LINEAR BIRTH PROCESS = YULE PROCESS = YULE-FURRY PROCESS

$\lambda_k = \beta \cdot k$, $\beta > 0$.

<u>Interpretation of β:</u> Consider a <u>single</u> <u>individual</u> in the population.

Assume $P\Big(1 \text{ birth to this individual in } (t, t+h] \,\Big|\, X(t)=k\Big) = \beta h + o(h)$

$P\Big(> 1 \text{ birth } '' \qquad '' \,\Big|\, X(t)=k\Big) = o(h)$

and $P\Big(0 \text{ births } '' \qquad '' \,\Big|\, X(t)=k\Big) = 1-\beta h + o(h)$.

Assume individuals give birth independently.

Then

$$P\Big(\underbrace{X(t+h)-X(t)=1}_{\substack{\text{one of the}\\ k \text{ individuals gives birth}}} \,\Big|\, X(t)=k\Big) = \underbrace{\binom{k}{1}\big(\beta h + o(h)\big)^1\big(1-\beta h+o(h)\big)^{k-1}}_{\substack{\text{binomial probability;}\\ k \text{ indep. trials with success}\\ \text{probability } \beta h + o(h)}} \quad (*)$$

Now $\big(1-\beta h + o(h)\big)^{k-1} = 1 + \underset{\substack{\uparrow\\ \text{"big oh of } h\text{"}}}{O(h)}$ ← stuff that is the same size as h, or smaller, like h^2, $o(h)$, $h\,o(h)$, ... $O(h) \to 0$ as $h \to 0$.

so $(*) = k \cdot \big(\beta h + o(h)\big)\big(1 + O(h)\big)$

$= k\beta h + k\beta h\, O(h) + k\, o(h) + k\, o(h) O(h)$

$= k\beta h + o(h)$, from which we get $\lambda_k = \beta \cdot k$.

Similarly, $P\big(X(t+h)-X(t)=0 \mid X(t)=k\big) = \binom{k}{0}\big(\beta h+o(h)\big)^0\big(1-\beta h+o(h)\big)^k$

$= 1 - k\beta h + o(h)$

So $\beta = \dfrac{\#\text{births}}{\text{unit time}}$ per individual. Again, β is not a probability.

INFERENCE FOR β IN THE YULE PROCESS

Assume you know n_0 = initial population size and $S_{0+n_0}, S_{1+n_0}, \ldots, S_{n-1+n_0}$ the times preceding each of n births.

likelihood of $\beta = \mathcal{L}(\beta) = \prod_{k=1}^{n} \left[\beta(n_0+k-1) \exp\left\{-\beta(n_0+k-1) S_{n_0+k-1}\right\} \right]$

and log-likelihood is $\ln \mathcal{L}(\beta) = \sum_{k=1}^{n} \left[\ln\beta + \ln(n_0+k-1) - \beta(n_0+k-1) S_{n_0+k-1} \right]$

$$= n \ln\beta + \text{constant} - \beta \sum_{k=1}^{n} (n_0+k-1) S_{n_0+k-1}$$

$$0 \overset{\text{set}}{=} \frac{\partial \ln \mathcal{L}}{\partial \beta} = \frac{n}{\beta} - \sum (n_0+k-1) S_{n_0+k-1} \Rightarrow \hat{\beta}_{MLE} = \frac{n}{\sum (n_0+k-1) S_{n_0+k-1}}.$$

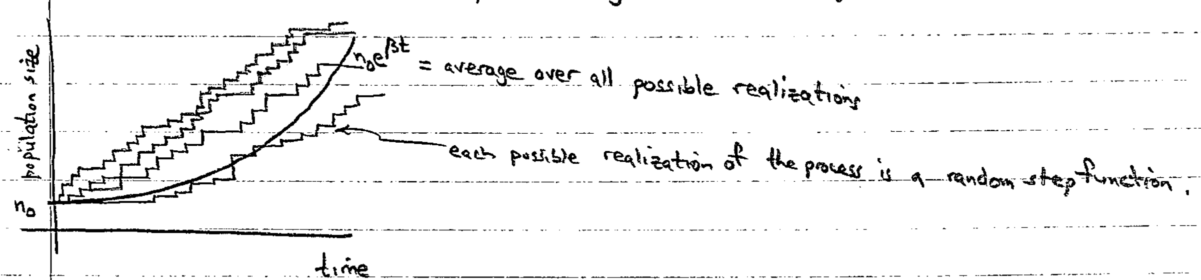
RELATION TO DETERMINISTIC MODEL

For the linear birth process, it can be shown that

$$P\left(X(t)=m \mid X(0)=n_0\right) = \binom{m-1}{n_0-1} e^{-\beta n_0 t}\left(1-e^{-\beta t}\right)^{m-n_0} \quad \text{for } m=n_0, n_0+1, \ldots$$

This is a negative binomial probability distribution, with

mean function $\underbrace{E\left[X(t)\right] = n_0 e^{\beta t}}_{\text{exponential growth on average.}}$ and variance $Var\left(N(t)\right)=n_0\left(1-e^{-\beta t}\right)e^{2\beta t}$.



Now $n_0 e^{\beta t}$ is the solution to $dy = \beta y\, dt$ subject to $y(0)=n_0$

↑ fixed, deterministic dy.

Random realizations can be thought of as

$$\text{random "}dy\text{"} = \begin{cases} 1 & \text{with probability } \beta y\, dt \\ 0 & \text{with probability } 1-\beta y\, dt \end{cases}$$

previously $\beta K h$ for $X(t)=K$.

so average "dy" $= (1)(\beta y\, dt) + (0)(1-\beta y\, dt) = \beta y\, dt$

PURE DEATH PROCESSES

Markov process that moves through states $N, N-1, \ldots, 2, 1, 0$ (extinction), spending a random time S_k sojourning in each state k, where $S_k \sim \text{Exp}(\mu_k)$, $\mu_k > 0$, and all sojourn times are independent.

Infinitesimal probabilities:

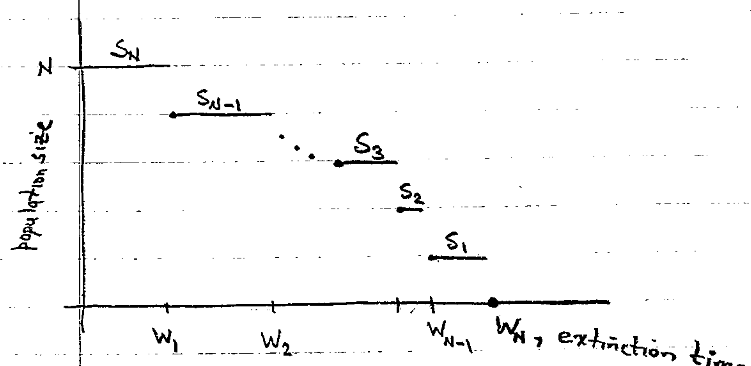
① $P(X(t+h)-X(t) = -1 \mid X(t)=k) = \mu_k h + o(h)$

② $P(X(t+h)-X(t) = 0 \mid X(t)=k) = 1-\mu_k h + o(h)$

③ $P(X(t+h)-X(t) > 0 \mid X(t)=k) = 0$.

SIMULATION OF A PURE DEATH PROCESS

- Start at time 0 with $X(0) = N$. Set the random number seed.
- Simulate $S_N \sim \text{Exp}(\mu_N)$ $W_1 = S_N$. $X(W_1) = X(0) - 1$.
- For $i = 2, 3, \ldots, N$:
    - Simulate $S_{N-i+1} \sim \text{Exp}(\mu_{N-i+1})$, independent of $S_N, S_{N-1}, \ldots, S_{N-i+2}$
    - Set $W_i = W_{i-1} + S_{N-i+1}$, $X(W_i) = X(W_{i-1}) - 1$.



LINEAR DEATH PROCESS $\mu_k = \mu \cdot k$

BIRTH AND DEATH PROCESSES

Markov process on the states $\{0, 1, 2, \dots\}$ or $\{0, 1, 2, \dots, N\}$ with stationary transition probabilities given via the following assumptions:

① $P\big(X(t+h) - X(t) = 1 \mid X(t) = k\big) = \lambda_k h + o(h)$ (birth)

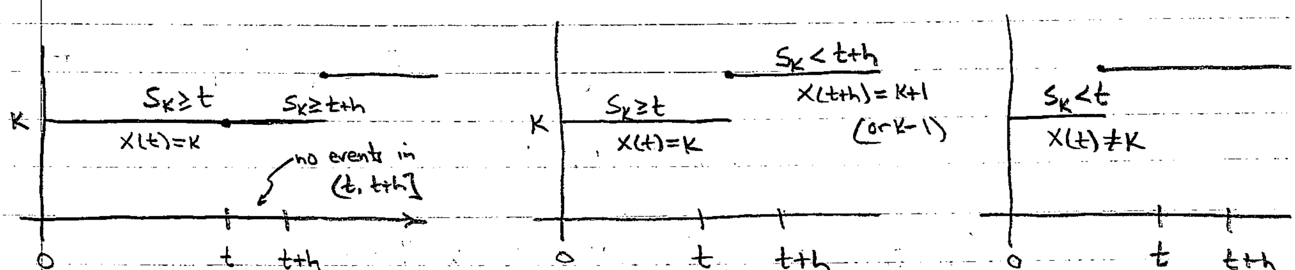
② $P\big(X(t+h) - X(t) = -1 \mid X(t) = k\big) = \mu_k h + o(h)$ (death)

③ $P\big(X(t+h) - X(t) = 0 \mid X(t) = k\big) = 1 - (\lambda_k + \mu_k)h + o(h)$

where $\mu_0 = 0$ (no deaths if nobody is alive)

SOJOURN TIMES

Let S_k = sojourn time of $X(t)$ in state k. Define $G_k(t) = P(S_k \ge t)$.



can use $0, t, t+h$ instead of $s, s+t, s+t+h$ by stationarity.

$$G_k(t+h) = P(S_k \ge t+h) = P\big(S_k \ge t, \text{ and no events in } (t, t+h]\big)$$

$$= P\big(X(t+h) - X(t) = 0 \mid S_k \ge t\big)\, P(S_k \ge t)$$

$$= P\big(X(t+h) - X(t) = 0 \mid X(t) = k\big)\, G_k(t)$$

$$= \big(1 - (\lambda_k + \mu_k)h + o(h)\big)\, G_k(t)$$

$$\Rightarrow \frac{G_k(t+h) - G_k(t)}{h} = \frac{-(\lambda_k + \mu_k)h\, G_k(t)}{h} + \frac{o(h)\, G_k(t)}{h}$$

$$\Rightarrow \quad G_k'(t) = -(\lambda_k+\mu_k)\, G_k(t) \quad \Rightarrow \quad G_k(t) = c\, e^{-(\lambda_k+\mu_k)t}.$$

But $G_k(0) = P(S_k \geq 0) = 1 \Rightarrow G_k(t) = e^{-(\lambda_k+\mu_k)t}$,

Exponential $(\lambda_k+\mu_k)$.

So <u>some</u> event (birth or death) ends the sojourn, where events arrive at rate $\lambda_k+\mu_k$. Which event is it?

birth with probability $\dfrac{\lambda_k}{\lambda_k+\mu_k}$, death w.p. $\dfrac{\mu_k}{\lambda_k+\mu_k}$.

(if $\lambda_k \equiv 0$ we get death w.p. 1 = pure death,

and if $\mu_k \equiv 0$ we get pure birth).

<u>SIMULATION OF A BIRTH AND DEATH PROCESS</u>

- Start at time 0 with $X(0) = n_0$. Set the random number seed.
- Simulate $S_1 \sim \text{Exp}(\lambda_1 + \mu_1)$. $W_1 = S_1$.
  - Simulate $U_1 \sim \text{Uniform}(0,1)$. If $U_1 < \dfrac{\lambda_1}{\lambda_1+\mu_1}$, $X(W_1) = X(0)+1$.
    Else $X(W_1) = X(0)-1$.
- For $i = 2, 3, \ldots$
  - Simulate $S_i \sim \text{Exp}(\lambda_i + \mu_i)$. $W_i = W_{i-1} + S_i$.
  - Simulate $U_i \sim \text{Unif}(0,1)$.

$$X(W_i) = X(W_{i-1}) + \begin{cases} 1, & \text{if } U_i < \dfrac{\lambda_i}{\lambda_i+\mu_i} \\ -1, & \text{otherwise.} \end{cases}$$

LINEAR GROWTH WITH IMMIGRATION $\lambda_k = \lambda \cdot K + a$, $\mu_k = \mu \cdot K$,

$\lambda, \mu, a > 0.$

$$E[X(t)] = \begin{cases} at + n_0, & \text{if } \lambda = \mu \\ \dfrac{a}{\lambda - \mu}\left\{e^{(\lambda-\mu)t} - 1\right\} + n_0 e^{(\lambda-\mu)t}, & \text{if } \lambda \neq \mu. \end{cases}$$

Infinitesimal probabilities: $P(X(t+h) - X(t) = +1 \mid X(t) = K) = (\lambda K + a)h + o(h)$

$$P(X(t+h) - X(t) = -1 \mid X(t) = K) = \mu K h + o(h)$$

Corresponding DE: $dy = (+1)(\lambda y + a)\,dt + (-1)(\mu y)\,dt$

or $\dfrac{dy}{dt} = (\lambda - \mu)y + a.$ Note that $E[X(t)]$ satisfies the DE in both cases.

Now suppose $a = 0$ so that $\lambda_0 = 0$, meaning state 0 is <u>absorbing</u>

$\Updownarrow$

extinct population with no chance of rescue by immigrants.

Can be shown that $P(\text{extinction} \mid X(0) = n_0) = \begin{cases} (\mu/\lambda)^{n_0}, & \text{if } \lambda > \mu. \\ 1, & \text{if } \lambda \leq \mu. \end{cases}$

- <u>Note</u> if $\lambda = \mu$, population is <u>certain</u> to go extinct, yet $E[X(t)] = n_0$!

  (mean value alone often fails to describe stochastic behavior.)