Consider the instantaneous probability of <u>some</u> transition for a generalized version of models we've considered:

$\underset{\sim}{X}(t) =$ state vector at time t $\quad$ ($K \times 1$ vector)

$Z = \{ \underset{K\times 1}{\underset{\sim}{z}_1}, \underset{K\times 1}{\underset{\sim}{z}_2}, \ldots, \underset{K\times 1}{\underset{\sim}{z}_m} \} =$ set of all possible state transitions (excluding $\underset{\sim}{0}$ = no change)

$$P\left(\underset{\sim}{X}(t+h) - \underset{\sim}{X}(t) \in \underset{\sim}{z} \;\middle|\; \underset{\sim}{X}(t) = \underset{\sim}{x}\right) = \begin{cases} \lambda(\underset{\sim}{x}, t)\,h + o(h) & \text{for } \underset{\sim}{z} \in Z \\ 1 - \lambda(\underset{\sim}{x}, t)\,h + o(h) & \text{for } \underset{\sim}{z} = \underset{\sim}{0} \\ 0 & \text{otherwise} \end{cases}$$

<u>EXAMPLES</u>

1. $X(t) =$ scalar, $Z = \{+1\}$.

$$P\left(X(t+h) - X(t) = +1 \mid X(t) = x\right) = \underbrace{\lambda}_{\lambda(x,t) = \text{constant}} h + o(h)$$

<u>HOMOGENEOUS POISSON PROCESS</u>

2. $X(t) =$ scalar, $Z = \{+1, -1\}$

$$P\left(X(t+h) - X(t) \in \underset{\sim}{Z} \mid X(t) = x\right) = \underbrace{(\beta x + \delta x)}_{\lambda(x,t) = \lambda(x)} h + o(h)$$

<u>LINEAR BIRTH AND DEATH</u>

3. $\underset{\sim}{X}(t) = \begin{bmatrix} S(t) \\ I(t) \end{bmatrix}$, $\quad Z = \left\{ \begin{bmatrix} +1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ +1 \end{bmatrix}, \begin{bmatrix} 0 \\ +1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \end{bmatrix} \right\}$

$$P\left(\underset{\sim}{X}(t+h) - \underset{\sim}{X}(t) \in Z \mid \underset{\sim}{X}(t) = \underset{\sim}{x}\right) = \underbrace{\left(\frac{mS}{N}(N-I-S) + x_S S + m\delta\frac{IS}{N} + gS + m\frac{I}{N}(N-I-S) + x_I I\right)}_{\lambda(\underset{\sim}{x}) = \lambda\left(\left[\begin{smallmatrix} S \\ I \end{smallmatrix}\right]\right)} h + o(h)$$

<u>GOG ET AL.</u>

(4) $X(t) =$ scalar, $\quad Z = \{+1\}$.

$$P\big(X(t+h) - X(t) = +1 \;\big|\; X(t) = x\big) = \lambda(t)\cdot h + o(h)$$

<u>INHOMOGENEOUS POISSON PROCESS</u>

(5) $X(t) =$ scalar, $\quad Z = \{+1, -1\}$

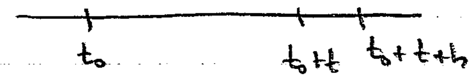
$$P\big(X(t+h) - X(t) \in Z \;\big|\; X(t) = x\big) = \Big(\underbrace{\underset{\text{nonseasonal birth rate}}{rx} + \underset{\text{seasonal death rate}}{r\frac{x^2}{K}\left(1 + a\cos(\omega t)\right)}}_{\lambda(x,t)}\Big)h + o(h)$$

<u>SEASONAL LOGISTIC</u>

* WHEN $\lambda(\underset{\sim}{x}, t)$ EXPLICITLY DEPENDS ON t, NOT JUST $\underset{\sim}{x}$,

THEN SOJOURN TIMES ARE <u>NO</u> <u>LONGER</u> <u>EXPONENTIAL</u>.

What is the sojourn time?



Let $f_{\underset{\sim}{x}}(t, t_0) = P\left(\text{no events in } (t_0, t_0+t) \mid \underset{\sim}{X}(t_0) = \underset{\sim}{x}\right) \leftarrow$ SOJOURN EXCEEDS t

Then $f_{\underset{\sim}{x}}(t+h, t_0) = P\left(\text{no events in } (t_0, t_0+t+h) \mid \underset{\sim}{X}(t_0) = \underset{\sim}{x}\right)$

$$= P\left(\text{none in } (t_0, t_0+t) \text{ and none in } (t_0+t, t_0+t+h) \mid \underset{\sim}{X}(t_0) = \underset{\sim}{x}\right)$$

$$= f_{\underset{\sim}{x}}(t, t_0)\left(1 - \lambda(\underset{\sim}{x}, t_0+t)h + o(h)\right)$$

$$\Rightarrow \frac{f_{\underset{\sim}{x}}(t+h, t_0) - f_{\underset{\sim}{x}}(t, t_0)}{h} = \frac{-\lambda(\underset{\sim}{x}, t_0+t) f_{\underset{\sim}{x}}(t, t_0) h}{h} + \frac{o(h)}{h}.$$

$$\Longrightarrow f'_{\underset{\sim}{x}}(t, t_0) = -\lambda(\underset{\sim}{x}, t_0+t)\, f_{\underset{\sim}{x}}(t, t_0)$$

$$\Longrightarrow f_{\underset{\sim}{x}}(t, t_0) = \exp\left\{-\int_0^t \lambda(\underset{\sim}{x}, t_0+s)\, ds\right\}.$$

$$= P\left(\begin{array}{l}\text{sojourn time starting from } t_0 \\ \text{exceeds } t\end{array}\right).$$

THIS IS NOT 1 − EXPONENTIAL CDF. $= \exp(-\text{constant} \cdot t)$

$\ast$ HOW TO SIMULATE? ONE WAY: DRAW $S^* \sim \text{Exp}(1)$.

Define $\Lambda(t) = \int_0^t \lambda(\underset{\sim}{x}, t_0+s)\, ds$, and $S = \Lambda^{-1}(S^*)$

Then $P(S \geq t) = P(\Lambda^{-1}(S^*) \geq t)$

$$= P(S^* \geq \Lambda(t))$$

$$= \exp\{-\Lambda(t)\}$$

$$= \exp\left\{-\int_0^t \lambda(\underset{\sim}{x}, t_0+s)\, ds\right\}, \text{ as required.}$$

This method assumes you know $\lambda(\underset{\sim}{x}, t_0 + s)$ for $s \in [0, T]$ where T is big enough to ensure $\Lambda(T) \geq S^*$.

If we have evaluations

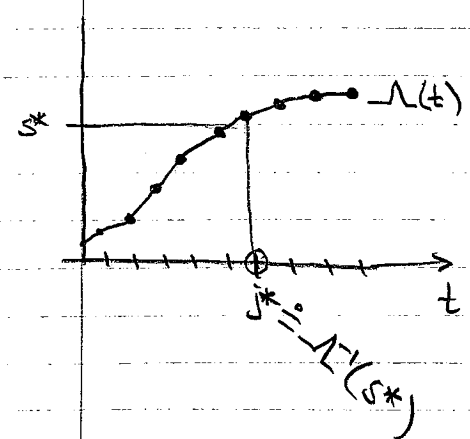
$$\left(\lambda(\underset{\sim}{x}, t_0+0),\ \lambda(\underset{\sim}{x}, t_0+\Delta t),\ \lambda(\underset{\sim}{x}, t_0+2\Delta t), \ldots,\ \lambda(\underset{\sim}{x}, t_0+T)\right)$$

then

$$\Lambda^{-1}(S^*) \doteq t_0 + j^* \Delta t \quad \text{where } j^* \text{ is}$$

first step among $j = 0, 1, 2, \ldots$

at which $\Lambda(t_0 + j\Delta t) > S^*$.



Ex 6: $\underset{\sim}{X}(t) = \begin{bmatrix} S_R(t) \\ I_R(t) \\ R_R(t) \end{bmatrix}$, $\quad Z = \left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \ldots, \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix} \right\}$

$$P\left(\underset{\sim}{X}(t+h) - \underset{\sim}{X}(t) \in Z \;\middle|\; \underset{\sim}{X}(t) = \begin{pmatrix} s \\ i \\ r \end{pmatrix} \right) =$$

$$\Bigg[\underbrace{f_0(s,i,r) + \left\{ \beta_R s \left(1 - \exp\left(-a(s+i+r)\right)\right) \Big/ (s+i+r) \right\} F(t)}_{\lambda(\underset{\sim}{x},\,t)} \Bigg] h + o(h)$$

KEELING AND GILLIGAN
WITH DETERMINISTIC FLEA DYNAMICS

$\lambda(\underset{\sim}{x}, t)$ explicitly depends on t through the free flea number, obtained as a solution to a differential equation.

Note that $\lambda(\underset{\sim}{x}, t) = f_0(\underset{\sim}{x}) + f_1(\underset{\sim}{x}) F(t)$, where $\underset{\sim}{x} = (s, i, r)$.

constants.

* So, solve the DE to get $F(t)$, then integrate $\lambda(\underset{\sim}{x}, t)$ and invert as in previous example.

This is one example of a PIECEWISE-DETERMINISTIC MARKOV PROCESS

(See Davis, M.H.A. (1984). J. R. Statist. Soc. B 46: 353–388.)

NOTES

- This is quite a lot of work! May not be a big time savings ← analytical or fast numerical solution of DE would be nice
  unless time scales of the two dynamic processes are quite different (eg, Salmonella in cows).

- Much simpler simulation routine is to make the rates piecewise constant function of time, changing at the event times. Doesn't make biological sense (the time of my death affects your death rate; if I had died an hour earlier, your survival probability would be higher!), but might be a useful, quick approximation.
  (Renshaw's book seems to use this method.)

- One other possibility: use the instantaneous probabilities to develop mean vector and covariance matrix (to terms of order $O(h)$) then approximate the system using a STOCHASTIC DIFFERENTIAL EQUATION

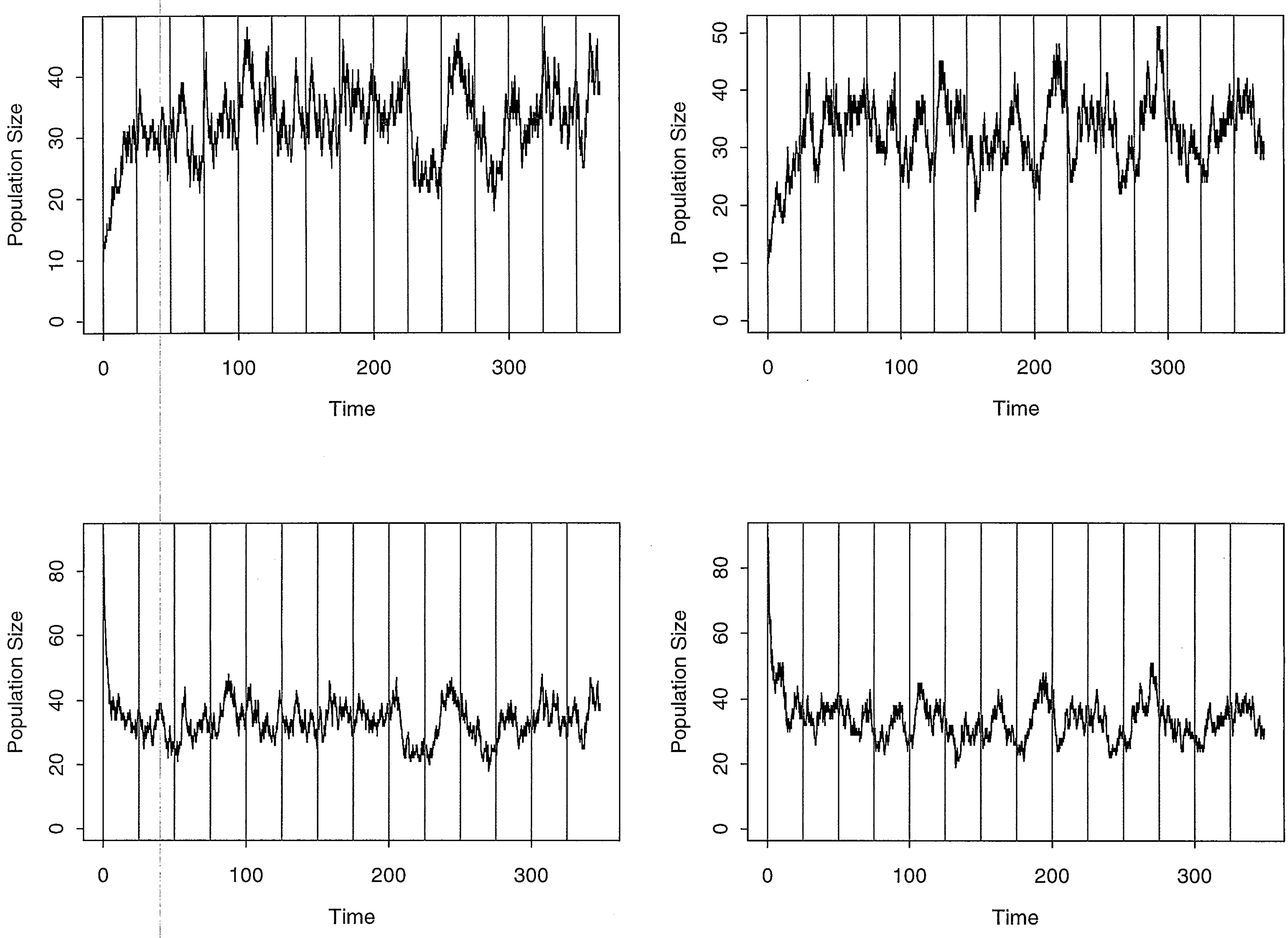
$$d\underset{\sim}{X}(t) = m(\underset{\sim}{X}, t)\,dt + v^{1/2}(\underset{\sim}{X}, t)\,dB(t),$$

where $B(t)$ is standard Brownian motion (Wiener process).

Renshaw, Ch. 8, sketches this idea. See Nisbet and Gurney (1982) for details.

```
"seasonal.logistic"<-
function(r=0.2,a=0.5,K=50,omega=2*pi/25,nsteps=1000,grid=10000,initial.X=10,iseed=123)
{
	#
	# Default values come from Renshaw, Modelling Biological Populations in Space and Time, p.
	227
	#
	#
	# Set the random number seed for iseed in [0,1023].
	set.seed(iseed)
	#
	#
	#
	X <- rep(0, nsteps)
	sojourn <- X
	W <- X
	X[1] <- initial.X
	W[1] <- sojourn[1]
	#
	#
	# Create a grid of time points covering over 99.99% of the range of Exponential(1)
	#
	tt<-(0:grid)*10/grid
	#
	#
	# Note that this is time = 0.
	#
	for(i in 2:nsteps) {
		# Model has a non-seasonal birth rate r*X and seasonal death rate (r*X^2/K)*(1+a*cos
	(omega*t)),
		# so it exhibits logistic growth with seasonal fluctuations.  Need to integrate the
	total
		# birth+death rate and apply the inverse of this integral to a standard exponential.
		#
		t0<-W[i-1]
		ff<-a*(sin(omega*(tt+t0))-sin(omega*t0))
		B<-r*X[i-1]
		tmp<-r*X[i-1]^2/K
		Lambda<-(B+tmp)*tt+tmp*ff
		#
		# Draw standard exponential to be transformed into sojourn time.
		#
		S.star<-rexp(1,1)
		#
		# Evaluate inverse of Lambda at S.star.
		#
		index<-grid+2-length(tt[Lambda>S.star])
		sojourn[i]<-tt[index]
		birth<-B
		death<-tmp*(1+a*cos(omega*tt[index]))
		#
		# In case the sojourn time was even longer than anticipated...
		#
		if(is.na(sojourn[i])){
			sojourn[i]<-10;
			birth<-B[grid+1];
			death<-tmp[grid+1]*(1+a*cos(omega*tt[grid+1]))
					}
		#
		# Note that the transition probabilities depend on the instantaneous rates,
		# not the integrated rates.
		#
		rates<-c(birth,death)
		#
		# Make sure all rates are non-negative.
		#
		rates[rates < 0] <- 0
		#
		# Check to see if the population is extinct.
		#
		overall.rate <- sum(rates)
		if(overall.rate <= 0 || X[i-1] == 0) {
```

```
            # population is extinct
            sojourn[i] <- 0
            W[i] <- W[i - 1]
        }
        #end if extinct
        if(overall.rate > 0) {
            W[i] <- W[i - 1] + sojourn[i]
            #
            # Determine the probabilities of each state transition and
            # draw a random state transition.
            #
            st <- sample(1:2, size = 1, prob = rates/overall.rate)
            if(st == 1) {
                # birth
                X[i] <- X[i - 1] + 1
            }
            if(st == 2) {
                # death
                X[i] <- X[i - 1] - 1
            }
        }
    }
    # end for loop on i
    plot(c(0,W),c(0,X), type = "n", xlab = "Time", ylab = "Population Size")
    segments(W[-1], X[ - nsteps], W[-1], X[-1])
    segments(W[ - nsteps], X[ - nsteps], W[-1], X[ - nsteps])
    grid<-(0:1000)*max(W)/1000
    top<-max(X)/5
    f<-1+a*cos(omega*grid)
    f<-(f-min(f))/max(f)
    #lines(grid,top*f,col=8)
    ncycles<-(max(W)*omega/(2*pi))%/%1
    abline(v=(0:ncycles)*2*pi/omega,col=8)
        return()
}
```


Population Size
Time
0
10
20
30
40
50
60
80
100
200
300