [1/2, 1], \zeta_{Ra\theta}(w) = 1 - \theta + w + \tilde{K}(\theta, w)$, where $\tilde{K}(\theta, w) = -(1-\sqrt{w})^2 < 0$ and is independent of θ. The other conditions are readily verified. Thus $\zeta_\theta(w)$ is an intermittent family of maps satisfying Definition 4.9 and Extension 1. On taking $\theta = 1$, $\zeta_\theta(w)$ reduces to the cusp map defined by (6.1), and it is an intermittency map by Extension 1.

It should be noted that we have restricted the upper limit of the admissible values of θ to $\tau = 16/15$. This is to ensure that $\zeta_\theta(w)$ is defined over $J = [-1, 1]$ for all values of $\theta \in [1, \tau]$. However, on letting $J(\theta) = [-\theta, \theta]$, where $J(\theta)$ depends on θ, the admissible values of θ may be increased to $[1, \tau]$, where $\tau > 1$ is a fixed constant. This further extension, while feasible, would nevertheless involve modifying Definition 4.10 for this change and we do not pursue it here.

The cusp map (6.1) is ergodic and its invariant density is given by $\chi(w) = 1/2$ for all $w \in J$, and which coincides with that of the uniform distribution, $U[-1, 1]$, on J.

The recurrence properties of this map have also been investigated by Balakrishnan et. al. [6, 5]. However, since its two intermediate fixed points coincide with the two end points of the interval, J, over which the map is defined, the definition of the set $A_{\delta_0}(\tilde{w})$, is modified to:

$$A_{\delta_0}(\tilde{w}) = [-1, -1+\delta] \cup [1-\delta, 1],$$

and which now defines the laminar region of the map (6.1). Moreover, in view of the fact that the invariant density of the map equals a constant for all $w \in J$, $G(n, C) = 2F(n, C)$, and these measures are related in an elementary manner. If $C \subset J - A_\delta(\tilde{w})$, denotes an interval inside the chaotic region of the map, Balakrishnan et. al. [6, 5] show that

$$F(n, C) = O(n^{-3}) \quad (n \to \infty); \tag{6.3}$$

and the time to return back to the chaotic region, having started from this region, decays like a cubic.

These authors also investigate the behaviour of the sojourn times for this map. Let $\Omega \subset J$ denote a subset of the interval, J, and let

$$Z(n, \Omega) = P\{w_1 \in \Omega, w_2 \in \Omega, \ldots, w_{n-1} \in \Omega | w_0 \in \Omega\}$$

denote the probability of the event that $w_1, w_2, \ldots, w_{n-1}$ remain in the set Ω, having started from this set, where the probability is evaluated with respect to the invariant density, $\chi(w)$, of the map ζ. These authors derive an explicit expression for $Z(n, \Omega)$, which is also the probability of a sojourn in Ω, and show that for all $\Omega \subset J$

$$Z(n, \Omega) = O(n^{-1}), \quad (n \to \infty).$$

Although an explicit expression for the behaviour of the correlation function for this map is as yet not available, it may be expected that the correlations display the characteristic power law decay of intermittency maps. Indeed, on appealing to the known relationship between the rate of decay of the correlation function and $F(n, C)$, see Young [49], it may be hypothesised that for all $\varphi, \psi \in \mathcal{H}$:

$$R_{\varphi,\psi} = O(n^{-1}), \quad (n \to \infty), \tag{6.4}$$

and hence that the spectral density function, $f(\lambda)$, for this map has a singularity at the origin, $f(\lambda) \to \infty$, as $\lambda \to 0$. It may be observed, however, that now $f(1/\lambda)$ is a log-like function as $\lambda \to 0$, and that the rate of decay of correlations specified in (6.4) above is on the boundary of that given in section 1 for distinguishing between an intermediate-memory process and a long-memory process; furthermore, it is not possible for the class of fractionally-differenced processes to have correlation and spectral density functions possessing these properties.

Balakrishnan et. al. [6, 5] also discuss the possibility of generalizing the intermittency map (6.1) so as to have a recurrence time distribution, $F(n, C)$, with a slower rate of convergence than that specified by (6.3). However, since a closed-form analytical expression for the resulting intermittency map does not exist, we do not consider this generalization here.

7 Applications of intermittent maps to packet traffic

In this section we give a brief review of how chaotic intermittency maps are applied to modelling the Ethernet packet traffic. A key device here is the *random wall* model, (see Mondragon et. al. [36]), and we explain also why this model is introduced. We begin by giving an outline of some of the characteristic features of Ethernet packet traffic, and then discuss how chaotic maps are used to model the traffic flow. We also review some work on chaotic models for the aggregated traffic.

Unlike voicemail traffic, which is typically Markovian, there is empirical work, see Leland et. al. [30], demonstrating that Ethernet packet traffic displays characteristics of self similarity and long range dependence. There are also heavy-tailed ON/OFF periods. A stochastic structural model for providing a theoretical explanation of the empirically observed long memory in Ethernet packet traffic has also been constructed by Leland, et. al. [30]. Their model is based on the specific physical features of communication networks and considers first an individual pair of host computers, one of which is the source, and the other the destination, of the packet traffic. Suppose that a source alternates between an active state, or ON period, and an idle state, or OFF period. They assume that during ON periods, packets are sent at a constant rate, and during OFF periods no packets are transmitted; the successive ON periods as well as the successive OFF periods are assumed to be stochastically independent and identically distributed, and independent of each other. Based on these assumptions, they establish a limit theorem which states that the superposition of many such ON/OFF sources will capture the empirically-observed long memory in the measured aggregate Ethernet traffic, provided that the distribution of either the ON periods or the OFF periods of an individual source-destination pair has infinite variance. For further information about the stochastic model outlined above and physical characteristics of Ethernet packet traffic, reference may be made to Park and Willinger [38], who have brought together in a single volume a number of articles on probabilistic, statistical and engineering aspects of network traffic.

Mondragon et. al. [36] consider instead a regular intermittency map, with $J = [0, 1]$, of the following form:

$$\zeta \equiv \zeta(w; a, m_1, m_2) = \begin{cases} \zeta_L(w) = w + (1-a)a^{-m_1}w^{m_1} & \text{if } w \in [0, a], \\ \zeta_R(w) = w - a(1-a)^{-m_2}(1-w)^{m_2} & \text{if } w \in (a, 1], \end{cases} \tag{7.1}$$

where $m_1, m_2 \in (1,2)$ and $a \in (0,1)$. Notice that this is a generalized version of the regular polynomial map discussed in Section 5.1, and like this map it has a neutral fixed point at $w = 0$. In addition, the generalized map (7.1) admits a neutral fixed point at $w = 1$, and the right hand branch, $\zeta_R(w)$, of this map is not uniformly expanding.

These authors model the behaviour of ON/OFF periods of an individual source-destination pair by introducing an indicator function, $y(w_n)$, defined as follows:

$$y_n = y(w_n) = \begin{cases} 0 & \text{if } w_n \in (0,a) \quad \text{OFF state}, \\ 1 & \text{if } w_n \in (a,1) \quad \text{ON state}, \end{cases} \tag{7.2}$$

where $w_n = \zeta^n(w_0)$, The traffic load, κ, is the average number of packets generated by the map, and is the frequency with which $w_n, n = 1, 2, \ldots$, visits the ON region:

$$\kappa = \int_a^1 \chi(w) dw,$$

where $\chi(w)$ is the invariant density of the map. For obtaining a tractable approximation for the invariant density, however, these authors make a simplifying assumption that the ON and OFF periods are stochastically independent, which leads to the random wall model referred to above; a similar assumption was made earlier, see Baladi et. al. [3], but in a slightly different context, namely for studying the behaviour of the correlation function of intermittent families of maps close to the boundary of ergodic and non-ergodic regions.

Mondragon et. al. [36] show that their random wall model gives rise to a heavy tail (Pareto) distribution for the length of both ON and OFF periods:

$$P(\text{ON} \geq w) = O\big(w^{\frac{m_2-2}{m_2-1}}\big) \quad P(\text{OFF} \geq w) = O\big(w^{\frac{m_1-2}{m_1-1}}\big) \quad (n \to \infty),$$

where $w > 0$ denotes the time spent in ON or OFF state.

These authors also establish that the aggregated traffic over N sources has long memory by demonstrating, under a Bernoulli trials assumption, that the behaviour of N independent traffic sources, each of load κ can be approximated by another intermittency map - called a *sawtooth map* - of the same form as (7.1), but whose parameters may be expressed in terms of the parameters of the original map. The work of these authors thus demonstrates that the sawtooth-intermittency maps can provide an alternative mathematical model for the observed self-similar behaviour of Ethernet traffic.

Appendix: induced maps and towers

In this technical appendix we give details of how the recurrence and correlation properties are obtained for families of intermittency maps, In an abstract set up that we describe below (in the terminology adopted by Young [49, 48]), a direct relationship between the recurrence properties and correlation decay rates will be given, with the recurrence properties being easier to analyse for the class of intermittency maps. For a fixed reference set $\mathcal{T}_0$, which can be regarded as a topological space, a set $\mathcal{T}$ and a map $\zeta : \mathcal{T} \to \mathcal{T}$, both depending on the first return time, $T_{\mathcal{T}_0}(w)$, are defined as follows:

$$\zeta(w,\ell) = \begin{cases} (w, \ell+1) & \text{if } \ell + 1 < T_{\mathcal{T}_0}(w), \\ (\zeta^{T_{\mathcal{T}_0}(w)} w, 0) & \text{if } \ell + 1 = T_{\mathcal{T}_0}(w), \end{cases} \tag{7.3}$$

and $\mathcal{T}=\{(w,\,\ell)\in\mathcal{T}_0\times\mathbb{N}:\,\ell<T_{\mathcal{T}_0}(w)\}$. The map $\zeta(w,\ell)$ is piecewise differentiable (but not necessarily invertible) and $\mathcal{T}$ is called a *tower* built over $\mathcal{T}_0$. To define $\mathcal{T}$ more precisely we assume that there exists a countable partition $\mathcal{P}^*=\{\omega_{0,i}\}_{i\in\mathbb{N}}$ on $\mathcal{T}_0$ and that the first return function $T_{\mathcal{T}_0}:\mathcal{T}_0\to\mathbb{N}$ is constant on each element of $\mathcal{P}^*$. It is assumed that $\zeta^{T_{\mathcal{T}_0}(w)}(w)$ maps each $\omega_{0,i}$ bijectively onto $\mathcal{T}_0$.

Recall that by definition $T_{\mathcal{T}_0}(w)\equiv\min\{n\geq 1\,:\,\zeta^n w\in\mathcal{T}_0,\,w\in\mathcal{T}_0\}$, and analogous to $T_{\mathcal{T}_0}(w)$ there is a return function $\hat{T}_{\mathcal{T}_0}:\mathcal{T}\to\mathbb{N}$ which is given by $\hat{T}_{\mathcal{T}_0}(w,\ell)\equiv\min\{n\geq 0\,:\,\zeta^n(w,\ell)\in\mathcal{T}_0,\,(w,\ell)\in\mathcal{T}\}$. Partitioning the ℓ'th level $\mathcal{T}_\ell$ into pieces $\{\omega_{\ell,i}\}$, the map $\zeta^\ell(w,0)$ takes $\omega_{0,i}$ bijectively onto $\omega_{\ell,i}$ when $\ell+1<T_{\mathcal{T}_0}(\omega_{0,i})$, and the function $\hat{T}_{\mathcal{T}_0}(w,\ell)$ is constant on each $\omega_{\ell,i}$. The partition $\mathcal{P}^*$ on $\mathcal{T}_0$ is extended so that it contains all partition elements $\{\omega_{\ell,i}\}$ in $\mathcal{T}$, and this extended partition is denoted by $\mathcal{P}$. To turn $\mathcal{T}$ into a measure space we introduce a reference measure m and a σ-algebra $\mathcal{B}$ generated by subsets of $\mathcal{T}$. If $\int_{\mathcal{T}_0}T_{\mathcal{T}_0}(w)\,dm(w)<\infty$ then m is a finite measure on $\mathcal{T}$. To get a notion of distance between two points $w_1,\,w_2\in\mathcal{T}_0$ we define their separation time $s(w_1,w_2)$ to be

$$s(w_1,w_2)=\inf\{n>0:(\zeta^{T_{\mathcal{T}_0}(w_1)})^n(w_1),\,(\zeta^{T_{\mathcal{T}_0}(w_2)})^n(w_2)\,\text{lie in distinct}\,\omega_{0,i}\text{'s}\}.$$

If $s(w_1,w_2)$ is large then w_1 and w_2 are considered to be fairly close. For two points $(w_1,\ell),(w_2,\ell)\in\omega_{\ell,i}$ their separation time is defined to be equal to the separation time of $\zeta^{-\ell}(w_1,\ell),\zeta^{-\ell}(w_2,\ell)\in\omega_{0,i}$.

Properties of $(\zeta,\mathcal{T})$ We now list three assumptions for $(\zeta,\mathcal{T})$.

(A1) Bounded distortion: There are constants $\beta\in(0,1)$ and $K>0$ such that for all $w_1,w_2\in\mathcal{T}_0$

$$\left|\frac{J\zeta^{T_{\mathcal{T}_0}(w_1)}(w_1)}{J\zeta^{T_{\mathcal{T}_0}(w_2)}(w_2)}-1\right|\leq K\beta^{s(\zeta^{T_{\mathcal{T}_0}(w_1)}(w_1),\zeta^{T_{\mathcal{T}_0}(w_2)}(w_2))}.$$

Here $J\zeta^{T_{\mathcal{T}_0}(w)}(w)$ denotes the Jacobian of $\zeta^{T_{\mathcal{T}_0}(w)}(w)$ with respect to m (which we assume to be nonsingular) and $J\zeta(w,\ell)\equiv 1$ on $\mathcal{T}-\zeta^{-1}\mathcal{T}_0$. In addition it is assumed that $J\zeta(w,\ell)>0$, m-a.e.

(A2) Topological mixing: let $T_i=T_{\mathcal{T}_0}(\omega_{0,i})\in\mathbb{N}$. Then $\gcd\{T_i\}=1$.

(A3) Finite reference measure: $\int_{\mathcal{T}_0}T_{\mathcal{T}_0}(w)\,dm(w)<\infty$.

Turning to the statistical properties of $(\zeta,\mathcal{T})$ it will be useful to define function spaces $\mathcal{H}_\beta$ and $\mathcal{H}_\beta^+$ where

$$\begin{aligned}\mathcal{H}_\beta&=\{\varphi:\mathcal{T}\to\mathbb{C}:|\varphi(x)-\varphi(y)|\leq\beta^{s(x,y)}\,\forall x,y\in\mathcal{T}\},\\ \mathcal{H}_\beta^+&=\{\varphi\in\mathcal{H}_\beta:\text{on each }\omega_{\ell,i}\text{ either }\varphi(x)\equiv 0\text{ or }\varphi(x)>0,\\ &\qquad\text{and }\exists\,K_\varphi>0:\left|\frac{\varphi(x)}{\varphi(y)}-1\right|\leq K_\varphi\beta^{s(x,y)}\;\forall x,y\in\mathcal{T}\}.\end{aligned}$$

Let us summarize the statistical properties of the map given by equation (7.3), assuming that (A1), (A2) and (A3) hold. Regarding the invariant measure, it is proved in Young [49] that $\zeta(w,\ell)$ admits a finite probability measure ν which is absolutely continuous with respect to m. Its density $\chi(w,\ell)$ with respect m is strictly positive and lies in $\mathcal{H}_\beta^+$. The measure ν also has the mixing property 2.10.

Regarding the rate of decay of correlations, the correlation function $R_{\varphi,\psi}(n)$ (for $\varphi,\psi\in\mathcal{H}_\beta$) depends on the asymptotics of $m\{w\in\mathcal{T}_0\,:\,T_{\mathcal{T}_0}(w)>n\}$ and $m\{(w,\ell)\in\mathcal{T}:\hat{T}_{\mathcal{T}_0}(w,\ell)>n\}$ as follows:

(i) Exponential decay. Suppose $m\{w \in \mathcal{T}_0 : T_{\mathcal{T}_0}(w) > n\} < \theta^n$ for some $\theta < 1$. Then there exists a $\tilde{\theta} < 1$ such that $R_{\varphi,\psi}(n) = O(\tilde{\theta}^n)$.

(ii) Polynomial decay. Suppose that $m\{w \in \mathcal{T}_0 : T_{\mathcal{T}_0}(w) > n\} = O(1/n^\beta)$ where $\beta > 0$. Then

$$R_{\varphi,\psi}(n) = \sum_{k>n} m\{w \in \mathcal{T}_0 : T_{\mathcal{T}_0}(w) > n\} \int \varphi(w)\, d\nu \int \psi(w)\, d\nu + O\big(\frac{1}{n^{\lfloor\beta\rfloor}}\big).$$

(iii) Slow decay. Suppose that $m\{(w,\ell) \in \mathcal{T} : \hat{T}_{\mathcal{T}_0}(w,\ell) > n\} = \rho(n)$ with $\rho : \mathbb{R}^+ \to \mathbb{R}^+$ a slowly varying function at ∞ with $\rho(w) \in C^\infty$ and monotonically decreasing to zero on $[M,\infty)$ for some $M > 0$. Then $R_{\varphi,\psi}(n) = O(\rho(n))$.

Remarks: For the upper bounds parts (i) and (ii) are proved in Young [49] while the optimal bounds where achieved by Sarig [44] (for $\beta > 2$) and extended to all $\beta > 0$ by Gouëzel [17]. Part (iii) is proved in Holland [22]. The return time functions $T_{\mathcal{T}_0}(w)$ and $\hat{T}_{\mathcal{T}_0}(w,\ell)$ are also related by the property:

$$m\{(w,\ell) \in \mathcal{T} : \hat{T}_{\mathcal{T}_0}(w,\ell) > n\} = \sum_{k\geq n} m\{w \in \mathcal{T}_0 : T_{\mathcal{T}_0}(w) > k\}.$$

In the specific examples that we consider, the reference measure m will be Lebesgue measure, and in this situation the quantity $m\{w \in \mathcal{T}_0 : T_{\mathcal{T}_0}(w) = n\}$ becomes the quantity $G(\mathcal{T}_0, n)$, defined in Section 4.

Constructing towers for intermittency maps. To deduce decay of correlations for the regular intermittency maps $\zeta : J \to J$, for $J = [0,1]$, as described in Section 5, one idea is to find a way of analysing the behaviour of the intermittency map using a map given by equation (7.3) acting on a given tower $\mathcal{T}$. Formally this is done by identifying a subinterval $\tilde{J}$, a return function $T_{\tilde{J}} : \tilde{J} \to \mathbb{N}$ (which is not necessarily the first return time to $\tilde{J}$), and a projection map $\pi : \mathcal{T} \to J$ defining a semi-conjugacy between the map $\zeta(w,\ell)$ on the tower, and the intermittency map $\zeta(w)$. Let us explain the main ingredients of the construction. Further details can be found in Young [49]. We remark that this construction can be used to analyse decay of correlations for many other types of systems, (for example logistic maps, Bruin et. al. [11]).

Pick a closed subinterval $\tilde{J} \subset [0,1]$ which does not contain the neutral fixed point (for example $\tilde{J} = [1/2,1]$) and find a countable partition $\mathcal{P}_{\tilde{J}}$ into a set of elements $\{\omega_i\}_{i\in\mathbb{N}}$, with the return function $T_{\tilde{J}} : \mathcal{P}_{\tilde{J}} \to \mathbb{N}$ constant on each ω_i. Moreover it is required that for each $\omega \in \mathcal{P}_{\tilde{J}}$, $\zeta^{T_{\tilde{J}}(\omega)}(\omega) = \tilde{J}$ bijectively and with bounded distortion (in the sense that $|\big(\zeta^{T_{\tilde{J}}(w)}\big)''(w)/\big(\big(\zeta^{T_{\tilde{J}}(w)}\big)'(w)\big)^2| < \infty$). If $\sum_{n=1}^{\infty}\sum_{k=n}^{\infty} G(\tilde{J},k) < \infty$, then a straightforward exercise (see Young [49]) implies that the correlation decay rates for the intermittency map can be determined using the model defined by the map (7.3) with tower

$$\mathcal{T} = \{(w,\,\ell) \in \tilde{J} \times \mathbb{N} : \,\ell < T_{\tilde{J}}(w)\}.$$

In particular the correlation decay then depends on the distribution of the recurrence times, and in the context of regular intermittency maps, the correlations depend on $G(\tilde{J},k)$.

Let us explain first how to estimate this quantity, and then explain how $R_{\varphi,\psi}(n)$ depends on $G(\tilde{J},k)$ for $\varphi,\psi \in \mathcal{H}$. For $\tilde{J} = [1/2,1]$, any point in $\tilde{J}$ that returns back to $\tilde{J}$ at time n must fall in $[w_n, w_{n-1}]$ just after leaving $\tilde{J}$. Here $w_0 = 1/2$ and $w_n = \zeta^{-1}(w_{n-1}) \cap [0,1/2]$. Hence $G(\tilde{J},n) = O(|w_{n-1} - w_n|)$, and to analyse the behaviour of w_n as $n \to \infty$ it is useful to appeal to the following result (see Chapter 4 of Aaronson [1] and de Bruijn [14]): let $f : \mathbb{R}^+ \to (0,1)$ be regularly varying at 0 with index $\alpha > 0$, and assume that $w^\alpha f(1/w) \in C^\infty$ on $[C_0, \infty)$ for some $C_0 > 0$. Let $w_0 = 1$ and let $w_n = g(w_{n+1})$ where $g(w) = w(1 + f(w))$. Then

$$w_n \sim \frac{1}{\alpha^{\frac{1}{\alpha}} h^{-1}(n)} \quad \text{where } h(w) = \frac{1}{f(1/w)}.$$

For the intermittency maps described by equation 5.1, it can be deduced that $w_n = O(n^{-1/\alpha})$ and therefore $G(\tilde{J}, n) = O(n^{-1-1/\alpha})$. A more involved calculation (see Holland [22]) reveals that for the intermittency maps described by equation (5.9), $w_n = O(1/(n(\log n)^{\beta+1}))$ and hence $G(\tilde{J}, n) = O(1/(n^2(\log n)^{\beta+1}))$. Moreover the correlations $R_{\varphi,\psi}(n)$ satisfy

$$R_{\varphi,\psi}(n) = O\left(\sum_{k \geq n} \sum_{j \geq k} G(\tilde{J}, j)\right) = O\left(\sum_{k \geq n} x_k\right)$$

when this quantity is finite, and this quantity is sharp if the asymptotics are polynomial in n as $n \to \infty$, Gouëzel [17] and Sarig [44]. For the regular intermittency maps (5.1) and (5.9) the correlation functions are given by $R_{\varphi,\psi}(n) = O(n^{1-1/\alpha})$ and $R_{\varphi,\psi}(n) \leq K(\log n)^{-\beta}$ respectively (for some constant K).

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