ST430

HW #2

Solutions

5.4.10

[redacted] $E(Y^2) = \int_0^\theta y^2 \frac{1}{\theta} dy = \frac{\theta^2}{3}$, so $3Y^2$ is unbiased.

5.4.14

[redacted] $f_{Y_{\min}}(y) = nf_Y(y)(1 - F_Y(y))^{n-1} = n\frac{1}{\theta}e^{-y/\theta}[1-(1-e^{-y/\theta})]^{n-1} = n\frac{1}{\theta}e^{-ny/\theta}$. Then

$f_{nY_{\min}}(y) = \frac{1}{n} f_{Y_{\min}}\left(\frac{y}{n}\right) = \frac{1}{n} n \frac{1}{\theta} e^{-n\frac{y}{n}/\theta} = \frac{1}{\theta}e^{-y/\theta}$. $E(nY_{\min}) = \theta$, so $nY_{\min}$ is unbiased. Also,

$\frac{1}{n}\sum_{i=1}^{n} Y_i$ is unbiased since each Y_i is (see Question 5.4.5).

5.4.15

[redacted] $\hat{\theta}_n = \frac{1}{n}\sum_{i=1}^{n}(Y_i - \bar{Y})^2$, so $E(\hat{\theta}_n) = \frac{n-1}{n}\sigma^2$. This estimator is asymptotically unbiased since

$\lim_{n\to\infty} E(\hat{\theta}_n) = \lim_{n\to\infty} \frac{n-1}{n}\sigma^2 = \sigma^2$.

5.4.20

[redacted] $\text{Var}(\hat{\lambda}_1) = \text{Var}(X_1) = \lambda$. $\text{Var}(\hat{\lambda}_2) = \text{Var}(\bar{X}) = \lambda/n$.

$\text{Var}(\hat{\lambda}_2)/\text{Var}(\hat{\lambda}_1) = (\lambda/n)/\lambda = 1/n$

5.5.2 $\ln f_Y(Y;\theta) = -\ln\theta - Y/\theta$

$$\frac{\partial \ln f_Y(Y;\theta)}{\partial\theta} = -\frac{1}{\theta} + Y/\theta^2$$

$$\frac{\partial^2 \ln f_Y(Y;\theta)}{\partial\theta^2} = \frac{1}{\theta^2} - 2Y/\theta^3$$

$E\left[\frac{\partial^2 \ln f_Y(Y;\theta)}{\partial\theta^2}\right] = \frac{1}{\theta^2} - 2\theta/\theta^3 = \frac{-1}{\theta^2}$, so the Cramer-Rao bound is θ^2/n. Also, $\text{Var}(\hat{\theta}) = Var(\overline{Y}) = \text{Var}(Y)/n = \theta^2/n$, so $\hat{\theta}$ is a best estimator.

5.5.4 $\ln f_Y(Y;\mu) = -\ln\sqrt{2\pi}\sigma - \frac{1}{2}\frac{(Y-\mu)^2}{\sigma^2}$

$$\frac{\partial \ln f_Y(Y;\mu)}{\partial\mu} = \frac{(Y-\mu)}{\sigma^2}$$

$$\frac{\partial^2 \ln f_Y(Y;\mu)}{\partial\mu^2} = \frac{-1}{\sigma^2}$$

$E\left[\frac{\partial^2 \ln f_Y(Y;\mu)}{\partial\mu^2}\right] = \frac{-1}{\sigma^2}$, so the Cramer-Rao bound is σ^2/n. Also, $\text{Var}(\hat{\mu}) = Var(\overline{Y}) = \text{Var}(Y)/n = \sigma^2/n$, so $\hat{\mu}$ is an efficient estimator.

5.5.6 a) Y is a gamma random variable with parameters r and $1/\theta$ so $E(Y) = r\theta$.

Let $\hat{\theta} = \frac{1}{r}\overline{Y} = \frac{1}{rn}\sum_{i=1}^{n} Y_i$. Then $E(\hat{\theta}) = \frac{1}{rn}\sum_{i=1}^{n} E(Y_i) = \frac{1}{rn}nr\theta = \theta$

b) $\ln f_Y(Y;\theta) = -\ln(r-1)! - r\ln\theta + (r-1)\ln Y - Y/\theta$

$$\frac{\partial \ln f_Y(Y;\theta)}{\partial\theta} = -r/\theta + Y/\theta^2$$

$$\frac{\partial^2 \ln f_Y(Y;\theta)}{\partial\theta^2} = r/\theta^2 - 2Y/\theta^3$$

$$E\left[\frac{\partial^2 \ln f_Y(Y;\theta)}{\partial\theta^2}\right] = r/\theta^2 - 2(r\theta)/\theta^3 = -r/\theta^2$$

The Cramer-Rao bound is θ^2/rn.

$\text{Var}(\hat{\theta}) = \left(\frac{1}{rn}\right)^2 \sum_{i=1}^{n} Var(Y_i) = \left(\frac{1}{rn}\right)^2 nr\theta^2 = \theta^2/rn$, so $\hat{\theta}$ is a minimum-variance estimator.