Research Article

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Testing for asymmetry in betas of cumulative returns: Impact of the financial crisis and crude oil price

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Abstract: We introduce a functional factor model to investigate the dependence of cumulative return curves of individual assets on the market and other factors. We propose a new statistical test to determine whether the dependence in two sample periods are equal. The statistical properties of the test are established by asymptotic theory and simulations. We apply this test to study the impact of the recent financial crisis and trends in oil price on individual stock and sector ETFs. Our analysis reveals the significance of the daily oil futures curves and their different impact on individual stocks and sector ETFs. It also shows that the functional approach has an information content different from that obtained from scalar factor models for point-to-point returns.

Keywords: Asymmetric beta, cumulative intraday return curves, functional data analysis, two sample test

MSC 2010: 62G10

1 Introduction

The contribution of this paper is twofold. At the methodological level, we propose a statistical inferential tool to test the equality of functional regression betas between two samples. At the empirical level, we utilize the new tool to investigate the impact of the 2007 to 2009 financial crisis, and the regime of increasing or decreasing oil prices on suitably defined betas of intraday cumulative returns. Our analysis uses cumulative intraday return (CIDR) curves computed from high frequency data. Such curves describe how the return on an investment evolves with time over a relatively short period. The information contained in them is different than the information provided by minute-by-minute or trade-by-trade returns, and may be more relevant in situations where positions are held for a period of few hours within a trading day. The two sample test we propose adds to the research on *asymmetric correlations*, a term referring to different correlations between security returns during market downturns and upturns. The research up to date is concerned with point-to-point returns on daily or longer time scales. We investigate this problem at the intraday level.

The study of asymmetric correlations is important for at least two reasons. First, hedge ratios rely crucially on the correlations between the assets to be hedged and the associated financial instruments. The presence of asymmetric correlations can impact hedging effectiveness. Asymmetric correlations require dynamic changing of hedge ratios conditional on market conditions. Second, portfolio diversification is the center piece of standard investment theory. It is found that correlations between stocks tend to be much higher as the market falls. Thus, the effectiveness of diversification might be questionable, particularly during "bear" markets. Asymmetric correlations of point-to-point stock returns with market indexes are well documented in the fi-

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nance literature. This paper focuses on an analogous problem in the setting of cumulative returns constructed from high frequency price data. This new setting is more relevant for traders who may change positions before the close of a trading day; studies based on daily closing prices are less relevant. At present, there are no statistical tools to study asymmetric correlations within the setting of cumulative returns because these are curves, one curve per day, rather than numbers, as the usual daily returns are. An asset's beta is the product of the correlation and the volatility of the asset relative to the volatility of the benchmark (the market). Therefore, asymmetric correlations and asymmetric betas contain similar information. However, betas are more closely related to asset pricing theories and more useful in understanding the riskiness of the associated assets. For these reasons, the extension we propose focuses on the betas. A natural framework to examine suitably defined betas is a two sample testing paradigm. For example, we want to compare the betas in "bull" vs "bear" markets, or in the regimes of declining vs falling oil prices. We therefore propose a statistical two sample test in the context of functional data analysis. The test can be broadly used to examine the statistical equality of the regression coefficients of the same model estimated from two samples. Although two sample problems in the context of functional data (curves) have been considered, e.g., [4, 12, 17, 24, 25], two sample inference for functional regression models have not been previously studied.

The new test is used to investigate whether the betas of individual assets are different before, during and after the financial crisis. The 30 components of the Dow Jones Industrial Average and the 9 selected sector ETFs are the assets used in the study. The results show that roughly one third of the individual stocks have statistically significantly different market betas before and during (10 stocks), during and after (11 stocks) and before and after (12 stocks) the financial crisis. For the ETFs, only one has a different beta before and during the crisis, five have different betas during and after, and five have different betas before and after the financial crisis.

To investigate the impact of the rising vs falling oil price, a crude oil price factor is added to the functional regression model. First, the impact of the crude oil factor is studied by using it as an additional factor in the comparison of the before, during and after the financial crisis. The results show that when the crude oil factor is added to the model, the betas for five, seven and nine out of the nine sector ETFs are different for the three paired samples, respectively. For the individual stocks, the number of significantly different betas are also dramatically increased for all of the three paired subsamples. Since the test is a joint test and, in the above mentioned results, we already tested the significance of the asymmetric market beta, the increases in the number of significantly different betas are due to the impact of the crude oil factor. Subsequently, different samples conditional on the trend of the crude oil prices are used to investigate the impact of the crude oil factor from a different angle. The test results show that during "bull" and "bear" crude oil prices, the betas for ETFs are different for six out of the nine sectors. For the individual stocks, the results show that the betas for most of the stocks are insignificant during the "bull" oil price subsample, and positive and significant during the "bear" oil price subsample. That is, negative crude oil price movements have a negative impact on intraday cumulative equity returns.

Furthermore, a comparison is made for the results of the functional factor model against the results of the point-to-point return model. The functional factor model, which considers intraday behavior, or in other words how the return accumulates within a trading day, reveals new information content regarding the relationship between equity price movement and the factors.

The remainder of the paper is organized as follows. In Section 2, we review finance research related to the subject matter of this paper. Section 3 introduces the statistical model. After a general framework has been established, Section 4 describes the test procedure and the properties of the test. Section 5 presents its application to cumulative intraday returns on blue chip stocks and sector ETFs, and a comparison of the findings to those obtained using point-to-point returns. Section 6 utilizes Monte Carlo simulations to examine the empirical size and power of the test, and Section 7 concludes the paper and proposes future research directions.

2 Research on asymmetric correlations

To provide some background, in this section we review the findings of the related research, but for point-to-point returns.

Using three alternative definitions of "bull" and "bear" markets, Fabozzi and Francis [10] found that the betas were not significantly different in the two types of markets for most of the 700 stocks studied (all the stocks traded at NYSE at that time). Kim and Zumwalt [22] showed that most of the tested 322 securities exhibited betas which were not significantly different in up and down markets. Clinebell, Squires and Stevens [7] revisited the procedures for new samples of [10]. Unlike the findings of [10], they found statistically different betas for stocks over "bull" and "bear" markets. More recently, Wardwood and Anderson [27] used a logistic smooth transition market (LSTM) model to investigate whether "bull" and "bear" market betas for Australian industry portfolios differ. Their results indicate that "bull" and "bear" betas are significantly different for most industries, and that up-market risk is not always lower than down-market risk. Other studies discover asymmetric betas on portfolios based on different criteria, such as size based portfolios (see [5, 19, 28]), risk based portfolios (see [26, 28]) and past performance based portfolios (see [9, 28]).

Ang and Chen [1] found strong asymmetric correlations between stock portfolios and the US market. In [2, 3, 6, 8], among others, asymmetries in betas of stock returns were documented, but no formal statistical tests were provided. Hong, Tu and Zhou [15] developed a model-free test for asymmetric correlations and provided formal tests for asymmetric betas and covariances. They sorted stocks by size, book-to-market and momentum, and found strong evidence of asymmetries for both size and momentum portfolios, but no evidence for book-to-market portfolios. They also showed that incorporating asymmetries into investment decisions can be of substantial importance for an investor with disappointment aversion. All the above studies use monthly closing price return data.

Studies in the finance literature have shown that crude oil prices is an important factor that impacts the economy as well as the equity markets. Hamilton [14] examined the correlation between the price of crude petroleum and the U.S. recessions. He found that oil shocks were a contributing factor in at least some of the U.S. recessions prior to 1972. Ferson and Harvey [11] used changes in the monthly average U.S. dollar price per barrel of crude oil as a factor for cross-sectional expected returns. Jones and Kaul [21] documented that in the postwar period, the reaction of the United States and Canadian stock prices to oil shocks can be completely accounted for by the impact of these shocks on real cash flows. After the financialization of commodity futures markets in 2004–05, oil volatility has become a strong predictor of returns and volatility of the overall stock market. Furthermore, stocks' exposure to oil volatility risk now drive the cross-section of expected returns. Gogineni [13] investigated the impact of daily oil price changes on the stock returns of a wide array of industries and found that in addition to the stock returns of industries that depend heavily on oil, stock returns of some industries that use little oil are also sensitive to oil prices perhaps because their main customers are impacted by oil price changes.

3 A functional factor model

Suppose $P(t_0)$ is the price of an asset at time t_0, a conveniently selected time within a trading day. For instance, t_0 can be the NYSE opening time, i.e., 9:30 am EST. The object of the study is the time series of Cumulative Intraday Returns (CIDRs) defined by

$$R_n(t) = \log P_n(t) - \log P_n(t_0), \quad t_0 < t \le t_0 + T,$$

where n indexes the trading day, t_0 is the opening time of the trading day, and T is the number of time units in a trading day. One minute is used as the sample frequency, so $T = 390$ minutes for a typical NYSE trading day. The CIDR curves $R_n(t)$, see Figure 1, have shapes almost identical to the intraday price curves. This follows from the approximation $R_n(t) \approx (P_n(t) - P_n(t_0))/P_n(t_0)$ and the observation that for any day n, $P_n(t_0)$ is a fixed number. The curves $R_n(t)$ start from zero. Even though for each day n, the curve $R_n(t)$ is a realization

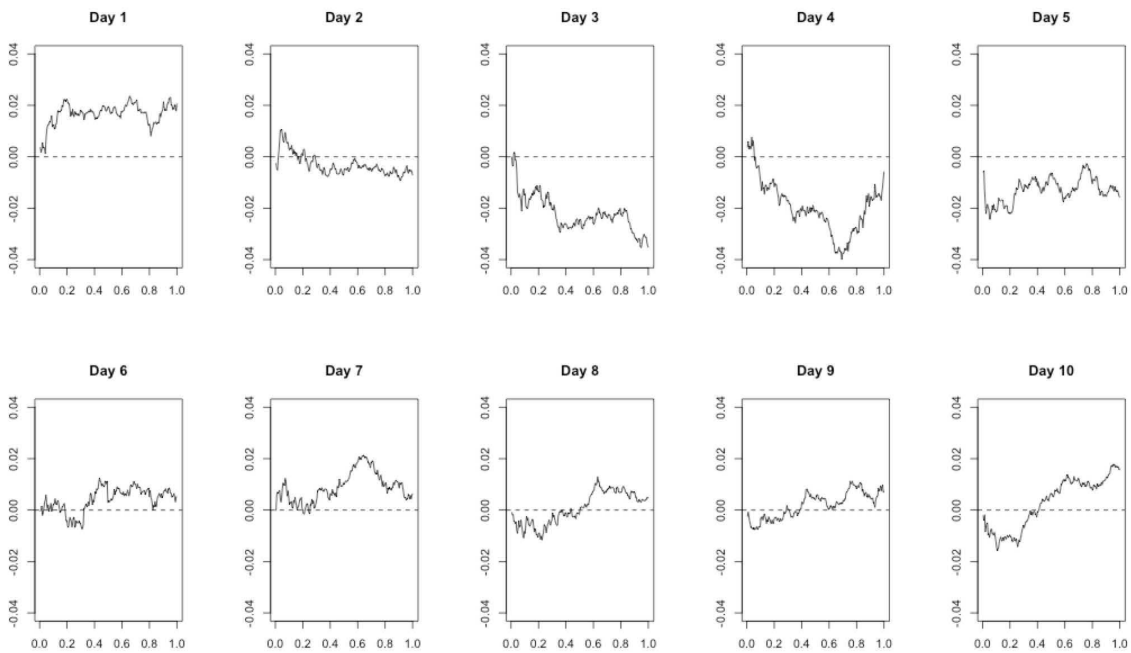

Figure 1. *Ten Consecutive CIDRs on JP Morgan Chase Stock.* This figure shows ten consecutive days CIDR on the JP Morgan Chase Stock. While each curve is a realization of a nonstationary stochastic process, the sequence of curves can be assumed to be a realization of a stationary time series with values in the space of square integrable functions. The curves have approximately the same distribution as the Brownian motion.

of a nonstationary stochastic process, basically a Brownian motion, the sequence of these curves is a stationary time series in a function space (cf. [18]). This is the main reason for studying the CIDRs instead of the price curves. At the methodological level, this allows us to focus on the shapes of intraday price curves by eliminating long term trends. The information contained in the shapes of the price curves can be expected to be different than in daily close-to-close returns or in high frequency returns.

To investigate the relationship between the cumulative intraday return curves of an asset and some factors, such as the intraday cumulative return curves of the market, Kokoszka, Miao and Zhang [23] postulated the following model:

$$R_n(t) = \beta_0(t) + \sum_{j=1}^{p} \beta_j F_{nj}(t) + \epsilon_n(t), \tag{3.1}$$

in which β_0 is an asset dependent intercept curve, F_{nj} is the CIDR curve of factor j (e.g., CIDR on a market index or oil price), and ϵ_n are error curves. This model is analogous to the commonly used factor models (i.e., CAPM), but the independent and the dependent variables are curves instead of scalar time series. The focus of this work is on the coefficients β_j, which describe the dependence of the CIDRs of an individual asset on the factors F_{nj}. The rationale for such a focus is that the intraday price curves of individual assets are generally proportional to the market index price curves. In the spirit of the CAPM, this proportionality should be even more pronounced when considered in the context of cumulative returns. In the application, the interest is in whether the betas β_j in (3.1) are different over different periods. This is not trivial since the dependent and independent variables are curves, and the standard t-test does not apply. Thus, in the next section, a two sample statistical test is developed applicable to the new functional settings.

4 Two sample test for functional factor models

Two nonoverlapping periods of time are considered, the first period includes N trading days, the second M trading days. The two samples are assumed to be independent. However, since each sample generally consists of data on consecutive trading days, serial independence within each sample is not assumed. Instead, a general nonparametric form of dependence, quantified in Section 4.1, is assumed. We then proceed with the definition of the test statistic in Section 4.2 and derive a consistent test in Section 4.3. A reader not interested in the details of the test may skip these subsections. They derive a test which allows us to test the null hypothesis that the β_j in (3.1) are the same for the two disjoint time periods. The test statistics is complex, but its distribution under the null hypothesis can be approximated by a standard chi-square distribution, so the application of the test is not difficult.

4.1 Dependence structure

We assume a very general form of dependence which has been extensively employed in time series research for over a decade. It relies on the representation of a stationary time series in an abstract space as a Bernoulli shift of independent and identically distributed (i.i.d.) errors. See [16, Chapter 16] for historical background, detailed discussion and additional references. Our results could also be proven using different dependence notions.

Suppose H is a separable Hilbert space, as shown in [20]. Let $p \geq 1$, and let L_H^p be the space of H-valued random elements X such that

$$\nu_p(X) = (E\|X\|^p)^{1/p} < \infty.$$

In our application, the Hilbert space in question is the space of square integrable functions on the interval $[0, 1]$, and so the norm is defined by $\|X\|^2 = \int_0^1 X^2(t)\,dt$.

Definition 4.1. A sequence $\{X_n\} \in L_H^p$ is called *L^p-m-approximable* if each X_n admits the representation

$$X_n = f(u_n, u_{n-1}, \ldots), \tag{4.1}$$

where the u_i are i.i.d. elements taking values in a measurable space S, and f is a measurable function $f\colon S^\infty \to H$. Moreover, we assume that if $\{u_i'\}$ is an independent copy of $\{u_i\}$ defined on the same probability space, then letting

$$X_n^{(m)} = f(u_n, u_{n-1}, \ldots, u_{n-m+1}, u_{n-m}', u_{n-m-1}', \ldots),$$

we have

$$\sum_{m=1}^{\infty} \nu_p(X_n - X_n^{(m)}) < \infty. \tag{4.2}$$

The gist of Definition 4.1 is that the dependence of f in (4.1) on the innovations far in the past decays so fast that these innovations can be replaced by their independent copies. Such a replacement is asymptotically negligible in the sense quantified by (4.2). The reader will notice that if $f(u_n, u_{n-1}, \ldots) = \sum_{j=0}^{\infty} c_j u_{n-j}$, a dependent linear model is recovered. A nonlinear f allows a fairly general nonlinear form of dependence.

4.2 The test statistic

For simplicity, we use the superscript * to denote the observations taken in the second period. Thus, the first sample follows model (3.1), and the second sample follows the model

$$R_m^*(t) = \beta_0^*(t) + \sum_{j=1}^{p} \beta_j^* F_{mj}^*(t) + \epsilon_m^*(t), \quad m = 1, 2, \ldots, M.$$

We wish to test the null hypothesis:

$$H_0: \quad \boldsymbol{\beta} = \boldsymbol{\beta}^*, \quad \text{where } \boldsymbol{\beta} = [\beta_1, \dots, \beta_p]^T, \ \boldsymbol{\beta}^* = [\beta_1^*, \dots, \beta_p^*]^T.$$

We want to determine whether the vectors of the coefficients of the factors are different over the two sample periods. In the simplest case of a single factor, which is the CIDR on a market index, this corresponds to checking whether the asset betas are statistically the same. We emphasize that even in this case our approach provides information which has not been available so far since we study the dependence of the shape of the intraday price curve of an asset on the shape of the intraday price curve of a market index, whereas previous research only considers point-to-point returns. In other words, we care about how and how much the price moves, whereas the traditional model only concerns how much the price changes.

The test is based on the comparison of suitable estimators $\hat{\boldsymbol{\beta}}$ and $\hat{\boldsymbol{\beta}}^*$. We now define the estimator $\hat{\boldsymbol{\beta}}$ introduced in [23]; $\hat{\boldsymbol{\beta}}^*$ is defined analogously. To simplify formulas, all functions (e.g., R_n, F_{nj}) appearing below are assumed to be defined on the unit interval $(0, 1]$. Thus, before performing the test, $t \in (t_0, t_0 + T]$ must be replaced by $(t - t_0)/T$. In other words, the CIDR functions are rescaled to the unit interval. To further lighten the notation, we introduce the inner product

$$\langle f, g \rangle = \int_0^1 f(t)g(t)\, dt.$$

With this notation, we define

$$\widehat{\mathbf{F}} = \left[N^{-1} \sum_{n=1}^{N} \langle F_{nj} - \bar{F}_j, F_{nk} - \bar{F}_k \rangle, \ j, k = 1, 2, \dots, p \right] \quad (p \times p),$$

$$\widehat{\mathbf{R}} = \left[N^{-1} \sum_{n=1}^{N} \langle R_n - \bar{R}, F_{nj} - \bar{F}_j \rangle, \ j = 1, 2, \dots, p \right]^T \quad (p \times 1),$$

where $\bar{R}(t) = N^{-1} \sum_{n=1}^{N} R_n(t)$ and $\bar{F}_j(t) = N^{-1} \sum_{n=1}^{N} F_{nj}(t)$, $(j = 1, \dots, p)$. The matrix $\widehat{\mathbf{F}}$ includes cross-sectional sample covariances of the factor curves, and the vector $\widehat{\mathbf{R}}$ consists of the sample covariances of the CIDRs on the asset of interest with the factor of CIDRs. The estimator of [23] is defined as

$$\widehat{\boldsymbol{\beta}} = \widehat{\mathbf{F}}^{-1} \widehat{\mathbf{R}}.$$

It is a hybrid method of moments/least squares estimator which is consistent and asymptotically normal. The intercept function is estimated by

$$\hat{\beta}_0(t) = \bar{R}(t) - \sum_{j=1}^{p} \hat{\beta}_j \bar{F}_j(t).$$

A natural test statistic would be $\|\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}^*\|^2 = \sum_{j=1}^{p} (\hat{\beta}_j - \hat{\beta}_j^*)^2$, but its asymptotic distribution is very complex, and even a Monte Carlo test based on it would be extremely time consuming to implement. However, a quadratic form of $\hat{\boldsymbol{\beta}} - \hat{\boldsymbol{\beta}}^*$ can be constructed whose limit distribution, as $N, M \to \infty$, is the standard chi-square distribution with p degrees of freedom. To define this quadratic form, we first introduce the long-run covariance matrices which describe serial dependence within each sample. We again focus on the first sample, with the understanding that everything is defined analogously for the second sample.

For $j = 1, \dots, p$, set $\mu_j(t) = EF_{nj}(t)$, and introduce the vectors

$$\boldsymbol{\xi}_n = [\xi_{n1}, \dots, \xi_{np}]^T = [\langle \epsilon_n, F_{n1} - \mu_1 \rangle, \dots, \langle \epsilon_n, F_{np} - \mu_p \rangle]^T.$$

Denote by $\boldsymbol{\Gamma}$ their long-run covariance matrix defined by

$$\boldsymbol{\Gamma} = \sum_{h=-\infty}^{\infty} E[\boldsymbol{\xi}_0 \boldsymbol{\xi}_h^T]. \tag{4.3}$$

Let $\widehat{\boldsymbol{\Gamma}}$ be a consistent estimator of $\boldsymbol{\Gamma}$. Many estimators are implemented in statistical packages. In our empirical application, we used the `R` function `lrvar` (with default tuning parameters) in the package `sandwich`.

An estimator must be applied to sample versions of the unobservable vectors $\boldsymbol{\xi}_n$. The jth component of $\hat{\boldsymbol{\xi}}_n$ is defined by

$$\hat{\xi}_{nj} = \langle \hat{\epsilon}_n, F_{nj} - \bar{F}_j \rangle, \tag{4.4}$$

$$\hat{\epsilon}_n(t) = R_n(t) - \hat{\beta}_0(t) - \sum_{j=1}^{p} \hat{\beta}_j F_{nj}(t). \tag{4.5}$$

Once the estimators $\widehat{\boldsymbol{\Gamma}}$ and $\widehat{\boldsymbol{\Gamma}}^*$ have been computed, we can calculate the matrix

$$\widehat{\boldsymbol{\Delta}}_{M,N} := \frac{N+M}{N} \widehat{\mathbf{F}}^{-1} \widehat{\boldsymbol{\Gamma}} \widehat{\mathbf{F}}^{-1} + \frac{N+M}{M} (\widehat{\mathbf{F}}^*)^{-1} \widehat{\boldsymbol{\Gamma}}^* (\widehat{\mathbf{F}}^*)^{-1}.$$

The test statistic is then defined as the quadratic form

$$T_{M,N} := (N+M)(\widehat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}^*)^T (\widehat{\boldsymbol{\Delta}}_{M,N})^{-1} (\widehat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}^*).$$

4.3 Derivation of the test and its properties

The previous subsection established a quantity to test the hypothesis. This subsection provides the properties of the test. For simplicity, we only state the assumptions and results for one sample, with the understanding that the same assumptions and results hold for the second sample with the index m and the superscript * being added. Assumption 4.2 states that the factor and error functions are general nonlinear moving averages of some abstract errors. Assumption 4.3 states usual moment and independence assumptions in the context of random functions.

Assumption 4.2. Suppose that there exist independent sequences $\{\delta_n\}$ and $\{\eta_n\}$ consisting of i.i.d. random variables taking values in the measurable spaces $\mathcal{S}_\delta$ and $\mathcal{S}_\eta$, respectively. Assume that there are measurable functions f_j and e defined on the appropriate product spaces, such that $F_{nj} = f_j(\delta_n, \delta_{n-1}, \dots)$, $\epsilon_n = e(\eta_n, \eta_{n-1}, \dots)$.

Assumption 4.3. The factors $\{F_{nj}, j = 1, \dots, p\}$ and the errors $\{\epsilon_n\}$ are random elements of the space $L^2([0,1])$ which satisfy $E\|F_{nj}\|^4 < \infty$, $E\|\epsilon_n\|^4 < \infty$, and $E\epsilon_n = 0$. Moreover, for each n, the error functions $\{\epsilon_n\}$ are independent of the vector factors $\{\mathbf{F}_n\}$.

Under Assumptions 4.2 and 4.3, letting $\mathbf{F}_n = [F_{n1}, F_{n2}, \dots, F_{np}]^T$, it follows that the vector factors $\{\mathbf{F}_n\}$ and the errors $\{\epsilon_n\}$ are stationary random elements of the spaces, respectively, $(L^2)^p$ and L^2.

The above assumptions must hold for both samples. Intuitively, these assumptions require that the CIDRs R_n and the factors F_{nj} form stationary sequences of functions (for each sample separately). As noted in the introduction, the above stationarity has been verified using the tests of [18] which are implemented in the `R` package `ftsa`. Examination of the graphs, cf. Figure 1, provides a visual justification.

Our last assumption requires the sample sizes to be comparable.

Assumption 4.4. Assume that, as $N, M \to \infty$,

$$\frac{N}{N+M} \to \theta \quad \text{for some } 0 < \theta < 1. \tag{4.6}$$

The following lemma, which follows from [23, Theorem 1.2], provides the first step in the derivation of the asymptotic distribution of the test statistic $T_{N,M}$.

Lemma 4.5. *Suppose that Assumptions 4.2 and 4.3 hold, the sequences $\{\epsilon_n\}$ and $\{\mathbf{F}_n\}$ are L^4-m-approximable, and the matrix $\mathbf{F}$, defined by*

$$\mathbf{F} = [E\langle F_{nj} - \mu_j, F_{nk} - \mu_k \rangle,\ j, k = 1, 2, \dots, p] \quad (p \times p), \tag{4.7}$$

is nonsingular. Then

$$\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) \xrightarrow{d} \mathbf{F}^{-1}\mathbf{W}, \tag{4.8}$$

where $\mathbf{W}$ is a p-dimensional mean zero Gaussian distribution with covariances $\mathrm{Var}(\mathbf{W}) = \boldsymbol{\Gamma}$, *and $\boldsymbol{\Gamma}$ is the long-run covariance matrix defined by* (4.3).

Relations leading to (4.8) also hold for the second sample. This is achieved by replacing all n by m and adding the superscript *. We thus have

$$\sqrt{M}(\widehat{\boldsymbol{\beta}}^* - \boldsymbol{\beta}^*) \xrightarrow{d} (\mathbf{F}^*)^{-1}\mathbf{W}^*, \tag{4.9}$$

where $\mathbf{W}^*$ is a p-dimensional mean zero Gaussian distribution with covariances $\boldsymbol{\Gamma}^*$.

The next theorem establishes the asymptotic distribution of $T_{N,M}$ under H_0.

Theorem 4.6. *Suppose that Assumptions 4.2–4.4 hold, the sequences $\{\epsilon_n\}$ and $\{\mathbf{F}_n\}$ are L^4-m-approximable, the matrix (4.7) is nonsingular, and the long-run covariance matrix $\boldsymbol{\Gamma}$ is nonsingular and is consistently estimated by $\widehat{\boldsymbol{\Gamma}}$. Also assume that the second sample satisfies analogous assumptions. Then, under H_0,*

$$T_{N,M} := (N+M)(\widehat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}^*)^T(\widehat{\boldsymbol{\Delta}}_{M,N})^{-1}(\widehat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}^*) \xrightarrow{d} \chi^2(p).$$

For the proof, see Appendix A.

Theorem 4.6 states that under suitable assumptions the asymptotic distribution of the statistic $T_{M,N}$ converges to the $\chi^2(p)$ distribution, if H_0 is true. We now establish the consistency of the testing procedure.

Theorem 4.7. *Suppose that the Assumptions of Theorem 4.6 hold, and $\boldsymbol{\beta}^* = \boldsymbol{\beta} + \boldsymbol{\eta}$ for some nonzero vector $\boldsymbol{\eta}$. Then, $T_{N,M} \xrightarrow{P} \infty$.*

For the proof, see Appendix B.

Theorem 4.7 implies that the power of the test approaches unity, as $N, M \to \infty$. That is, for large N and M, the probability of rejection of the null hypothesis if alternative is true converges to 100%.

5 Empirical application

In this section, we apply the test introduced in Section 4 in several settings. First the data, the assets and the factors are summarized. Next, Section 5.2 presents the results of the application of our test to periods before, during and after the financial crisis. Section 5.3 reports a similar analysis for the "bull" and "bear" crude oil price periods. Finally, in Section 5.4, the results are compared with those obtained by applying a simplified form of our test to point-to-point returns on the same assets and over the same time periods to compare the information contents of the cumulative return curves and the point-to-point returns.

5.1 Data

We are interested in examining whether the betas for individual stocks and "portfolios" are time-varying. We also want to show that our approach extracts different information than the classic factor models. Thus, our approach is applied to 30 stocks from the Dow Jones Industrial Average components during the study period. As proxies for "portfolios", the nine Select Sector SPDR ETFs are used. The nine Select Sector SPDR ETFs are Exchange Traded Funds that track the nine S&P 500 sector indexes. The ETFs hold individual stocks within the corresponding sectors. For ease of reference, Table 1 lists the companies and the Select Sector SPDR ETFs.

The factors include the market factors and the crude oil factor. The Dow Jones Industrial Average (DJ) is used as the market factor for individual stocks, and the S&P 500 Index (SP) as a proxy for the market for the ETFs. The crude oil factor is represented by the nearest term crude oil futures contract (CL). To construct continuous time series, we roll the nearest contracts to the next contract 20 calendar days before the expiration of the contracts (roughly at the beginning of each month). The Crude Oil Futures contracts are traded at the CME group.

The one minute frequency stock and ETF price data is obtained from Quantquote whereas the price data for the indexes and futures is from TickData.

Ticker	Company/Sector	Ticker	Company/Sector
Panel A: Individual Stocks			
AA	Alcoa Inc.	AXP	American Express Company
BA	Boeing Co.	BAC	Bank of America Corp.
CAT	Caterpillar	CSCO	Cisco Systems Inc.
CVX	Chevron Corporation	DD	E.I. Du Pont De Nemours & Co.
DIS	Walt Disney Co.	GE	General Electric Company
HD	Home Depot Inc.	HPQ	Hewlett-Packard Company
IBM	International Business Machines Corp.	INTC	Intel Corporation
JNJ	Johnson & Johnson	JPM	JPMorgan Chase & Co.
KFT	Kraft Foods Inc.	KO	The Coca-Cola Co.
MCD	McDonald's Corporation	MMM	3M Co.
MRK	Merck & Co. Inc.	MSFT	Microsoft Corporation
PFE	Pfizer	PG	Procter & Gamble Co.
T	AT&T Inc.	TRV	Travelers Companies Inc.
UTX	United Technologies Co.	VZ	Verizon Communications Inc.
WMT	Wal-Mart Stores Inc.	XOM	Exxon Mobil Corporation
Panel B: Sector ETFs			
XLY	Consumer Discretionary	XLP	Consumer Staples
XLE	Energy	XLF	Financials
XLV	Health Care	XLI	Industrials
XLB	Materials	XLK	Technology
XLU	Utilities		

Table 1. *List of Tickers.* This table presents the tickers of the assets used in this study.

5.2 Asymmetric CIDR betas around the financial crisis

In this subsection, we apply the statistical test to responses and factors listed in Section 5.1 around the recent financial crisis. Our objective is to determine whether there are pair-wise statistically significant differences for betas during periods before and during, during and after, and before and after the financial crisis, respectively. Comparing betas for periods around the financial crisis is particularly interesting, as it allows us to determine whether the crisis has changed the relationship between the factors and individual assets reflected by the betas defined in this paper. The periods are defined as: "Before" (06/13/2006–09/27/2007), "During" (11/01/2007–02/27/2009) and "After" (01/04/2010–12/30/2011) the financial crisis. The number of sample days for the three periods are 326, 328 and 503 days, respectively. The sample periods are selected after a preliminary analysis. We also applied the models to different periods. The different sample selection does not influence the conclusions materially. The conclusions do not depend on the specific proxy for the efficient market index either. We ran the analysis using SP500 in place of DJIA and obtained almost identical results.

We first run the one factor model and apply the test to the models for the three periods. The model is of the following form:

$$R_n(t) = \beta_0(t) + \beta_1 M_n(t) + \epsilon_n(t), \tag{5.1}$$

where $R_n(t)$ is the CIDR (a function of t) on day n, $M_n(t)$ is the CIDR of the S&P 500 index for the ETFs, and the CIDR of the Dow Jones Industrial Average Index for individual stock.

Table 2 presents the estimated results of the model of equation (5.1) for the three periods. The table includes the estimated betas, the standard error of the beta estimates and the P-values of the pair-wise difference tests using the above developed statistic test. The results show that all the betas are highly significantly different from zero, as indicated by the fact that the ratios of the estimators and the corresponding errors are between 5.00 (XLP for the "During" period) and 16.25 (XLE for the "After" period). In general, the curve betas are consistent with common sense. For instance, for all the three periods, the economically sensitive sectors

Symbol	$\hat{\beta}_{M1}$	SE_{M1}	$\hat{\beta}_{M2}$	SE_{M2}	$\hat{\beta}_{M3}$	SE_{M3}	$P_{1,2}$	$P_{2,3}$	$P_{1,3}$
XLB	0.87	0.07	0.76	0.05	0.61	0.04	0.21	0.02*	0.00***
XLE	0.78	0.09	0.81	0.07	0.65	0.04	0.78	0.04*	0.17
XLF	0.62	0.10	0.83	0.12	0.62	0.04	0.17	0.10	0.97
XLI	0.68	0.07	0.66	0.05	0.62	0.04	0.83	0.56	0.48
XLK	0.86	0.08	0.61	0.05	0.47	0.03	0.01**	0.02*	0.00***
XLP	0.37	0.05	0.30	0.06	0.23	0.03	0.35	0.23	0.01**
XLU	0.49	0.07	0.45	0.07	0.28	0.03	0.66	0.03*	0.00***
XLV	0.45	0.06	0.38	0.05	0.38	0.03	0.38	0.98	0.30
XLY	0.64	0.07	0.66	0.09	0.49	0.04	0.89	0.08*	0.05*

Table 2. *Asymmetric beta around the financial crisis (One Factor) – ETFs.* This table presents the estimation results for the model: $R_n(t) = \beta_0(t) + \beta_{Mi}M_n(t) + \epsilon_n(t)$ for the sector ETFs. Here, i is the ith period, and 1, 2 and 3 refer to the "Before", "During" and "After" financial crisis periods, respectively. The periods are defined as: "Before" (06/13/2006–09/27/2007), "During" (11/01/2007–02/27/2009) and "After" (01/04/2010–12/30/2011) the financial crisis. SE_{Mi} refers to the standard error of the estimates. $P_{i,j}$ is the P-value of the statistical test with H_0: $\beta_{Mi} = \beta_{Mj}$. The superscripts ***,** ,* and * indicate significant at 0.1%, 1%, 5% and 10%, respectively.

of XLE (energy), XLB (materials), XLI (industrials), XLF (financials), and XLK (technology) have higher betas for all periods than the other sectors.

The P-values show that for eight out of the nine sector ETFs, only XLK has significantly different betas before and during the financial crisis (0.86 vs 0.61). For the other ETFs, although the estimated betas are much higher for XLF during the financial crisis (0.83) than before (0.62), the difference is not statistically different. For all the others, even the magnitudes of the betas are very similar before and during the financial crisis. In summary, the results show that the curve betas of the sector ETFs do not increase during the market downward or "bear" market.

Comparing the betas during and after the financial crisis, the picture is quite different. For five XLB, XLE, XLK, XLU (utilities) and XLY (discretionary) out of the nine sector ETFs, the betas after the crisis are statistically significantly smaller than during the crisis at the 10% level. That is, the correlations between the asset and the market cumulative return curves generally decrease after the downward market.

Similarly, the results show that for five out of the nine sectors, the curve betas are significantly smaller after the financial crisis than before. For instance, the betas for XLB, XLK and XLU are 0.87, 0.86 and 0.49, respectively, before the financial crisis. The corresponding betas are only 0.61, 0.47 and 0.28 during the "After" period. Interestingly, these are 30%, 45% and 43% smaller although the market conditions were similar during the two periods. However, betas for those sectors are dramatically different. It might indicate that investors' behavior changes or may simply due to the portfolios holding different assets before and after the financial crisis. This issue might be partially revealed by testing individual stocks.

Table 3 reports the results for the 30 components of the Dow Jones Industrial Average Index. First, notice that all but one are statistically significantly different from zero with the estimate to standard error ratio ranging from 2.29 to 19.33 (the Beta of PG for the "Before" period has an estimate of 0.18 with standard error of 0.11). Second, 10 stocks have significantly different betas before and during the financial crisis. Among the 10 stocks, only two (AXP and BAC) have higher betas whereas the other eight have significantly lower betas during than before the financial crisis. This is consistent with the results for the ETFs. Third, one stock (MMM) has a significantly higher curve beta in the "After" period than the "During" period, 10 stocks have significantly lower curve betas in the "After" period than the "During" period, whereas betas for the other 19 do not show statistically significant differences between the "After" and "During" periods. Fourth, a direct comparison of the "before" and "after" periods show that only one stock (GE) has a higher beta, 10 stocks have lower betas, and the betas of the other 19 are not significantly different between the two periods. In summary, the results for both the ETFs and the individual stocks indicate that the betas for most of those assets do not increase during the financial crisis. However, the betas of some of the assets do decrease after the financial crisis. Therefore, the results are partially different from those reported in the literature for point-to-point returns.

Symbol	$\hat{\beta}_{M1}$	SE_{M1}	$\hat{\beta}_{M2}$	SE_{M2}	$\hat{\beta}_{M3}$	SE_{M3}	$P_{1,2}$	$P_{2,3}$	$P_{1,3}$
AA	0.94	0.12	0.83	0.14	0.71	0.06	0.55	0.42	0.09*
AXP	0.76	0.10	0.99	0.09	0.56	0.06	0.08*	0.00***	0.08*
BA	0.77	0.10	0.51	0.09	0.66	0.04	0.05*	0.10	0.33
BAC	0.41	0.09	0.90	0.21	0.56	0.09	0.03*	0.14	0.23
CAT	0.79	0.10	0.65	0.07	0.73	0.06	0.24	0.42	0.55
CSCO	0.68	0.10	0.58	0.07	0.52	0.04	0.45	0.41	0.14
CVX	0.46	0.09	0.62	0.05	0.46	0.04	0.11	0.01**	0.99
DD	0.69	0.15	0.59	0.06	0.58	0.05	0.55	0.94	0.51
DIS	0.61	0.12	0.76	0.06	0.56	0.04	0.29	0.01**	0.68
GE	0.49	0.05	0.64	0.14	0.62	0.05	0.29	0.89	0.05*
HD	0.68	0.09	0.62	0.09	0.37	0.06	0.64	0.01**	0.00***
HPQ	0.68	0.08	0.47	0.07	0.57	0.06	0.04*	0.25	0.26
IBM	0.39	0.08	0.49	0.06	0.38	0.03	0.31	0.10	0.94
INTC	0.74	0.08	0.51	0.08	0.41	0.04	0.04*	0.26	0.00***
JNJ	0.26	0.05	0.24	0.05	0.28	0.04	0.84	0.62	0.79
JPM	0.65	0.12	0.60	0.20	0.56	0.05	0.83	0.83	0.46
KFT	0.41	0.11	0.21	0.05	0.23	0.05	0.10	0.84	0.14
KO	0.23	0.07	0.29	0.06	0.28	0.04	0.43	0.82	0.50
MCD	0.52	0.11	0.35	0.05	0.19	0.03	0.17	0.00***	0.00***
MMM	0.56	0.09	0.36	0.06	0.58	0.03	0.06*	0.00***	0.84
MRK	0.59	0.07	0.42	0.06	0.41	0.04	0.08*	0.85	0.03*
MSFT	0.44	0.06	0.52	0.06	0.37	0.04	0.36	0.04*	0.34
PFE	0.60	0.08	0.35	0.06	0.36	0.04	0.01**	0.88	0.01**
PG	0.18	0.11	0.31	0.04	0.20	0.03	0.26	0.03*	0.87
T	0.47	0.07	0.61	0.06	0.29	0.03	0.13	0.00***	0.02*
TRV	0.49	0.14	0.32	0.14	0.30	0.04	0.37	0.90	0.18
UTX	0.72	0.09	0.54	0.06	0.56	0.04	0.08*	0.76	0.09*
VZ	0.48	0.11	0.56	0.06	0.22	0.03	0.53	0.00***	0.02*
WMT	0.54	0.10	0.31	0.07	0.17	0.04	0.07*	0.08*	0.00***
XOM	0.50	0.09	0.47	0.07	0.41	0.04	0.75	0.47	0.33

Table 3. *Asymmetric Beta around the financial crisis (One Factor) – Stocks.* This table presents the estimation results for the model: $R_n(t) = \beta_0(t) + \beta_{Mi} M_n(t) + \epsilon_n(t)$ for the 30 Stocks. Here, i is the ith period, and 1, 2 and 3 refer to the "Before", "During" and "After" financial crisis periods, respectively. The periods are defined as: "Before" (06/13/2006–09/27/2007), "During" (11/01/2007–02/27/2009) and "After" (01/04/2010–12/30/2011) the financial crisis. $P_{i,j}$ is the P-value of the statistical test with H_0: $\beta_{Mi} = \beta_{Mj}$. The superscripts ***,** ,* and * indicate significant at 0.1%, 1%, 5% and 10%, respectively.

Next we introduce the crude oil factor into the model and run a two factor model of the following form:

$$R_n(t) = \beta_0(t) + \beta_1 M_n(t) + \beta_2 C_n(t) + \epsilon_n(t), \tag{5.2}$$

where, $C_n(t)$ is the CIDR of the Crude Oil factor (CL).

Table 4 reports the results for the nine sector ETFs. In the "Before" period, the crude oil betas are only significantly different from zero for five ETFs. Except the beta for XLE (the Energy Sector ETF) which is 0.45, betas for all the other three ETFs are very small, even though they are statistically significantly different from zero (0.06, –0.04, –0.04 and –0.04 for XLB, XLP, XLV and XLY, respectively). For the "During" period, betas of the crude oil factor are significant for 6 sectors. The crude oil beta for XLE (0.44) is almost the same as for the "Before" period. The magnitudes of the other significant crude betas are much higher than before (0.21, 0.16, 0.07, 0.12 and 0.09 for the XLB, XLF, XLI, XLK and XLU). For the "After" period, all the crude oil betas are significant. The P-values of the pairwise joint tests show that jointly, 5, 7 and 9 ETFs have different betas for the three pair periods, respectively.

Comparing the results with Table 2, we observe that the additional significance is due to the significant differences of the crude oil factor. After introducing the crude oil factor, the number of significant P-values for the "Before" and "During" periods increases from 1 to 5. This means that at least for four of the ETFs (XLB, XLF, XLP and XLU), the crude oil betas are different between the two periods. On the other hand, crude oil

Symbol	$\hat{\beta}_{M1}$	SE_{M1}	$\hat{\beta}_{O1}$	SE_{O1}	$\hat{\beta}_{M2}$	SE_{M2}	$\hat{\beta}_{O2}$	SE_{O2}	$\hat{\beta}_{M3}$	SE_{M3}	$\hat{\beta}_{O3}$	SE_{O3}	$P_{1,2}$	$P_{2,3}$	$P_{1,3}$
XLB	0.86	0.07	0.06	0.03	0.68	0.05	0.21	0.04	0.47	0.04	0.34	0.04	0.01**	0.00***	0.00***
XLE	0.71	0.07	0.45	0.04	0.65	0.06	0.44	0.05	0.46	0.03	0.44	0.03	0.76	0.01**	0.00***
XLF	0.62	0.10	−0.01	0.02	0.77	0.12	0.16	0.07	0.51	0.04	0.27	0.04	0.01**	0.13	0.00***
XLI	0.68	0.07	0.01	0.02	0.63	0.05	0.07	0.03	0.54	0.04	0.21	0.04	0.15	0.01**	0.00***
XLK	0.86	0.08	−0.03	0.02	0.56	0.06	0.12	0.04	0.40	0.03	0.17	0.03	0.00***	0.02*	0.00***
XLP	0.38	0.05	−0.04	0.02	0.28	0.06	0.07	0.04	0.19	0.03	0.09	0.02	0.02*	0.42	0.00***
XLU	0.49	0.07	0.00	0.03	0.41	0.07	0.09	0.03	0.24	0.03	0.10	0.03	0.07*	0.07*	0.00***
XLV	0.46	0.06	−0.04	0.01	0.37	0.06	0.02	0.03	0.32	0.03	0.13	0.03	0.19	0.05*	0.00***
XLY	0.65	0.07	−0.04	0.02	0.64	0.09	0.06	0.06	0.40	0.04	0.20	0.04	0.19	0.05*	0.00***

Table 4. *Asymmetric Beta around the financial crisis (Two Factors) – ETFs.* This table presents the estimation results for the model: $R_n(t) = \beta_0(t) + \beta_{Mi}M_n(t) + \beta_{Oi}CL_n(t) + \epsilon_n(t)$ for the sector ETFs. Here, i is the ith period, and 1, 2 and 3 refer to the "Before", "During" and "After" financial crisis periods, respectively. The periods are defined as: "Before" (06/13/2006–09/27/2007), "During" (11/01/2007–02/27/2009) and "After" (01/04/2010–12/30/2011) the financial crisis. SE refers the standard error. $P_{i,j}$ is the P-value of the statistical test with H_0: $\boldsymbol{\beta}_i = \boldsymbol{\beta}_j$. The superscripts ***,** ,* and * indicate significant at 0.1%, 1%, 5% and 10%, respectively.

betas for XLK are different (−0.03 with standard error 0.02 for the "Before" period and 0.12 with standard deviation of 0.04 for the "During" period). Since the test is a joint test and the test is significant in the one factor model setting, we can not directly conclude that the crude oil betas for XLK are different for the two periods. However, considering that the crude oil beta is not significant for the "Before" period and it is significant for the "During" period, we can safely conclude that the crude oil betas for the two periods are statistically different. Similarly, comparing the results for the other two paired periods, the results show that adding the crude oil factor increases the number of significant results for all two paired periods. Interestingly, the crude oil betas for XLE are all significantly different from zero and consistent across all periods at roughly 0.44–0.45. The tests show they are not statistically different for all the paired periods. Broadly specking, however, the results confirm that crude oil is an important factor for the sector ETFs, and crude oil betas are time varying for most of the sectors.

Table 5 presents the results for the individual stocks. The results are consistent with the results for the ETFs. In general, observe that for most of the individual stocks except the three energy stocks (AA, CVX, XOM), the crude oil betas are insignificant for the "Before" period. For the majority of the stocks, the crude oil betas are positive and significant for the "During" period. For the "After" period, the magnitudes of the crude oil betas increase further from the "During" period. The number of significant P-values of the test increased from 9, 11, 11 to 21, 13 and 29, respectively, and these findings support the conclusions obtained from the ETFs.

In summary, tests of the one factor and two factor models show that for the ETFs and individual stocks, some have time-varying market betas and some have time-varying crude oil betas. However, in general, the results are not consistent with the findings in the literature that correlations between individual assets and the market increases during crisis periods. For most of the tested assets, market betas do not increase during the crisis period. We emphasize that the betas we study refer to intraday cumulative returns which can be impacted differently by a "bear" market than daily or monthly returns.

5.3 Asymmetric CIDR betas for "bull" and "bear" oil price periods

In this subsection, we consider only two samples corresponding to periods of increasing and decreasing oil prices. The two periods are based on crude oil price trends and are defined as: "Bull" (05/01/2007–06/30/2008) and "Bear" (07/01/2008–01/30/2009). These two periods mostly occur during the financial crisis period, and thus the confounding of the two effects is reduced. The purpose of this section is to provide further insights on the effect of oil prices on cumulative intraday returns and demonstrate our methodology in a different context.

Symbol	$\hat{\beta}_{M1}$	SE_{M1}	$\hat{\beta}_{O1}$	SE_{O1}	$\hat{\beta}_{M2}$	SE_{M2}	$\hat{\beta}_{O2}$	SE_{O2}	$\hat{\beta}_{M3}$	SE_{M3}	$\hat{\beta}_{O3}$	SE_{O3}	$P_{1,2}$	$P_{2,3}$	$P_{1,3}$
AA	0.93	0.12	0.18	0.05	0.71	0.12	0.43	0.07	0.53	0.06	0.48	0.06	0.01**	0.40	0.00***
AXP	0.76	0.10	0.01	0.03	0.94	0.09	0.16	0.07	0.46	0.06	0.26	0.05	0.04*	0.00***	0.00***
BA	0.77	0.10	0.03	0.04	0.47	0.09	0.14	0.06	0.57	0.05	0.25	0.04	0.07*	0.02*	0.00***
BAC	0.41	0.10	0.00	0.03	0.80	0.21	0.36	0.11	0.38	0.09	0.48	0.07	0.00***	0.17	0.00***
CAT	0.79	0.10	0.09	0.04	0.59	0.07	0.23	0.05	0.58	0.06	0.38	0.05	0.04*	0.06*	0.00***
CSCO	0.68	0.10	0.03	0.04	0.55	0.07	0.14	0.05	0.43	0.04	0.23	0.05	0.19	0.30	0.00***
CVX	0.44	0.08	0.40	0.04	0.53	0.05	0.33	0.04	0.33	0.03	0.33	0.03	0.29	0.00***	0.05*
DD	0.69	0.15	−0.01	0.03	0.55	0.06	0.15	0.04	0.48	0.05	0.29	0.04	0.01**	0.08*	0.00***
DIS	0.61	0.12	−0.04	0.04	0.72	0.06	0.12	0.04	0.52	0.04	0.11	0.04	0.01**	0.02*	0.01**
GE	0.49	0.05	−0.02	0.03	0.60	0.14	0.15	0.06	0.53	0.05	0.25	0.04	0.00***	0.38	0.00***
HD	0.67	0.09	0.01	0.04	0.60	0.09	0.06	0.06	0.28	0.06	0.22	0.05	0.69	0.01**	0.00***
HPQ	0.68	0.08	0.01	0.04	0.43	0.07	0.14	0.05	0.51	0.06	0.15	0.05	0.02*	0.57	0.04*
IBM	0.39	0.09	−0.01	0.03	0.45	0.06	0.13	0.04	0.33	0.04	0.14	0.03	0.00***	0.13	0.00***
INTC	0.74	0.08	−0.04	0.04	0.47	0.08	0.14	0.06	0.35	0.05	0.17	0.05	0.01**	0.37	0.00***
JNJ	0.26	0.05	−0.04	0.02	0.24	0.05	0.03	0.02	0.24	0.04	0.10	0.03	0.09*	0.10	0.00***
JPM	0.65	0.12	−0.02	0.03	0.56	0.21	0.17	0.09	0.42	0.05	0.36	0.05	0.14	0.19	0.00***
KFT	0.41	0.11	−0.01	0.04	0.19	0.05	0.07	0.03	0.19	0.06	0.11	0.03	0.05*	0.78	0.01**
KO	0.23	0.07	−0.05	0.03	0.29	0.06	0.02	0.03	0.25	0.04	0.08	0.03	0.16	0.38	0.00***
MCD	0.52	0.11	−0.01	0.03	0.34	0.05	0.03	0.03	0.14	0.03	0.12	0.03	0.28	0.00***	0.00***
MMM	0.56	0.09	0.01	0.04	0.34	0.06	0.06	0.04	0.51	0.03	0.19	0.04	0.10	0.00***	0.00***
MRK	0.59	0.07	−0.07	0.04	0.41	0.06	0.05	0.04	0.35	0.04	0.17	0.04	0.04*	0.12	0.00***
MSFT	0.44	0.06	0.02	0.03	0.49	0.06	0.08	0.04	0.31	0.04	0.16	0.05	0.31	0.05*	0.01**
PFE	0.60	0.08	−0.06	0.03	0.32	0.07	0.09	0.04	0.30	0.04	0.16	0.04	0.00***	0.41	0.00***
PG	0.18	0.11	−0.04	0.03	0.30	0.04	0.03	0.03	0.19	0.04	0.03	0.03	0.06*	0.08*	0.14
T	0.47	0.07	−0.03	0.04	0.57	0.06	0.17	0.05	0.25	0.03	0.09	0.03	0.00***	0.00***	0.00***
TRV	0.49	0.14	0.02	0.04	0.29	0.14	0.09	0.07	0.25	0.04	0.14	0.03	0.43	0.81	0.01**
UTX	0.72	0.09	0.01	0.04	0.51	0.06	0.10	0.04	0.51	0.04	0.15	0.04	0.04*	0.71	0.01**
VZ	0.48	0.11	−0.01	0.04	0.52	0.07	0.12	0.04	0.18	0.04	0.10	0.03	0.07*	0.00***	0.01**
WMT	0.54	0.10	−0.08	0.03	0.31	0.07	0.01	0.04	0.16	0.05	0.04	0.03	0.04*	0.19	0.00***
XOM	0.49	0.08	0.38	0.03	0.39	0.07	0.28	0.04	0.31	0.04	0.27	0.03	0.04*	0.46	0.00***

Table 5. *Asymmetric Beta around the financial crisis (Two Factors) – Stocks.* This table presents the estimation results for the model: $R_n(t) = \beta_0(t) + \beta_{Mi}M_n(t) + \beta_{Oi}CL_n(t) + \epsilon_n(t)$ for the sector ETFs. Here, i is the ith period, and 1, 2 and 3 refer to the "Before", "During" and "After" financial crisis periods, respectively. The periods are defined as: "Before" (06/13/2006–09/27/2007), "During" (11/01/2007–02/27/2009) and "After" (01/04/2010–12/30/2011) the financial crisis. *SE* refers the standard error. $P_{i,j}$ is the P-value of the statistical test with H_0: $\boldsymbol{\beta}_i = \boldsymbol{\beta}_j$. The superscripts ***,** ,* and * indicate significant at 0.1%, 1%, 5% and 10%, respectively.

Table 6 exhibits the estimated results for the ETFs. Panel A presents the results for the one factor model of equation (5.1). The P-values of the test show that for all the ETFs, the market betas are not significantly different. When the oil factor is added, the P-values for six ETFs (XLF, XLI, XLK, XLP, XLU and XLV) are significantly different. For XLV, the crude oil beta for the first period is significant and negative, whereas for the second period it is insignificant. For the other five ETFs, $\hat{\beta}_{O2}$ is positive and statistically significantly different from zero whereas $\hat{\beta}_{O1}$ is statistically insignificant. For XLE, the crude oil factor is positive, stable and not different for the two periods ($\hat{\beta}_{O1} = 0.46$ and $\hat{\beta}_{O2} = 0.47$). For XLB, $\hat{\beta}_{O1} = 0.14$ and $\hat{\beta}_{O2} = 0.24$, and both of them are statistically different from zero, but are not statistically different from each other.

Table 7 shows analogous results for the individual stocks. Similarly, these results show that without the crude oil factor, only seven out of the 30 stocks have statistically asymmetric market beta during the two periods. When the crude oil factor is added, the number of significant P-values increases to 23. For most of them, the crude oil price betas are negative and insignificant for the first period but positive and significant for the second period. The magnitudes of the oil factor also increase dramatically. For instance, for BAC, $\hat{\beta}_{O1} = -0.07$ and $\hat{\beta}_{O2} = 0.45$.

The results show that the crude oil beta for the majority of the assets is asymmetric for the two periods. For the "Bear" crude oil period, most of the crude oil betas are positive, significant with larger magnitudes than

Symbol	$\hat{\beta}_{M1}$	SE_{M1}	$\hat{\beta}_{O1}$	SE_{O1}	$\hat{\beta}_{M2}$	SE_{M2}	$\hat{\beta}_{O2}$	SE_{O2}	$P_{1,2}$
Panel A: Single Factor Model									
XLB	0.73	0.06			0.77	0.06			0.64
XLE	0.67	0.06			0.82	0.08			0.15
XLF	0.80	0.10			0.77	0.15			0.86
XLI	0.60	0.07			0.65	0.06			0.55
XLK	0.68	0.07			0.59	0.07			0.35
XLP	0.26	0.04			0.30	0.08			0.67
XLU	0.31	0.07			0.49	0.09			0.11
XLV	0.38	0.06			0.37	0.07			0.93
XLY	0.65	0.07			0.64	0.11			0.96
Panel B: Two Factor Model									
XLB	0.73	0.07	0.14	0.04	0.68	0.06	0.24	0.06	0.43
XLE	0.66	0.05	0.46	0.04	0.64	0.08	0.47	0.07	0.95
XLF	0.80	0.10	−0.10	0.06	0.67	0.16	0.24	0.10	0.01**
XLI	0.60	0.07	−0.02	0.03	0.61	0.07	0.10	0.04	0.01**
XLK	0.68	0.07	−0.03	0.04	0.52	0.07	0.18	0.06	0.01**
XLP	0.26	0.04	−0.04	0.04	0.26	0.08	0.10	0.05	0.03*
XLU	0.31	0.07	−0.01	0.04	0.45	0.09	0.11	0.04	0.06*
XLV	0.38	0.06	−0.09	0.03	0.34	0.07	0.07	0.05	0.01**
XLY	0.65	0.07	−0.07	0.04	0.60	0.12	0.10	0.08	0.11

Table 6. *Bull vs Bear Oil Price Periods: ETFs* . This table presents the estimation results for the ETFs. The periods are defined as Bull oil price (05/01/2007–06/30/2008) and Bear oil price (07/01/2008–01/30/2009). The number of sample days for the two are 294 and 145 days, respectively. $P_{1,2}$ is the P-value of the statistical test with H_0: $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$. The superscripts ***, **, * and * indicate significant at 0.1%, 1%, 5% and 10%, respectively.

the "Bull" period. Since both of the "Bull" and "Bear" periods are within the downward stock market, the tests show that the crude oil factor is more important than the stock market. However, it can also be the case that during the second period, both the stock market and the crude oil price trended downward. Plausibly, the correlation between the crude oil and market factors are higher, leading to the higher crude oil betas.

5.4 Comparison to point-to-point betas

Traditional financial models deal with point-to-point returns. Factor models for such returns are essentially scalar regression models. Our functional factor model reveals characteristics of intraday price curves which cannot be inferred using traditional factor models. It has a different information content which focuses on the behavior of price curves within a trading day. The objective of this section is to use a quantitative analysis to show that the information contained in point-to-point returns is often quite different from that obtained using the CIDRs and our functional approach. Therefore, in this section, betas obtained from point-to-point returns are estimated and compared with the CIDR betas. We consider the scalar model

$$R_n = \beta_0 + \sum_{j=1}^{p} \beta_j F_{nj} + \epsilon_n, \tag{5.3}$$

where the intercept β_0, the factors F_{nj} and the errors ϵ_n are all scalars. It can be verified that the entire test procedure described in Section 4 still holds for the scalar model (5.3), with obvious modifications. Two types of point-to-point returns are used. The first type are returns between the open and close of the market. These returns ignore the intraday evolution of the price curves and the overnight price change. The second type are the usual daily returns based on closing prices, i.e., returns between the close of trading day $n-1$ and the close of trading day n. These returns take into account the overnight price change. To keep the length of the paper within limits, we report the results of the test for the ETFs only.

Symbol	$\hat{\beta}_{M1}$	SE_{M1}	$\hat{\beta}_{O1}$	SE_{O1}	$\hat{\beta}_{M2}$	SE_{M2}	$\hat{\beta}_{O2}$	SE_{O2}	$P_{1,2}$
Panel A: Single Factor Model									
CVX	0.50	0.06			0.65	0.07			0.09*
DIS	0.50	0.07			0.83	0.08			0.00***
HD	0.80	0.13			0.51	0.10			0.08*
INTC	0.72	0.07			0.43	0.10			0.01**
MMM	0.53	0.07			0.26	0.07			0.01**
PFE	0.48	0.06			0.30	0.08			0.07*
PG	0.15	0.06			0.34	0.05			0.02*
Panel B: Two Factor Model									
AA	0.93	0.08	0.23	0.07	0.57	0.15	0.46	0.09	0.03*
AXP	1.07	0.12	−0.08	0.06	0.81	0.12	0.26	0.10	0.01**
BAC	0.63	0.14	−0.07	0.07	0.61	0.24	0.45	0.15	0.00***
CAT	0.66	0.08	0.07	0.05	0.50	0.10	0.25	0.06	0.05*
CSCO	0.58	0.08	−0.04	0.06	0.48	0.09	0.21	0.07	0.01**
DIS	0.50	0.07	−0.03	0.04	0.78	0.08	0.19	0.06	0.00***
GE	0.63	0.05	−0.08	0.04	0.51	0.18	0.24	0.09	0.00***
HD	0.80	0.13	−0.08	0.06	0.47	0.11	0.13	0.07	0.04*
HPQ	0.60	0.07	−0.00	0.06	0.38	0.09	0.19	0.07	0.03*
IBM	0.49	0.08	−0.03	0.04	0.39	0.07	0.17	0.05	0.01**
INTC	0.72	0.07	−0.05	0.05	0.37	0.10	0.20	0.07	0.00***
JNJ	0.19	0.04	−0.07	0.03	0.24	0.07	0.05	0.03	0.01**
JPM	0.72	0.13	−0.13	0.07	0.41	0.28	0.27	0.13	0.02*
KFT	0.34	0.07	−0.01	0.04	0.15	0.06	0.12	0.04	0.03*
KO	0.23	0.06	−0.10	0.04	0.29	0.07	0.05	0.05	0.04*
MMM	0.53	0.07	−0.05	0.03	0.23	0.07	0.11	0.05	0.00***
MRK	0.38	0.07	−0.07	0.05	0.41	0.08	0.13	0.06	0.03*
MSFT	0.47	0.06	−0.04	0.04	0.50	0.09	0.14	0.06	0.01**
PFE	0.48	0.05	−0.10	0.03	0.25	0.08	0.17	0.05	0.00***
PG	0.14	0.06	−0.07	0.03	0.32	0.05	0.04	0.04	0.00***
T	0.58	0.07	−0.05	0.04	0.54	0.08	0.23	0.07	0.00***
VZ	0.54	0.09	−0.05	0.05	0.48	0.08	0.17	0.06	0.01**
WMT	0.35	0.08	−0.10	0.04	0.24	0.09	0.04	0.05	0.08*

Table 7. *Bull vs Bear Oil Price Periods: Individual Stocks.* This table presents the estimation results for the individual stocks. The periods are defined as: Bull oil price (05/01/2007–06/30/2008) and Bear oil price (07/01/2008–01/30/2009). The number of sample days for the two are 294 and 145 days, respectively. $P_{1,2}$ is the P-value of the statistical test with H_0: $\boldsymbol{\beta}_1 = \boldsymbol{\beta}_2$. The superscripts ***, **, * and * indicate significant at 0.1%, 1%, 5% and 10%, respectively. To save space, only the significant results are reported.

Table 8 reports the results for the one factor model. Comparing the results with Table 2, it is observed that the CIDR curve betas are smaller than the corresponding open-to-close betas which are smaller than the close-to-close return betas. For instance, for XLB, $\hat{\beta}_{M1}$ is 0.87, 0.91 and 1.23, for the curve, the open-to-close returns and the close-to-close returns, respectively. This observation is consistent across all periods and for all the ETFs. Second, for some ETFs, the pair-wise test P-values for the estimated betas are different. For instance, for the "Before" and "During" periods, the test show that only one ETF has a statistically different CIDR beta ($\hat{\beta}_{M1} \neq \hat{\beta}_{M2}$), for the open-to-close return, three ETFs have different betas, and for the close-to-close beta, eight ETFs have significantly different betas. However, for the other two pair of periods, the differences are not that dramatic. For the "During" and "After" periods, the number of significant P-values are five for all three different return measures. Third, for some ETFs, the results with different returns can be very different. For instance, the P-values of the curve beta for XLI for all the three pair-wise periods are insignificant. The close-to-close betas are statistically different for the "Before" and "During" as well as the "During" and "After" periods.

Table 9 reports the results for the two factor models for the three periods around the financial crisis. The results show similar patterns as the one factor model.

Symbol	$\hat{\beta}_{M1}$	SE_{M1}	$\hat{\beta}_{M2}$	SE_{M2}	$\hat{\beta}_{M3}$	SE_{M3}	$P_{1,2}$	$P_{2,3}$	$P_{1,3}$
Panel A: Open to Close Return									
XLB	0.91	0.07	0.88	0.03	0.83	0.04	0.66	0.39	0.34
XLE	0.94	0.07	0.99	0.06	0.82	0.04	0.62	0.02*	0.13
XLF	0.86	0.07	1.19	0.10	0.92	0.04	0.01**	0.01**	0.53
XLI	0.79	0.05	0.76	0.03	0.76	0.03	0.56	1.00	0.57
XLK	0.89	0.06	0.74	0.03	0.60	0.03	0.02*	0.00***	0.00***
XLP	0.50	0.03	0.43	0.04	0.35	0.02	0.19	0.09*	0.00***
XLU	0.74	0.07	0.57	0.04	0.41	0.03	0.04*	0.00***	0.00***
XLV	0.57	0.04	0.54	0.04	0.50	0.03	0.51	0.46	0.13
XLY	0.79	0.05	0.83	0.07	0.67	0.04	0.56	0.02*	0.04*
Panel B: Close to Close Return									
XLB	1.23	0.05	1.08	0.03	1.25	0.02	0.01**	0.00***	0.76
XLE	1.11	0.06	1.27	0.07	1.25	0.03	0.08*	0.78	0.04*
XLF	1.22	0.04	1.57	0.10	1.35	0.03	0.00***	0.03*	0.02*
XLI	1.02	0.03	0.92	0.02	1.15	0.01	0.01**	0.00***	0.00***
XLK	1.02	0.04	0.93	0.02	0.94	0.01	0.04*	0.59	0.07*
XLP	0.68	0.02	0.57	0.03	0.57	0.01	0.00***	0.96	0.00***
XLU	0.91	0.06	0.72	0.05	0.64	0.02	0.02*	0.08*	0.00***
XLV	0.73	0.03	0.66	0.03	0.76	0.02	0.06*	0.00***	0.42
XLY	1.05	0.03	1.01	0.03	1.02	0.02	0.29	0.77	0.32

Table 8. *Two Sample Test Results – ETFs point-to-point.* This table presents the estimation results for the point-to-point return series of the ETFs. The periods are defined as: "Before" (06/13/2006–09/27/2007), "During" (11/01/2007–02/27/2009) and "After" (01/04/2010–12/30/2011) the financial crisis. The number of sample days for the three periods are 326, 328 and 503 days, respectively. $P_{i,j}$ is the P-value of the statistical test with H_0: $\boldsymbol{\beta}_i = \boldsymbol{\beta}_j$. The superscripts ***, **, * and * indicate significant at 0.1%, 1%, 5% and 10%, respectively.

In general, the comparisons show that cumulative intraday returns and functional factor models reveal features different from point-to-point returns. The analysis based on functional regression models considers the relationship between the intraday prices of individual assets and factors. Consequently, these models are more relevant to investors taking positions within a trading day, i.e., day traders or high-frequency traders.

6 Finite sample performance

This section examines the test's finite sample performance using sample sizes similar to those encountered in Section 5.2, and data generating processes that produce curves similar to the CIDR. The correct empirical sizes and sufficiently high power lend confidence to the findings reported in Section 5. The data generating processes are described and the empirical rejection rates are displayed.

The building block of artificial data is Brownian motion. Brownian motion is the standard model for price curves over short time intervals. In the simulations, Brownian motion is used to imitate the general shape of the CIDRs, as well as the error curves. We use B_n or B_m^* to denote the Brownian motions that are used to generate factors, and ϵ_n, ϵ_m^* to denote the Brownian motions used to generate errors. Serial and cross-sectional dependence is generated by forming linear combinations of these processes. Specifically, we consider the case of $p = 2$, and set

$$F_{n1}(t) = B_n^{(1)}(t) + 0.5B_{n-1}^{(1)}(t) + 0.25B_{n-2}^{(1)}(t), \tag{6.1}$$

$$F_{n2}(t) = B_n^{(2)}(t) + 0.5B_{n-1}^{(2)}(t) - 0.25B_{n-2}^{(1)}(t), \tag{6.2}$$

$$F_{m1}^*(t) = B_m^{*(1)}(t) + 0.5B_{m-1}^{*(1)}(t) + 0.25B_{m-2}^{*(1)}(t), \tag{6.3}$$

$$F_{m1}^*(t) = B_m^{*(2)}(t) + 0.5B_{m-1}^{*(2)}(t) - 0.25B_{m-2}^{*(1)}(t), \tag{6.4}$$

Symbol	$\hat{\beta}_{M1}$	SE_{M1}	$\hat{\beta}_{O1}$	SE_{O1}	$\hat{\beta}_{M2}$	SE_{M2}	$\hat{\beta}_{O2}$	SE_{O2}	$\hat{\beta}_{M3}$	SE_{M3}	$\hat{\beta}_{O3}$	SE_{O3}	$P_{1,2}$	$P_{2,3}$	$P_{1,3}$
Panel A: Open to Close Return															
XLB	0.90	0.07	0.05	0.03	0.83	0.03	0.15	0.04	0.70	0.04	0.27	0.04	0.10	0.02*	0.00***
XLE	0.87	0.07	0.39	0.04	0.88	0.06	0.33	0.05	0.65	0.03	0.34	0.03	0.67	0.00***	0.00***
XLF	0.87	0.07	−0.03	0.02	1.16	0.10	0.10	0.06	0.83	0.05	0.17	0.04	0.00***	0.02*	0.00***
XLI	0.79	0.05	0.00	0.02	0.75	0.03	0.02	0.03	0.69	0.04	0.13	0.03	0.75	0.02*	0.00***
XLK	0.90	0.06	−0.04	0.02	0.72	0.03	0.06	0.03	0.55	0.03	0.10	0.03	0.00***	0.00***	0.00***
XLP	0.51	0.03	−0.04	0.02	0.43	0.04	0.02	0.03	0.33	0.03	0.04	0.02	0.14	0.15	0.00***
XLU	0.74	0.07	0.00	0.03	0.56	0.04	0.04	0.03	0.38	0.03	0.07	0.03	0.06*	0.00***	0.00***
XLV	0.58	0.04	−0.04	0.01	0.55	0.04	−0.03	0.03	0.46	0.03	0.08	0.03	0.79	0.01**	0.00***
XLY	0.80	0.05	−0.05	0.02	0.83	0.07	0.00	0.05	0.60	0.04	0.13	0.04	0.34	0.02*	0.00***
Panel B: Close to Close Return															
XLB	1.22	0.06	0.05	0.02	1.01	0.03	0.12	0.03	1.19	0.03	0.07	0.02	0.00***	0.00***	0.86
XLE	1.04	0.06	0.40	0.03	1.08	0.07	0.35	0.04	1.06	0.03	0.20	0.02	0.58	0.00***	0.00***
XLF	1.23	0.05	−0.06	0.02	1.65	0.10	−0.15	0.04	1.40	0.04	−0.05	0.02	0.00***	0.01**	0.00***
XLI	1.02	0.03	−0.02	0.01	0.93	0.02	−0.02	0.02	1.16	0.02	−0.01	0.01	0.05*	0.00***	0.00***
XLK	1.03	0.04	−0.06	0.01	0.93	0.02	0.00	0.01	0.97	0.02	−0.03	0.01	0.01**	0.12	0.35
XLP	0.68	0.02	−0.04	0.01	0.60	0.03	−0.05	0.01	0.59	0.02	−0.02	0.01	0.03*	0.31	0.00***
XLU	0.91	0.07	0.00	0.02	0.72	0.05	0.01	0.03	0.62	0.02	0.01	0.01	0.06	0.17	0.00***
XLV	0.74	0.03	−0.05	0.01	0.69	0.03	−0.06	0.02	0.78	0.02	−0.02	0.01	0.31	0.00***	0.03*
XLY	1.06	0.03	−0.05	0.01	1.04	0.03	−0.06	0.02	1.06	0.02	−0.04	0.01	0.77	0.61	0.95

Table 9. *Two sample test for the comparison two Factor Models -ETFs- Point to Point.* This table presents the estimation results for the point-to-point return series of the ETFs. The periods are defined as: "Before" (06/13/2006–09/27/2007), "During" (11/01/2007–02/27/2009) and "After" (01/04/2010–12/30/2011) the financial crisis. The number of sample days for the three periods are 326, 328 and 503 days, respectively. $P_{i,j}$ is the P-value of the statistical test with H_0: $\boldsymbol{\beta}_i = \boldsymbol{\beta}_j$. The superscripts ***, **, * and * indicate significant at 0.1%, 1%, 5% and 10%, respectively.

where the superscript $^{(1)}$ and $^{(2)}$ on B_n and B_m^* represent different Brownian motions. The factors are thus both serially and cross-sectionally dependent. Letting $\beta_0(t) = 0$, which corresponds to the empirically observed value for most assets, we generate R_n and R_m^* using the following equations:

$$R_n(t) = \beta_1 F_{n1}(t) + \beta_2 F_{n2}(t) + \epsilon_n(t), \tag{6.5}$$

$$R_m^*(t) = \beta_1^* F_{m1}^*(t) + \beta_2^* F_{m2}^*(t) + \epsilon_m^*(t). \tag{6.6}$$

Under H_0, we set $\beta_1 = \beta_1^*$ and $\beta_2 = \beta_2^*$. Under H_A, the data generation process for the second sample is modified by setting $\beta_1 \neq \beta_1^*$ or $\beta_2 \neq \beta_2^*$. To obtain relevant values of these parameters, three ETFs (XLB, XLK and XLU) are selected to be used in model (5.2). Crude oil prices are used as the additional factor for the comparison between the during and after the crisis periods. For each of the three ETFs, we generate 328 market factor curves M_n^G, 328 crude oil factor curves C_n^G and 328 error curves $\hat{\epsilon}_n^G$ such that they have approximately the same magnitude as the empirical curves M_n, C_n and $\hat{\epsilon}_n$ for the period during the crisis, where $\hat{\epsilon}_n$ is the residual function that can be calculated from the data. Analogously, 503 M_n^G, C_n^G and $\hat{\epsilon}_n^G$ curves are generated for the period after the crisis. To assess the empirical size of test, we generate the ETF curves as

$$R_n^G = \hat{\beta}_0(t) + \hat{\beta}_1^{\text{ave}} M_n^G(t) + \hat{\beta}_2^{\text{ave}} C_n^G(t) + \hat{\epsilon}_n^G(t)$$

for each period, where $\hat{\beta}_1^{\text{ave}}$ and $\hat{\beta}_2^{\text{ave}}$ are the averages of the coefficient estimates for the two periods. To assess the empirical power, we use the data generation process

$$R_n^G = \hat{\beta}_0(t) + \hat{\beta}_1 M_n^G(t) + \hat{\beta}_2 C_n^G(t) + \hat{\epsilon}_n^G(t)$$

for each period. We replicate this procedure $R = 3000$ times and obtain Monte Carlo empirical size and power. The simulation results exhibited in Table 10, demonstrate that the test exhibits excellent calibration size and has a very high power in practically relevant settings.

Sig. Level	XLB	XLK	XLU
Panel A: Empirical Size			
10.0%	11.3%	9.8%	11.3%
5.0%	5.9%	5.6%	5.4%
2.5%	2.8%	2.7%	2.6%
1.0%	1.0%	1.2%	1.2%
Panel B: Empirical Power			
10.0%	100.0%	100.0%	97.4%
5.0%	100.0%	99.8%	94.4%
2.5%	100.0%	99.7%	90.8%
1.0%	100.0%	99.2%	85.3%

Table 10. *Monte Carlo simulation results of the test statistics $T_{M,N}$.* This table presents the Monte Carlo simulation results using three ETFs as examples.

The empirical size evaluates the performance of a test when the null hypothesis is true, i.e., if there is no difference between the two samples. To illustrate, if we set the significance level at 5%, then for a perfect test we expect acceptance of the null in 95% of the simulation runs and rejection in 5% of them. This is never the case in practice. The actual percentage of rejections of the null is called the empirical size or the empirical type I error. Our test has a slight tendency to over reject, i.e., the probability of a type I error is generally slightly higher than the nominal level of the test.

Due to the slightly overinflated empirical size, when interpreting the P-values, one may expect that the true P-value may be slightly larger that the computed P-value. This distortion is, however, very small. For example, if a computed P-value is 0.045, the actual P-value can be slightly above 0.05. It must, however, be kept in mind that the traditional cut-off percentages, like 5% or 10%, are merely conventions; there is practically no difference between the statistical interpretation of a P-value of 4.5% or 5.5%. Comparing this test to other tests for complex data structures, the empirical size is remarkably close to the nominal size, and the computed P-values can be trusted.

The power of the test evaluates its performance if an alternative is true. In our case, power is evaluated for data simulated in such a way that the regression coefficients are different in the two samples. Power is the probability of rejecting the null hypothesis if an alternative is true. It depends on the alternative and on the significance level used to perform the test. If the difference between the samples is smaller, power will be smaller. If the significance level decreases, it becomes more and more difficult for the test to reject the null, so the power declines. We would like the power to be as close to 100% as possible. This is never the case in theory, because there is always a positive probability of accepting the null due to chance. If only a finite number of simulations are used to assess power, one can observe empirical power of 100% (the null is not accepted in any simulation run). Our test has excellent power. If it is applied at the usual significance level of 5%, one can be almost certain to detect the difference between the two samples, if it exists.

7 Concluding remarks

We first summarize the main technical contributions and findings of the paper, and then place it in a broader perspective of finance research.

We propose functional factor models to study the relationship between the cumulative intraday return curves of individual stocks and factor curves that may drive price movement. To examine whether the betas change from period to period, we develop a statistical test. Based on general assumptions, we prove that the asymptotic distribution of the test statistic is a chi-square distribution. We also prove theoretically that the power of the test approaches 100% asymptotically.

We apply the test to empirically investigate whether asymmetric betas exist for downward versus upward market conditions. We use the 30 stocks of the Dow Jones Industrial Average index and the Select Sector ETFs as the base assets. The tests on the one factor model show mixed results. For some assets betas are time varying and market betas of most assets do not increase during the financial crisis. However, most of the assets have lower market betas after the financial crisis.

When crude oil factor is added in the model, some assets exhibit asymmetric crude oil factor betas. Interestingly, the crude oil betas are not time varying for energy stocks and XLE, the energy ETF. This means that these assets are very strongly tied to the underlying commodity, and this connection is not influenced by the trends in the factor. We further examine the betas for the "Bull" and "Bear" crude oil price periods, and the results also show that some assets have asymmetric crude oil factor betas.

A comparison of the results of the functional model to scalar models for point-to-point returns shows that the new return measure contains different information about the relationship between individual assets and the market factors. Considering the nature of the return curves, the models we propose and the results of our analysis are more useful for day traders and other higher frequency traders.

Since the Capital Asset Pricing Model (CAPM) was introduced in the 1960s, it has been the center piece of modern finance theory. Although its empirical power has been debated for decades, it is still used both in academia and the investment world. Financial data, at the tick level, have been available for decades, but at the high frequency levels, the time series are very noisy, hence it is almost impossible to use traditional approaches such as OLS regression to examine the relations between high frequency intraday return series. Thus, the lack of proper tools puts a limit on studies in this area. The introduction of Functional Data Analysis (FDA) techniques into the finance area could be a potential solution and a new direction of financial research. FDA is about the analysis of information contained in shapes of curves. It is particularly applicable in high-frequency studies, since statistical tools of FDA typically rely on some form of smoothing to transform high dimensional or incomplete data building into a smoother curve that can be more readily analyzed using statistical tools.

In the traditional CAPM world, both researchers and practitioners are aware of the time varying beta problem. This has been investigated by previous studies. Since the CAPM model is defined in a OLS environment, the study of asymmetric or time varying beta is straightforward. However, in the FDA world, the test is not straightforward. In this paper, we develop statistical methodology for testing asymmetry of betas. Since the FDA approach utilizes high frequency data at any high frequency, the approach could incorporate a lot more information in the intraday price movements into the betas and thus, hopefully, characterizing the relations better. If so, the test for asymmetry in the FDA framework could potentially help investors and portfolio managers control risk and construct portfolios more effectively.

A Proof of Theorem 4.6

Proof. Since we assume that the two samples are independent, the limits in (4.8) and (4.9) are independent. Combining (4.8), (4.9) and (4.6), we obtain

$$\sqrt{N+M}(\widehat{\boldsymbol{\beta}}-\widehat{\boldsymbol{\beta}}^*) = \sqrt{\frac{N+M}{N}}\sqrt{N}(\widehat{\boldsymbol{\beta}}-\boldsymbol{\beta}) - \sqrt{\frac{N+M}{M}}\sqrt{M}(\widehat{\boldsymbol{\beta}}^*-\boldsymbol{\beta}) \xrightarrow{d} \frac{1}{\sqrt{\theta}}\mathbf{F}^{-1}\mathbf{W} + \frac{1}{\sqrt{1-\theta}}(\mathbf{F}^*)^{-1}\mathbf{W}^*. \tag{A.1}$$

Define

$$\widehat{\boldsymbol{\Delta}}_{N,M} = \frac{N+M}{N}\widehat{\mathbf{F}}^{-1}\widehat{\boldsymbol{\Gamma}}\widehat{\mathbf{F}}^{-1} + \frac{N+M}{M}(\widehat{\mathbf{F}}^*)^{-1}\widehat{\boldsymbol{\Gamma}}^*(\widehat{\mathbf{F}}^*)^{-1}.$$

It was shown, in [23, Lemma 4], that $\widehat{\mathbf{F}} \xrightarrow{\text{a.s.}} \mathbf{F}$. By assumption, $\widehat{\boldsymbol{\Gamma}} \xrightarrow{P} \boldsymbol{\Gamma}$. With analogous relations for the second sample, we obtain

$$\widehat{\boldsymbol{\Delta}}_{N,M} \xrightarrow{P} \frac{1}{\theta}\mathbf{F}^{-1}\boldsymbol{\Gamma}\mathbf{F}^{-1} + \frac{1}{1-\theta}(\mathbf{F}^*)^{-1}\boldsymbol{\Gamma}^*(\mathbf{F}^*)^{-1}. \tag{A.2}$$

Observe that $\mathbf{W}$ and $\mathbf{W}^*$ are two p-dimensional random vectors satisfying

$$\begin{bmatrix} \mathbf{W} \\ \mathbf{W}^* \end{bmatrix} \sim \mathbf{N}\left(\mathbf{0}, \begin{bmatrix} \mathbf{\Gamma} & \mathbf{0} \\ \mathbf{0} & \mathbf{\Gamma}^* \end{bmatrix}\right).$$

Since the matrices $\mathbf{F}$ and $\mathbf{F}^*$, defined by (4.7), are nonsingular, we can conclude that

$$\frac{1}{\sqrt{\theta}}\mathbf{F}^{-1}\mathbf{W} + \frac{1}{\sqrt{1-\theta}}(\mathbf{F}^*)^{-1}\mathbf{W}^* \sim N\left(\mathbf{0},\ \frac{1}{\theta}\mathbf{F}^{-1}\mathbf{\Gamma}\mathbf{F}^{-1} + \frac{1}{1-\theta}(\mathbf{F}^*)^{-1}\mathbf{\Gamma}^*(\mathbf{F}^*)^{-1}\right), \tag{A.3}$$

where the variance matrix of $\frac{1}{\sqrt{\theta}}\mathbf{F}^{-1}\mathbf{W} + \frac{1}{\sqrt{1-\theta}}(\mathbf{F}^*)^{-1}\mathbf{W}^*$ has rank p, and also $\mathbf{\Gamma}$ and $\mathbf{\Gamma}^*$ are assumed to be full rank.

Now Let

$$\begin{aligned} \mathbf{Y}_p &:= \frac{1}{\sqrt{\theta}}\mathbf{F}^{-1}\mathbf{W} + \frac{1}{\sqrt{1-\theta}}(\mathbf{F}^*)^{-1}\mathbf{W}^*, \\ \mathbf{\Sigma}_p &:= \frac{1}{\theta}\mathbf{F}^{-1}\mathbf{\Gamma}\mathbf{F}^{-1} + \frac{1}{1-\theta}(\mathbf{F}^*)^{-1}\mathbf{\Gamma}^*(\mathbf{F}^*)^{-1}, \end{aligned}$$

where $\mathbf{\Sigma}_p$ is nonsingular. Then (A.3) becomes $\mathbf{Y}_p \sim N(\mathbf{0}, \mathbf{\Sigma}_p)$. This follows from the identity $\mathbf{Y}_p^T\mathbf{\Sigma}_p^{-1}\mathbf{Y}_p \sim \chi^2(p)$. By Slutsky's theorem, the claim of the theorem follows by combining (A.1) and (A.2). □

B Proof of Theorem 4.7

Proof. Let

$$\begin{aligned} \mathbf{B}_{N,M} &:= \sqrt{\frac{N+M}{N}}\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) - \sqrt{\frac{N+M}{M}}\sqrt{M}(\widehat{\boldsymbol{\beta}}^* - \boldsymbol{\beta}^*), \\ \mathbf{B} &:= \frac{1}{\sqrt{\theta}}\mathbf{F}^{-1}\mathbf{W} + \frac{1}{\sqrt{1-\theta}}(\mathbf{F}^*)^{-1}\mathbf{W}^*, \\ \mathbf{\Delta} &:= \frac{1}{\theta}\mathbf{F}^{-1}\mathbf{\Gamma}\mathbf{F}^{-1} + \frac{1}{1-\theta}(\mathbf{F}^*)^{-1}\mathbf{\Gamma}^*(\mathbf{F}^*)^{-1}. \end{aligned}$$

Then, by (A.1) and (A.3), $\mathbf{B}_{N,M} \xrightarrow{d} \mathbf{B}$, where $\mathbf{B} \sim N(\mathbf{0}, \mathbf{\Delta})$. Since $\boldsymbol{\beta}^* = \boldsymbol{\beta} + \boldsymbol{\eta}$, we have

$$\begin{aligned} \sqrt{N+M}(\widehat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}^*) &= \sqrt{\frac{N+M}{N}}\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) - \sqrt{\frac{N+M}{M}}\sqrt{M}(\widehat{\boldsymbol{\beta}}^* - \boldsymbol{\beta}^* + \boldsymbol{\eta}) \\ &= \sqrt{\frac{N+M}{N}}\sqrt{N}(\widehat{\boldsymbol{\beta}} - \boldsymbol{\beta}) - \sqrt{\frac{N+M}{M}}\sqrt{M}(\widehat{\boldsymbol{\beta}}^* - \boldsymbol{\beta}^*) - \sqrt{N+M}\boldsymbol{\eta} \\ &= \mathbf{B}_{N,M} - \sqrt{N+M}\boldsymbol{\eta}. \end{aligned}$$

Let $\widetilde{T}_{N,M} := \mathbf{B}_{N,M}^T\widehat{\mathbf{\Delta}}_{N,M}^{-1}\mathbf{B}_{N,M}$. Then, by Theorem 4.6, $\widetilde{T}_{N,M} \xrightarrow{d} \chi^2(p)$. Now we have

$$\begin{aligned} T_{N,M} &= (N+M)(\widehat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}^*)^T(\widehat{\mathbf{\Delta}}_{M,N})^{-1}(\widehat{\boldsymbol{\beta}} - \widehat{\boldsymbol{\beta}}^*) \\ &= (\mathbf{B}_{N,M} - \sqrt{N+M}\boldsymbol{\eta})^T(\widehat{\mathbf{\Delta}}_{M,N})^{-1}(\mathbf{B}_{N,M} - \sqrt{N+M}\boldsymbol{\eta}) \\ &= \mathbf{B}_{N,M}^T(\widehat{\mathbf{\Delta}}_{M,N})^{-1}\mathbf{B}_{N,M} - 2\sqrt{N+M}\boldsymbol{\eta}^T(\widehat{\mathbf{\Delta}}_{M,N})^{-1}\mathbf{B}_{N,M} + (N+M)\boldsymbol{\eta}^T(\widehat{\mathbf{\Delta}}_{M,N})^{-1}\boldsymbol{\eta} \\ &= \widetilde{T}_{N,M} - 2\sqrt{N+M}\boldsymbol{\eta}^T(\widehat{\mathbf{\Delta}}_{M,N})^{-1}\mathbf{B}_{N,M} + (N+M)\boldsymbol{\eta}^T(\widehat{\mathbf{\Delta}}_{M,N})^{-1}\boldsymbol{\eta}. \end{aligned}$$

It follows that

$$\begin{aligned} (N+M)^{-1}T_{N,M} &= (N+M)^{-1}\widetilde{T}_{N,M} - 2(\sqrt{N+M})^{-1}\boldsymbol{\eta}^T(\widehat{\mathbf{\Delta}}_{M,N})^{-1}\mathbf{B}_{N,M} + \boldsymbol{\eta}^T(\widehat{\mathbf{\Delta}}_{M,N})^{-1}\boldsymbol{\eta} \\ &\xrightarrow{d} (N+M)^{-1}\chi^2(p) - 2(\sqrt{N+M})^{-1}\boldsymbol{\eta}^T\mathbf{\Delta}^{-1}\mathbf{B} + \boldsymbol{\eta}^T\mathbf{\Delta}^{-1}\boldsymbol{\eta} \\ &\xrightarrow{d} \boldsymbol{\eta}^T\mathbf{\Delta}^{-1}\boldsymbol{\eta}, \end{aligned}$$

where $\boldsymbol{\eta}^T\mathbf{\Delta}^{-1}\mathbf{B} \sim N(0, \boldsymbol{\eta}^T\mathbf{\Delta}^{-1}\boldsymbol{\eta})$. This implies $T_{N,M} \xrightarrow{P} \infty$. □

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