ST 430
HW #1
Solution Sheet

5.2.2 $L(p) = p(1-p)(1-p)p(1-p) = p^2(1-p)^3$

$L(1/3) = \left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 = \frac{8}{243}$ is greater than

$L(1/2) = \left(\frac{1}{2}\right)^5 = \frac{1}{32}$, so $\hat{p} = 1/3$.

5.2.4 $L(\theta) = \prod_{i=1}^{n} \frac{\theta^{2k_i} e^{-\theta^2}}{k_i!} = \frac{\theta^{2\sum_{i=1}^{n} k_i} e^{-n\theta^2}}{\prod_{i=1}^{n} k_i!}$.

$\ln L(\theta) = \left(2\sum_{i=1}^{n} k_i\right)(\ln \theta) - n\theta^2 + \ln \prod_{i=1}^{n} k_i!$

$\frac{d \ln L(\theta)}{d\theta} = 0$ implies $\frac{2\sum_{i=1}^{n} k_i}{\theta} - 2n\theta = \frac{2\sum_{i=1}^{n} k_i - 2n\theta^2}{\theta} = 0$

or $\hat{\theta} = \sqrt{\frac{\sum_{i=1}^{n} k_i}{n}}$

5.2.6 $L(\theta) = \prod_{i=1}^{4} \frac{\theta}{2\sqrt{y_i}} e^{-\theta\sqrt{y_i}} = \frac{\theta^4}{16\prod_{i=1}^{4}\sqrt{y_i}} e^{-\theta\sum_{i=1}^{4}\sqrt{y_i}}$

$\ln L(\theta) = 4\ln\theta - \ln\left(16\prod_{i=1}^{4}\sqrt{y_i}\right) - \theta\sum_{i=1}^{4}\sqrt{y_i}$

$\frac{d \ln L(\theta)}{d\theta} = \frac{4}{\theta} - \sum_{i=1}^{4}\sqrt{y_i}$.

$\frac{d \ln L(\theta)}{d\theta} = 0$ implies $\hat{\theta} = \frac{4}{\sum_{i=1}^{4}\sqrt{y_i}} = \frac{4}{8.766} = 0.456$

5.2.10 $L(\theta) = \prod_{i=1}^{6} \frac{2y_i}{1-\theta^2} = \frac{64\prod_{i=1}^{n} y_i}{(1-\theta^2)^6}$, if $\theta \le y_1, y_2, \ldots, y_n \le 1$ and 0 otherwise. If $\theta > y_{\min}$, then $L(\theta) = 0$. So $\hat{\theta} \le y_{\min}$. Also, to maximize $L(\theta)$, minimize the denominator, which in turn means maximize θ. Thus $\hat{\theta} \ge y_{\min}$. We conclude that $\hat{\theta} = y_{\min}$, which for these data is 0.92.

5.2.18 For Y exponential, $E(Y) = 1/\lambda$. Then $1/\lambda = \bar{y}$ implies $\hat{\lambda} = 1/\bar{y}$.

5.2.20 $E(Y) = \int_k^\infty y\theta k^\theta \left(\frac{1}{y_i}\right)^{\theta+1} dy = \theta k^\theta \int_k^\infty y^{-\theta} dy = \frac{\theta k}{\theta - 1}$

Setting $\frac{\theta k}{\theta - 1} = \bar{y}$ gives $\hat{\theta} = \bar{y}/(\bar{y} - k)$

5.2.22 $E(Y) = \mu$, so $\hat{\mu} = \bar{y}$. $E(Y^2) = \sigma^2 + \mu^2$. Then substitute $\hat{\mu} = \bar{y}$ into the equation for $E(Y^2)$ to obtain $\hat{\sigma}^2 + \bar{y}^2 = \frac{1}{n}\sum_{i=1}^n y_i^2$ or $\hat{\sigma}^2 = \frac{1}{n}\sum_{i=1}^n y_i^2 - \bar{y}^2$

5.4.2 $f_{\hat{\theta}}(u) = n\frac{1}{\theta}\left(\frac{u}{\theta}\right)^{n-1} I_{(0,\theta)}(u) = \frac{n}{\theta^n} u^{n-1} I_{(0,\theta)}(u)$

$P(\hat{\theta} \le x) = F_{\hat{\theta}}(x) = \int_{-\infty}^x f_{\hat{\theta}}(u)\,du = \int_{-\infty}^x \frac{n}{\theta^n} u^{n-1} I_{(0,\theta)}(u) = \frac{u^n}{\theta^n}\Big|_0^x I_{(0,\theta)}(x) + I_{[\theta,\infty)}(x) = \frac{x^n}{\theta^n} I_{(0,\theta)}(x) + I_{[\theta,\infty)}(x)$

$\theta = 3$

$P(|\hat{\theta} - 3| < 0.2) = P(2.8 < \hat{\theta} < 3.2) = P(\hat{\theta} < 3.2) - P(\hat{\theta} \le 2.8) = P(\hat{\theta} \le 3.2) - P(\hat{\theta} \le 2.8) = F_{\hat{\theta}}(3.2) - F_{\hat{\theta}}(2.8) =$

$1 - F_{\hat{\theta}}(2.8) = 1 - \frac{2.8^n}{3^n} = 1 - \left(\frac{2.8}{3}\right)^n$

a. $n = 6$ $P(|\hat{\theta} - 3| < 0.2) = 1 - \left(\frac{2.8}{3}\right)^6 \doteq 0.339$

b. $n = 3$ $P(|\hat{\theta} - 3| < 0.2) = 1 - \left(\frac{2.8}{3}\right)^3 \doteq 0.187$

5.4.6 $f_{Y_{\min}}(y) = n\frac{1}{\theta}\left(1 - \frac{y}{\theta}\right)^{n-1}$, so $E(Y_{\min}) = n\frac{1}{\theta}\int_0^\theta y\left(1 - \frac{y}{\theta}\right)^{n-1} dy$

Integration by parts yields $E(Y_{\min}) = \frac{\theta}{n+1}$. An unbiased estimator would be $(n + 1)Y_{\min}$.

5.4.8 a) $f_{Y_3'} = 12\left(\frac{Y}{\theta}\right)^2\left(1 - \frac{y}{\theta}\right)\frac{1}{\theta} = \frac{12}{\theta^4}[y^2(\theta - y)]$

$E(Y_3') = \frac{3}{5}\theta$, so the unbiased estimator is $\frac{5}{3}Y_3'$.

b) $\frac{5}{3}Y_3' = \frac{5}{3}18 = 30$

c) Suppose the sample were 10, 14, 18, 31. The estimate for θ is 30, but the largest observation 31 falls outside of the [0, 30] interval.